FORMULARIO MATHEMATICO

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EDITO PER

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EDITIO V

(Tomo V de Formulario completo)

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PRÆFATIONE.

Omni progressu de Mathematica responde ad introductione de signos ideographico vel symbolos.

Symbolos plus antiquo, hodie adoptato, es cifras Indo-Arabico, 0, 1, 2... 9, facto Europæo in anno 1200 circa.

Utilitate plus evidente de cifras es brevitate in scriptura.

In secundo loco, cifras reduce vocabulario. Nam numeratione per cifras introduce nullo novo symbolo pro vocabulos « decem, viginti... centum, mille... » que es expresso per symbolos præcedente.

Systema de cifras Indo-Arabo-Europæo resulta ex systema de cifras Græco, per tres operatione:

a) Variatione de signos $\alpha\ \beta\ \gamma\ ...\ \vartheta$, in 1 2 3 ... 9, quod es indifferente;

b) additione fundamentale de symbolo 0, que nos lege per vocabulo Arabo « zero »;

c) suppressione de $\iota\quad \varkappa\quad \lambda\quad ...\quad \varrho\quad \sigma\ ...$
que resulta expresso per 10 20 30 ... 100 200 ...
ut combinatione de symbolos præcedente. (*)

Inter duo systema symbolico, illo que contine minore numero de symbolos es, in generale, plus perfecto.

Sed utilitate fundamentale de cifras es facilitate in calculos.

Archimede, per cifras Græco, post magno labore, calcula duo cifra decimale de π. Usu de cifras nostro duce Aryabhata, mathematico Indo de anno 500, ad calculo de 4 cifra, et mathematicos Europæo, in anno 1600 circa, ad calculo de 15 et 32 cifra decimale de π. (**)

Rationes de utilitate nunc exposito pro cifras, subsiste pro omni systema symbolico.

(*) Vide Formul. t. 3, p. 76.

(**) Vide Formul. t. 5, p. 255; cifras successivo es calculato per evolutione in serie.

Signos $+$, $-$ (a. 1500), $\times$ (a. 1600), $=$ (a. 1550), $>$ (a. 1650), e, π (a. 1700), Σ, Π (a. 1800), constitue calculo algebrico, et nos non pote concipe Algebra sine signos præcedente.

In realitate, magno parte de Algebra elementare es scripto in libros VII, VIII, IX, X de Euclide. (*)

Introductione de symbolos moderno redde libros de Euclide multo plus breve, elimina enorme vocabulario, mortuo in Algebra moderno; redde theorias præcedente plus facile, et permitte constructione de numeroso novo theoria.

Algebra moderno exprime ideas et in symbolos, et in lingua commune. Si nos fixa correspondentia univoco inter symbolos et vocabulos, per ex., si nos lege singulo symbolo

$$2 \quad + \quad 3 \quad = \quad 5 \qquad (1)$$

per duo plus tres æqua quinque (2),

tunc propositione symbolico (1) pote es lecto per (2). Vocabulos (2) es symbolos phonetico æquivalente ad symbolos graphico (1).

Sed, in tractatos commune, systema de vocabulos es multo plus numeroso que systema de symbolos. Introducto p. ex., signos $+$ (plus), et $\times$ (multiplicato per), Algebra habe constructo nullo symbolo speciale pro « summa, additione, termine, producto, factore, coefficiente... », et alio vocabulo, que in aliquo tractato es multo numeroso et inutile. Suppressione de isto vocabulos supprime in idem tempore definitione de illos et obscuritate in definitione. Libro fi plus facile et plus præciso.

Non solo plure vocabulo adoptato in tractatos responde ad nullo symbolo, sed sæpe non pote es repræsentato per symbolo, nam vocabulo repræsenta nullo ente reale. Per ex. post definitiones conveniente, resulta $2/3 = 4/6$. Nunc $2/3$ es fractione irreductibile, $4/6$ es reductibile; ergo expressione « fractione irreductibile » habe forma de classe de fractiones, sed non es classe de fractiones, nam non satisfac proprietate fundamentale de æqualitate (Formul. p. 8, p. 16).

(*) Libros VII, VIII, IX de Euclide es traducto in symbolos in RdM. t. 1 p. 10, et libro X in t. 2 p. 7; et es transcripto in præsente Formul., in pauco linea. Vide, in Bibliographia, citatione de Euclide relativo ad pag. 40-121.

Nos habe $2+3 = 5$; primo membro es binomio, secundo es monomio; ergo isto vocabulos indica pseudo-entes.

Nos habe $2^4 = 4^2$, exponente in primo membro es 4, in secundo es 2, ergo vocabulo « exponente » non indica ente reale. In generale, vocabulos relativo ad proprietates formale non pote es transformato in symbolos.

Si lingua commune, pro repræsentatione de symbolos, habe vocabulos in excessu, sæpe defice vocabulo proprio.

P. ex., symbolos graphico D (*Cauchy*), S (*Leibniz*), et symbolos phonetico correspondente « derivata, integrale », habe forma de literas de alphabeto, et de vocabulos de lingua commune; sed habe valore ideologico respondente in modo exacto ad nullo vocabulo de lingua vulgare; valore de vocabulos mathematico « derivata, integrale », non resulta ex etymologia, sed ex definitione scripto in libros de Calculo.

In anno 1800 circa incipe constructione de Calculo geometrico, hodie satis diffuso in libros de theoria et de applicationes, et si non omni libro adopta illo. (*) Si nos considera ut symbolos, vocabulos « recta, plano, sphæra, cylindro... » et innumero alio de Geometria antiquo et moderno, nos non construe Calculo geometrico. Illo resulta per introductione de novo ente, hodie vocato « vectore », que responde in modo exacto ad nullo vocabulo de Geometria præcedente, et que redde non necessario et sæpe inutile toto nomenclatura de Geometria.

Theoria de vectores exige nullo studio anteriore de Geometria, ut usu de cifras moderno non exige studio anteriore de cifras græco.

Hodie existe aliquo varietate in notationes et in nomenclatura de Calculo geometrico, ut jam in calculo algebrico. Ratione principale es in differente systema de operationes considerato.

Per ex. existe plure operatione distributivo super vectores, et omni pote es vocato producto, distincto per nomen « interno, alterno, quaternione... ». Ratione secundario es que plure Auctore puta licito introductione de symbolo novo, sine studio de

(*) Vide Formul. pag. 167.

notationes jam vigente. Diffusione de studio comparativo de notationes parallelo es facturo uniformitate. (*)

Ultimo in tempore, et non minus interessante de præcedentes, es Calculo logico. Incepto ab LEIBNIZ, (**) et secuto per BOOLE, SCHRÖDER, et plure alio, Calculo logico stude proprietates de ideas repræsentato in Formulario per signos $\cap \cup - = \supset$ ...

Ad theoria præcedente me adde distinctione inter symbolos ε, pro propositione individuale, et $\supset$, pro propositione universale, distinctione jam facto ab Logicos scholastico, et non ab Auctores de Algebra de Logica. Me supprime numeroso symbolo de nullo aut pauco utilitate, et da forma commodo ad symbolos que remane.

Combinatione de signos de Logica-Mathematica cum signos plus diffuso de Arithmetica, Algebra, Calculo infinitesimale, Calculo geometrico, da expressione symbolico completo de omni theoria de Mathematica.

Me habe reducto in symbolos aliquo theoria in:

Arithmetices principia, nova methodo exposita, Torino a. 1889.

Principii di Geometria, logicamente esposti, Torino a. 1889.

Plure Auctore, in *Revista de Mathematica*, a. 1891 et sequentes, applica idem methodo ad analysi de vario theoria.

Tunc me publica *Formulario Mathematico*, que es collectione de propositiones, expresso per solo symbolos.

Introductione, *Notations de Logique mathématique*, a. 1894, contine formulas de Logica-Mathematica.

Formulaire Mathématique, tome I, a. 1895, es composito in collaboratione cum professores F. CASTELLANO in Torino (formulas de Algebra); G. VAILATI in Roma (indicationes historico de Logica-Mathematica); C. BURALI-FORTI in Torino (formulas de Arithmetica et theoria de magnitudines); G. VIVANTI (theoria de classes), R. BETTAZZI (limites), F. GIUDICE (serie), G. FANO (numeros algebrico). Tomo II a. 1897-99, III a. 1901, IV a. 1903 reproduce theorias de Arithmetica, et de Algebra, cum numeroso additione de plure collaboratore, citato in præfatione ad

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(*) Vide C. Burali-Forti e R. Marcolongo, *Per l'unificazione delle notazioni vettoriali*. Palermo R. a. 1907-08.

(**) Vide Formul. t. 5, p. 16, et in modo speciale t. 3.

tomo IV, et reduce in symbolos Geometria, et elementos de Calculo infinitesimale. Me mentiona in modo speciale labore de D.r G. VACCA, in Sina, et de Prof. A. PADOA, in Cagliari.

Symbolos de logica produce brevitate. Formulario, post plure propositione scripto per symbolos, reproduce forma originale. Vide pag. 40, 41, 42, 54, 58, 59, 112, 122, 124, 174, 175, 215, 224, 226...

Resulta que scriptura in symbolos es circa decem vice plus breve que scriptura per lingua commune. Publicatione in lingua vulgare de Formulario amplo ut præsente, es in praxi quasi impossibile, ut publicatione de tabula de logarithmos in lingua commune, aut in cifras Romano.

Systema de symbolos logico es multo minus numeroso que systema de vocabulos æquivalente in lingua commune. Symbolos $= \supset \varepsilon\ \exists$ exprime ideas « es, habe, omni, aliquo, nullo, si, tunc, existe, coexiste, compatibile, independente, possibile, necessario, sufficiente, vero, falso, contrario, contradictorio, dato, fixo, determinato, arbitrario, constante, variabile, æquale, diverso, æqualitate, æquatione, identitate, generale, particulare, universale, pertine, contine... ». (*)

Si ad omni symbolo de Logica responde in lingua commune plure expressione approximato, viceversa non existe vocabulo cum valore exacto de symbolo.

Ita vocabulo « es » responde ad symbolos ε, $=$, $\supset$, $\exists$:

(7 *è* un numero primo) $= (7\ \varepsilon\ \mathrm{Np})$.

(13 *è la* somma di due quadrati) $= (13\ \varepsilon\ \mathrm{N}_1^2+\mathrm{N}_1^2)$.

(*Ogni* multiplo di 4 *è un* multiplo di 2) = (*Tutti* i multipli di 4 *sono* multipli di 2) $= (4\mathrm{N}_1 \supset 2\mathrm{N}_1)$.

(13 *è la* somma di 4 e 9) $= (13 = 4 + 9)$.

(I multipli comuni a 4 e a 6 *sono i* multipli di 12) $= (4\mathrm{N}_1 \frown 6\mathrm{N}_1 = 12\mathrm{N}_1)$.

(*Sonvi* quadrati somme di due quadrati) $= [\exists\ \mathrm{N}_1^2 \frown (\mathrm{N}_1^2+\mathrm{N}_1^2)]$.

Vocabulo « et » habe valore de $\frown$ aut de $\smile$:

(I numeri multipli di 4 *e* multipli di 6 sono multipli di 12) $= (4\mathrm{N}_1 \frown 6\mathrm{N}_1 \supset 12\mathrm{N}_1)$.

(*) Vide RdM. t. 7, pag. 160-172.

(I multipli di 4 *e* i multipli di 6 sono multipli di 2) =
$(4N_1 \cup 6N_1 \supset 2N_1)$.

Vocabulo « existe » responde ad $\exists$ aut ad ε. (Vide pag. 341).

In generale, valore de symbolos de Mathematica non es dato per lingua commune; illo resulta ex definitiones nominale, aut ex systema de propositiones primitivo.

Sed utilitate fundamentale de symbolos de Logica es in rigore et præcisione. Nam illos reduce ratiocinio ad calculo algebrico et multo plus simplice. LEIBNIZ dice:

« Itaque profertur hic calculus quidam novus et mirificus, « qui in omnibus nostris ratiocinationibus locum habet, et qui « non minus accurate procedit quam Arithmetica aut Algebra. « Quo adhibito semper terminari possunt controversiæ quantum « ex datis eas determinari possibile est, manu tantum ad ca- « lamum admoto, ut sufficiat duos disputantes omissis verborum « concertationibus sibi invicem dicere: *calculemus*, ita enim « perinde ac si duo Arithmetici disputarent de quodam calculi « errore ».

Juxta opinione universale, Mathematica excelle inter scientias, pro exactitudine et veritate absoluto de propositiones. Ita propositiones: $2 + 3 = 5$, $3+1/7 > \pi > 3+10/71$, ...
dato valore de singulo symbolo, es vero in modo absoluto; non existe plus vero et minus vero. Propositiones præcedente es jam expresso in symbolos.

In propositiones hodie scripto in parte aut toto, per lingua commune, sæpe vocabulos habe valore de symbolos. Tunc reductione totale in symbolos non es difficile. Ideographia redde evidente, in modo mechanico, que definitiones es justo, que demonstrationes es rigoroso.

Per exemplo, es regula fundamentale pro definitiones, que symbolo que nos defini, debe es expresso per symbolos præcedente. Tunc, si nos considera per ex. definitione de numero primo, pag. 58, nos vide illo expresso per $-$ 1 $+$ $\times$ N_1,
signos introducto in pag. 10 29 29 32 37,
ubi plure inter ce signos es definito per signos præcedente, et ita porro, usque ad decompositione in ideas primitivo, determinato per propositiones primitivo.

In punctos obscuro et incerto, introductione de symbolos es impossibile, ante quam nos habe eliminato omni obscuritate et dubio.

Punctos obscuro, in libros commune, non es raro. In generale, quæstiones tractato in modo differente ab differente Auctores, non habe stabilitate et claritudine perfecto.

In Arithmetica et Algebra, ipso introductione de numeros naturale, negativo, fracto, irrationale, imaginario, es, in plure libro, obscuro. Nos vide definitiones, ubi vocabulo noto es explicato per vocabulos minus noto. Es dato ut definitione, propositione in se contradictorio; etc. (*)

In Geometria, objectos primo « puncto, linea, superficie, recta, plano » es sæpe definito per vocabulos minus claro. Analysi de principios de Geometria, enumeratione de ideas non definibile, et de propositiones non demonstrabile, es facto, et ne pote es facto, que per ideographia. (**)

Historia de Calculo infinitesimale, scientia mirabile in theoria et in applicationes, contine numeroso exemplo de definitiones non præciso, de demostrationes incompleto.

EULERO, in *Introductio in analysin infinitorum*, a. 1748, et in alio libros, institue Calculo super ideas de infinitesimo et de limite.

Definitiones de Eulero non satisfac LAGRANGE, que publica *Théorie des fonctions analytiques*, a. 1813, ubi principios es « dégagés de toute considération d'infiniment petits, d'évanouissans, de limites », et funda theoria super Algebra de serie.

Tunc CAUCHY in libro de a. 1821, dice: « Les raisons de cette espece s'accordent peu avec l'exactitude si vantée des sciences mathématiques », et affer magno rigore in theoria de series.

(*) Vide RdM. t. 8, p. 85, etc. Plure theoria exacto jam es dato ab differente Auctores citato in Formulario. Tunc ideographia es utile per distinctione de tractatione exacto ab inexacto.

(**) M. Pieri, *Della Geometria elementare, come sistema ipotetico deduttivo*, Mem. Ac. Torino, a. 1898-99, t. 49, p. 173, reduce ideas non definito in Geometria ad duo: puncto et motu. Omni reductione facto ante et post scripto de Prof. Pieri, contine numero excessivo de ideas non definito.

Sed ABEL in a. 1826 (Œuvres p. 219) dice: « Si l'on fait subir au raisonnement dont on se sert en général quand il s'agit des séries infinies, un examen plus exact, on trouvera qu'il est, à tout prendre, peu satisfaisant ».

Et ita porro.

Sed rigore non procede per gradu, usque ad infinito. Libros de uno generatione non destrue, sed completa, libros de generatione præcedente. Solutione de aliquo puncto obscuro non es dato per magno libro, sed per aliquo novo combinatione de ideas noto.

Definitione rigoroso de vocabulo *limite* (et æquivalentes, *infinitesimo, evanescente*...) id es, suo expressione per solo ideas de algebra elementare, es dato in anno 1871 circa (vide pag. 232). Principios de Calculo infinitesimale accipe forma stabile per opere de WEIERSTRASS, G. CANTOR, DINI, DARBOUX, et plure alio. Introductione de ideographia non duce ad novitates multo importante. Propositiones fi plus facile et plus claro; conditiones non necessario es suppresso; hypothesi tacito et necessario, debe es scripto in modo explicito.

Tamen me cita aliquo resultatu.

Idea de *limite*, secundo libros moderno, indicato in Formulario per symbolo « lim », es plus restrictivo que idea de limite, secundo Cauchy, indicato per symbolo « Lm », que occurre in plure quæstione (vide p. 211, 214, 215, 224, ...).

Necessitate de convergentia de omni serie, que occurre in calculo, es nimis restrictivo. Serie de Taylor pote es considerato ut serie asymptotico (p. 298, 303).

Definitione de integrale non exige consideratione de « limite de functione » (symbolo: lim), sed solo de « limite supero de classe » (symbolo: l′; vide p. 339).

Idem, pro arcu de curva (p. 370).

Demonstratione de integrabilitate de æquationes differentiale (p. 416), non pote, in praxi, es dato in modo completo per lingua commune. Etc.

In puro campo de logica, ideographia duce ad regulas pro definitiones in Mathematica; ad analysi in formas simplice de ratiocinios mathematico, que non es reductibile ad solo syllo-

gismo; ad studio de indipendentia de aliquo systema de propositiones primitivo.

Præsente editione, vel tomo V de Formulario, contine, in parte I, Logica-Mathematica, reducto ad symbolos, et ad regulas de ratiocinio, que occurre in theorias sequente.

Ergo tomo V es intelligibile sine auxilio de tomos præcedente. Theoria plus amplo de Logica-Mathematica es in tomos II et III.

Arithmetica (parte II), et Algebra (parte III), habe pauco explicatione in lingua commune. Expositione plus diffuso es in meo libro:

Aritmetica generale e Algebra elementare. Torino, Paravia, a. 1902.

Calculo differentiale et integrale, in parte elementare, nunc es exposito in modo satis completo.

D.r PAGLIERO adde « Theoria de curvas » (p. 389-407).

Formulario contine historia de omni symbolo, formas que illo habe apud differente auctores, et in diverso tempore, et rationes historico et logico pro symbolo adoptato.

De omni propositione importante es scripto historia. Bibliographia, composito per D.r VACCA, tomo IV de Formulario, et posito in correspondentia cum tomo V per D.r PAGLIERO, es compendio breve, sed præciso, de historia de Mathematica.

Formulario, satis completo pro mathematica de seculos præterito, es multo incompleto pro auctores moderno et vivente. Nam reductione in symbolos de aliquo theoria exige analysi de omni idea, enunciatione de omni hypotesi, quod es longo et sæpe difficile. Plure theoria moderno non es satis rigoroso.

Formulario non contine omni propositione jam reducto in symbolos; existe numeroso alio applicatione de Logica-Mathematica ad differente quæstiones, per plure Auctore, que adopta symbolos, vel methodos de Logica-Mathematica. Nam symbolos graphico es utile, quasi necessario in longo theoria; sed pote es expresso per symbolos phonetico, putato plus commodo ad publico profano.

Lectore pote stude progressu de Logica - Mathematica, in scriptos infra citato:

Bibliographia de Logica-Mathematica post anno 1900.

RdM. t. 7, p. 3-5, contine bibliographia de 67 scriptos, ante anno 1900.
* indica que propositiones es scripto in ideographia.

Max. Bôcher, *The fundamental conceptions and methods of mathematics*, AmericanB. a. 1904, t. 11 p. 115-135.

C. Burali-Forti, *Sur les différentes méthodes logiques pour la définition du nombre réel*, Congrès intern. de Phil., Paris, a. 1900, p. 289-307.
— *Sur l'égalité, et sur l'introduction des éléments dérivés dans la science*, Enseign. math., a. 1901, p. 246-261.
— **Teoria generale delle grandezze e dei numeri*, TorinoA. a. 1904.

S. Catania, *Trattato di aritmetica ed algebra*, Catania, 2.a ed., 1908.
— *Aritmetica razionale*, Catania, a. 1908.

M. Cipolla, **Theoria de congruentias intra numeros integro*, RdM. a.1905.
— **Specimen de calculo arithmetico integrale*, RdM. a.1908.

L. Couturat, **Les principes des mathématiques*. RMM., a. 1904-05.
— *Sur l'utilité de la logique algorithmique*. Comptes rendus II Congrès intern. de philos., Genève, a. 1904.
— *L'Algèbre de la logique*. Paris, a. 1905.
— **Les définitions mathématiques*. Enseign. math., p. 27, a. 1905.
— **Définitions et démonstrations mathématiques*. Enseign. math., a. 1905.
— *La philosophie des mathématiques de Kant*. RMM.
— *Pour la logistique*. RMM., a. 1906.

E. Huntington, *A complete set of postulates for the theory of absolute continuous magnitude*, AmericanT. a. 1902, t.3 p. 264-184.
— *Two definitions of an Abelian group by independent postulates*, id. 1903.
— *Complete sets of postulates for the theory of real quantities*, id. 1903.
— *Set of independent postulates for the algebra of logic*, Americ.T. a.1904.
— **A set of postulates for the real algebra* ..., id. 1905.
— *Note on the definitions of abstract groups* ..., id. 1905.
— *A set of postulates for ordinary complex algebra*, id. 1905.
— *The continuum as a type of order* ..., Ann. of M. a. 1905.
— *The fundamental laws of addition and multiplication in elementary algebra*, id. a. 1906.
— et plure alio.

P. H. Jourdain, **De infinito in mathematica*. RdM., t. 8, p. 121, a. 1905.

E. H. Moore, *On the projective axioms of geometry*. AmericanT., a. 1902.
— **On a form of general analysis with application to differential and integral equations*, IV congr. intern. mat., Roma 1908.

A. Padoa, *Essai d'une théorie algébrique des nombres entiers*... Congrès international de philosophie, Paris, a. 1900, p. 309-365.
— *Un nuevo sistema de definiciones para la geometría Euclídea*, a. 1900.
— **Numeri interi relativi*, RdM. t. 7, p. 73, a. 1901.
— **Théorie des nombres entiers absolus*, RdM. t. 8 p. 45 a. 1902.
— *Le problème N. 2 de M. David Hilbert*. Enseign. math., a. 1903.
— *Un nuovo sistema di definizioni per la geometria euclidea*. LazzeriP. 1903.
— **Che cos'è una relazione?* TorinoA., a. 1906.
— **Ideografia logica*, Ateneo Veneto, a. 1906.

G. Pagliero, **Applicationes de calculo infinitesimale*, Torino, Paravia, 1907.

G. Peano, *Les définitions mathématiques*, Congrès de phil., Paris 1900.
— *Definicie w matematyce*, Prz. Krygowski, Warszawa 1902.
— **La geometria basata sulle idee di punto e di distanza*, TorinoA. 1902.
— **Aritmetica generale e algebra elementare*, Torino, Paravia 1902.
— **Sulle differenze finite*, Roma Lincei, 1907.
— **Super theorema de Cantor-Bernstein*, RdM. t.8, a. 1906.

M. Pieri, *Sur la géométrie envisagée comme un système purement logique*. Congrès intern. de philos., p. 367-404, Paris, a. 1900.
— *La geometria delle rette*. TorinoA., a. 1901.
— *Circa il teorema fondamentale di Staudt*. TorinoA., a. 1904.
— *Nuovi principii di geometria projettiva complessa*. TorinoM., a. 1905.
— *Breve aggiunta* ..., TorinoA., a. 1905.
— **Sur la compatibilité des axiomes de l'arithmétique*. RMM, a. 1906.
— **Sopra una definizione aritmetica degli irrazionali*. Accad. Gioenia di Catania, a. 1906.
— *Uno sguardo al nuovo indirizzo logico-matematico delle scienze deduttive*. Discorso letto inaugurandosi l'anno accademico 1906-07 nella R. Università di Catania.

B. Russell, **Sur la logique des relations*... RdM. t. 7, p. 115, a. 1901.
— **Théorie générale des séries bien ordonnées*. RdM. t. 8, p. 12, a. 1902.
— *The principles of mathematics*. Cambridge, a. 1903.

O. Veblen, *A system of axioms for geometry*. AmericanT., a. 1904.
— *The foundations of geometry*. The popular science monthly, a. 1906.

A. N. Whitehead, *Memoir on the Algebra of symbolic Logic*, Am.J. t. 23, pag. 139, 297, a. 1901.
— **On cardinal numbers*, Am.J. t. 24, p. 307, a. 1903.
— **The logic of relations, logical substitution groups, and cardinal numbers*, Am.J. t. 25, p. 157, a. 1903.
— **On mathematical concepts of the material world*. LondonT., t. 205, p. 465-525, a. 1906.
— *The axioms of projective geometry*. Cambridge, a. 1906.
— *The axioms of descriptive geometry*. Cambridge, a. 1907.

Wilson, *The foundations of mathematics*. AmericanB., p. 74, a. 1904.

Præsente tomo de Formulario, de principale vocabulos mathematico, da extensione in linguas moderno, et etymologia, id es, suo origine, que in generale es in Latino-Græco, aut in Indo-Europæo; es dato analysi de vocabulos composito.

Resulta que vocabulos de Mathematica, et in generale, de omni scientia, es internationale, vel commune ad linguas de Europa, ab Italo ad Anglo, ab Hispano ad Russo.

Notas et explicationes, in editiones præcedente scripto in Latino scholastico, in Italo, et in Franco, nunc es scripto in *Latino sine flexione*. Omni elemento es Latino. Vocabulos es reducto ad thema. Non existe grammatica. Vide RdM. t.8 p.74.

Omni vocabulo adoptato in Formulario es commune ad Anglo Franco Hispano et Italo, excepto circa 10 vocabulo, que es minus internationale. Sæpe vocabulo es Germano et Russo.

Latino sine flexione, non solo es hodie plus intellecto que omni lingua nationale, sed, sine vinculo de grammatica, habe majore libertate pro elige forma plus præciso.

Pro scribe in *Latino sine flexione*, et in omni systema intelligibile sine studio, nos debe cognosce vocabulario internationale. Vocabulario plus commodo es :

HEMME, *Das lateinische Sprachmaterial im Wortschatze der deutschen, französischen und englischen Sprache*, 1904.

Me spera de publica, inter pauco tempore, studio plus amplo de isto interessante problema.

Torino, Junio 1908.

G. PEANO.

TABULA DE SYMBOLOS.

Numero indica pagina. * significa « definitione ».

Signos de forma speciale.

Literas Graeco.

Literas Latino.

Indice alphabetico, et abbreviationes.

e indica *etymologia* de vocabulo. Vide Tabula de signos, et Form. t.4 p.393.

Publicationes periodico citato per abbreviatione in Formulario.

AErud. = Acta Eruditorum, Lipsiae a.1682-1757
AJ. = American Journal of Mathematics, Baltimore a.1878...
AM. = Acta Mathematica, Stockholm a.1882...
AmericanB. = Bulletin of the American Math. Society, New-York.
AmericanT. = Transactions of the American Mathematical Society, New-York a.1900...
Amsterdam Ak. = Versl. d. k. Akad. v. W. te Amsterdam
AnnN. = Nouvelles annales de Mathématiques, Paris, a.1840...
Annali di Matem. = Annali di Matematica pura ed applicata (Tortolini, ecc). Roma a.1845...
Ann(als) of M(athematics), Harvard University, Cambridge U. S. A.
BBonc. = Bullettino di bibliografia etc., di B. Boncompagni, Roma a.1868-87
BD. = Bulletin des Sciences mathématiques, par Darboux, Paris a.1870...
BelgiqueM. = Mémoires publiés par l'Académie R. des sciences de Belgique a.1818...
BerlinM. = Mémoires de l'Académie de Berlin, a. 1745 ...

Berol(inensia) Misc(ellanea)
BM = Bibliotheca mathematica, par Eneström, a. 1884 ...
BsF. = Bulletin de la Societé math. de France. Paris a.1873 ...
Cambridge Journ = Cambridge Mathematical Journal a.1839 ...
CorrM. = Correspondance Mathématique etc. publiée par P. H. Fuss, St Petersbourg a.1843.
CorrN. = Nouvelle correspondance Mathématique, a.1878...
CR. = Comptes rendus de l'Académie des Sciences, Paris a.1835 ...
DarbouxB = BD.
Encyklopädie der Mathematischen Wissenschaften, Leipzig a.1898...
Formul(aire mathématique), vide pag. viii.
GergonneA. = Annales de Mathématiques, a.1811-29.
IdM. = Intermédiaire des Mathématiciens, Paris a.1894...
JdM. = Journal de Mathématiques, Paris a.1836 ...
JfM. = Journal für die reine und ang. Math., Berlin a.1826...
JP. = Journal de l'École Polytechnique, Paris a.1795
LazzeriP. = Periodico di Matematica, Livorno
LinceiR. = Rendiconti della R. Accademia dei Lincei, Roma.
LondonT. = Philosophical Transactions of the R. Society, London a.1666...
LondonP. = Proceedings of the R. Society. London
LoriaB. = Bollettino di bibliografia ..., Genova a.1898 ...
MA. = Mathematische Annalen, Leipzig a.1869...
Mathesis, Recueil Math. publié par P. Mansion, Gand a.1881...
Mm. = The Messenger of mathematics, London, a.1871...
MünchenA. = Abhandlungen der Königl. Bayerischen Akademie der Wissenschaften zu München a.1832...
Monh. = Monatshefte für Mathematik, Wien a.1889...
NapoliR. = Rendiconti della Accad. delle scienze di Napoli
ParisM. = Mémoires de l'Acad. des Sciences de Paris, a.1666...
ParisSE. = Memoires presentés par divers savants à l'Académie des sciences de Paris (Savants Etrangers) a.1805...
PetrC. = Commentarii Academiæ Scientiarum Petropolitanæ, a.1726-1746.
PetrNC. = Novi Commentarii Academiæ Scient. Petropolitanæ, a.1747-1776.
PetrA. = Acta Academiæ Scientiarum Petropolitanæ, 1777-1782.
PetrNA. = Nova Acta Ac. Sc. Petropolitanæ, a.1783...
PetrB. = Bulletin de l'Ac. des Sc. de St. Petersbourg.
QJ. = Quarterly Journal of Mathematics. Cambridge a.1857...
RdM. = Rivista di Matematica, t.1-5 a.1891-95.
= Revue de Mathématiques t.6 a.1896-99, t.7 a 1900-01, t.8 a.1092-3
RMM. = Revue de Métaphysique et de Morale, Paris a.1893...
TorinoA. = Atti della R. Accademia delle Scienze di Torino a.1865...
TorinoM. = Memorie » » » » » a.1759...
WarszawaP. = Prace Matematyczno-fiziczne, Warszawa.
ZeuthenT. = Tidsskrifft for Mathematik, Copenhague.
Zm. = Zeitschrift für Mathematik und Physik, Leipzig a.1856...

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ABÛ'LWÉFA n. Bouzdjan (Persia) a.940: astronomo in Bagdad: traduce operas de Diophanto in arabo: m. Bagdad a.998. 251, 266

Adams 132, 242, 408.

AHAMESU = filio de luna auctore, aut copista, de papyro « Rhind », conservato in British Museum, publicato per

EISENLOHR, *Ein Mathematisches Handbuch der alten Aegypter*, Leipzig a.1877.

Ce papyro, que repræsenta quasi toto scientia mathematica de antiquo Ægypto que ad nos perveni, videre de a.—1700; putare copia de tractato de a.—2200. 95, 254

ALBATEGNIO = AL BATTÂNI n. Battân (Mesopotamia) a. 850 ca. astronomo in Raqqa et in Bagdad: m. Tikrît a.929. 266

ALQÂCHÂNÎ, medico, vocato El-Kasi, astronomo ad observatorio de Samarqand: m. a.1436ca.

— *La clé du calcul*, trad. par Woepcke, d'une copie datée a.1589, Annali di Matem. a.1864 t.6 p.225. 121, 122

ALCHODSCHANDÎ Muhammed, astronomo arabo a.992. 40

Ampère, a. 1775-1836 306

ANTHONISZ A. a.1527—1607 255

APOLLONIO PERGAEO = Ἀπολλώνιος ὁ Περγαῖος a.—250 ca., nato in Perga (Pamphilia): stude in Alexandria (Egypto) sub discipulos de Euclide: post —200 reside in Pergamo. Magno parte de suo opera es perdito.

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174, 175, 390, 391

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ARBOGAST L. F. A., *Du calcul des dérivations*, a.1800.
a.1759 n.Mutzig, prof. ad Universitate de Strassburg: a.1803 m.Strassburg. 278, 304

ARCHIMEDE = Ἀρχιμήδης, a.—286 n. Syracusa, filio de astronomo Pheidias, parente de rege Hierone: stude in Alexandria (Ægypto) sub successores de Euclide: a.—216 necato per milite romano, post captu de Syracusa pro defensu de que illo inveni machinas de bello.

— *Opera omnia*, Edid. Heiberg, Lipsiæ a.1880.
122, 172, 215, 252, 255, 342, 351, 383, 385, 390, 392, 399, 446 v

Argand 94.

ARISTOTELE = Ἀριστοτῆλης, Ἀναλύτικα πρότερα. 5, 14, 15
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n. a.476: a.500-550 doce in Pataliputra super Gange. 121, 255

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BACHET, C. G. a.1587 ca. n.Bourgi in Bresse: jesuita, prof. de rethorica ad Milano, postea membro de Academia de Paris: a.1638 d.59 m.Paris.
— *Commentaria in Diophantum*, a.1621. 40

Barrieu 104 Bellavitis 167, 185.

BERNOULLI Daniel a. 1700 d.29 n.Groningen, secundo filio de Joh: a.1725-33 prof. in Acad. de Petroburgo: a.1750 ad Univ. de Basilea: a.1782 d.76 m.Basilea. 219

BERNOULLI Jacobo a.1654 d.351 n.Basilea: suo familia ex Antwerpen migra in Helvetia verso fine de XVI seculo, pro fuge persecutiones de religione. a.1687 post itinere in Europa es nominato prof. ad Universitate de Basilea. Cum suo fratre Johanne uno ex primos discipulo de Leibniz. a.1705 d.228 m.Basilea.
— Opera, Genevæ a.1744.
122, 124, 132, 133, 256, 310, 342, 396, 398, 400, 432

BERNOULLI Johanne a.1667 d.208 n.Basilea, fratre de Jacobo: a.1691-1705, prof. in Groningen; postea ad Univ. de Basilea, post morte de suo fratre: a.1748 d.1 m.Basilea.
— Opera, a.1742.
126, 129, 218, 223, 226, 248, 258, 260, 262, 264, 301, 303, 304, 365, 396, 398, 400, 432, 451

BERNOULLI Johanne II, a.1710 n.Basilea: ultimo filio de Joh.: a.1748 prof. de math. in loco de suo patre: a.1790 m.Basilea. 248

BERTRAND Joseph, a.1822 d.70 n.Paris: a.1839 doctore in scientias: a.1844-95 prof. ad Polytechnico: a.1900 d.93 m.Paris.
40, 61, 92, 100, 127, 262, 363

BINET Jacques, a.1786 d.33 n.Rennes: prof. de astron. ad Collège de France: a.1856 d.133 m.Paris. 132, 147, 357

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BOLZANO Bernard, *Rein analytischer Beweis...* Prag a.1817.
a.1781 n.Prag: prof. de philosophia ad Universitate de Prag: a.1848 m.Prag. 215, 412

BOMBELLI Rafael, *L'Algebra*, Bologna, a.1579 152

BONNET Ossian a.1819 n. : a.1878 professore de astronomia ad Sorbonne. a.1892 m.Paris. 232, 289, 348

CESÀRO Ernesto, a.1859-1906, prof. ad Universitate de Napoli. 126, 127, 219, 240, 299, 398

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CHUQUET Nicolas, « parisien, Bachelier en medecine ». 3

Cipolla XIV Clausen 132, 256 Clifford 456 Coqué 433

COTES Roger, a.1682 d.183 n.apud Leicester: a.1699-1707 stude in Trinity College, Cambridge: discipulo de Newton, editore de 2º editione de suos *Principia*: a.1707 prof. de astron.: a.1716 d.157 m. Cambridge.

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DAHSE Zacharias, a.1824 d.175 n.Hamburg: calculatore prodigio. 62, 256, 263

DARBOUX Gaston, a.1842 d.226 n.Nimes: prof. ad Coll. de France et ad Sorbonne. 118, 296, 343

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DEGEN C. F., a.1766 d.305 n.Braunschweig: prof. ad Univ. de Köbenhavn: a.1825 d.98 m.Köbenhavn. 90

de LAGNY Thomas Fantet a.1660-1734. 174, 256

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DELAMBRE J. B. Jos., a.1747 d.262 n.Amiens: prof. de astron. ad Collège de France: a.1822 d.231 m.Paris. 263, 266

DE L'HOSPITAL G. F., a.1661 n.Paris: a.1704 m.Paris. 294, 391

DE MORGAN Augustus, a.1806 d.150 ca. n.Madura (Madras): a.1828-67 prof. ad Univ. de London: a.1871 d.78 m.London. 5, 16, 34

DESCARTES René, a.1596 d.90 n.Lahaye (Touraine): ex patre deputato de Bretania: a.1604-12 stude in collegio de la Flêche sub Jesuitas: a.1617 fi militare sub Nederlandia: post a.1619 visita plure regione de Europa: a.1629 habita apud Amsterdam: a.1649 vocato ad Stockholm per Regina de Sverige: a.1650 d.42 m.Stockholm.

— Œuvres, ed. Ch. Adam et P. Tannery, Paris a.1897... 34, 195, 313, 332, 406

DINI Ulisse, a.1845 n.Pisa: a.1866 prof. ad Universitate de Pisa. 226, 234, 295

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DIOPHANTO, *Διοφάντου Ἀλεξανδρέως Ἀριθμητικῶν*. vive inter +200 et +400. 52, 85, 89, 101, 113

LEGENDRE A. M., a.1752 d.261 n. Paris: a.1775-80 prof. ab Schola militare in Paris: a.1783 membro de Academia de Scientias: a. 1824 perde functione pro politica: a.1833 d.10 m. Paris.
43, 60, 87, 93, 102, 127, 174, 219, 251, 257, 310, 357, 358

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LEONARDO PISANO, de filiis Bonaccii, filio de notario de mercatantes de Pisa: habita longo tempore in Bugia de Barberia; postea i in Ægypto et in Oriente. 59, 112, 272

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LIOUVILLE Joseph, a.1809 d.83 n.: prof. ad Sorbonne: a.1882 d.251 m. Paris. 63, 242

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LORIA Gino, a.1862 n.Mantova, prof. ad Universitate de Genova.
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a.1842 d.94 n.Amiens: a.1891 d.281 m.Paris. 53, 57, 61, 62, 64

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MACLAURIN Colin, a.1698 d.45ca. n.Kilmoddan (Inverary): a.1717 prof. in Aberdeen: a.1725-46 prof. ab Univ. de Edinburgh: a.1746 d.165 m.York, ubi fuge revolutione. 41, 303, 304, 355

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Marcolongo, n. a. 1862, prof. Univ. Napoli. 456, viii

MASCHERONI Lorenzo, a.1750 d.134 n.Castagnetto (Bergamo): abbate professore in lycæo de Bergamo, postea prof. in Univ. de Pavia: a.1800, d.212 m.Paris. 255, 359, 408

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MERCATOR Nicolao (Kaufmann), a.1620ca. n.Cismar (Holstein): stude in Kopenhagen: a.1160ca. visita London, postea habita in Francia: a.1187 d.40ca. m.Paris.

— *Logarithmo-technia*, Londini a.1668. 131, 246, 293

MERTENS F. C. J., a.1840 d.80 n.Wreschen (Posen): a.1865 prof. ad Univ. de Krakau. 60, 225

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PTOLEMÆO Claudio = Πτολεμαῖος Κλαυδίος, a.150. 251, 255

PYTHAGORA = Πυθαγόρας n.Samos a.—570ca.: stude in Ægypto: a.—490 doce in Crotona ubi funda schola: m. a.—470ca. 174

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RECORDE Robert, n. Wales a.1500ca.: medico: prof. de math. in Oxford: a.1558 m. London 3

REGIOMONTANO (JOANNES DE REGIO MONTE, Müller), a.1436 d.158 n.Königsberg: a.1476 d.188 m.Roma.

— *De Triangulis... libri quinque*, Norimbergæ a.1533 266

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RIEMANN Bernhard, *Werke*, Leipzig, a.1876.
a.1826 d.260 n.Breselenz (Hannover): prof. Göttingen, Univ.: a.1866 d.201 m.Selasca (Lago Maggiore). 225, 354, 375, 441

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de SAINT-VENANT, a.1797 n.Portoiseau (Melun): a.1813 stude in Ecole pol.; ingeniero: a.1886 m.St. Ouen (Vendôme). 188, 316

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STAUDT K. G., a.1798 d.24 n.Rothenburg: a.1835 prof. ad Universitate de Erlangen: a.1867 d.152 m.Erlangen. 132, 175, 199

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STEWART Matthew, a.1717 n.Rothsay, Scotland: a.1747-75 prof. Edinburgh, Univ.: a 1785 d.23 m.Edinburgh. 176

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STIELTJES Thomas Jean, *Essai sur la th. des nombres*, a.1895.
a.1856 d.364 n.Zwolle (Over-Yssel, Nederland): a.1877 assistente ad observatorio de Leyden: a.1885 prof. ad Universitate de Toulouse: a.1894 d.365 m.Toulouse. 57, 223, 306

STIFEL Michael, a.1487ca. n.Esslingen (Würt.): monaco augustino; a.1523 seque Luther ad Wittemberg: a.1559 prof. de math. Jena, Univ.: a.1567 d.109 m.Jena. 125

STIRLING James, a.1696ca. n.St. Ninians (Scotland): a.1770 d.339 m. Leadhils. 260, 303, 357

— *Methodus differentialis*, Londini a.1730.

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TARTAGLIA Nicolò, n.Brescia a.1506ca.: doce mathematica in Venezia: a.1557 d.347 m.Venezia. 114, 125

TAYLOR Brook, a.1685 d.230 n.Edmonton (Middlesex): stude in Cambridge: a.1731 d.363 m.London. 303, 304

— *Methodus incrementorum directa et inversa*, Londini a.1715.

TCHEBYCHEF P. L., a.1821 d.147 n.Borowsk (Moscov): a.1859 prof. Petroburgo, Univ.: a.1894 d.343 m. Petroburgo.

— Œuvres, St. Petersbourg a.1899 t.1 61, 219, 249

THALETE = *Θαλῆς*, n.Mileto (Asia Minore) a.—640ca.: stude in Ægypto funda schola ionico: m. a.—548ca. 175

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TORRICELLI Evangelista, a.1608 n.Piancaldoli: stude in Roma sub B. Castelli, discipulo de Galileo: a.1647 d.298 m.Firenze. 355

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VANDERMONDE C. A., a.1735 n.Paris: a.1796 d.1 m.Paris. 125, 147

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VEGA Georg, a.1756 — 1802.

— *Thesaurus logarithmorum* a.1794. 242, 256, 263

VIETA Francisco, a.1540ca. n.Fontenay-le-Comte: protestante: a.1580 Magistro de requisitiones de Rege: a.1603 d.347 m.Paris.
3, 113, 251, 255, 256, 262

VIVANTI Giulio, a.1859 d.144 n.Mantova prof. ad Universitate de Messina.

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VIVIANI Vincenzo a.1622 d.95 n. Firenze: discipulo de Galileo et de Torricelli: ingeni-ero et mathematico de Granduca de Toscana: a.1703 d.265 m. Firenze 445

WALLIS John, a.1616 d.327 n.Ashford (Kent): stude Theologia in Cambridge: a.1649 prof. Oxford, Univ.: a.1703 d.301 m.Oxford.

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103, 124, 126, 129, 215, 259, 272, 342, 355-357, 397, 399, 401, 406

WARING Edward, a.1736ca. n.Shrewsbury: a.1760 prof. Cambridge, Univ.: a.1798 d.227 m.Plealey (Shrewsbury). 62, 306

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WEIERSTRASS Karl, *Werke* a.1894.
a.1815 d.303 n.Ostenfelde (Kreis Münster): a.1834-38 stude jure in Bonn: a.1838-40 stude *privatim* mathematica in Munster sub Gudermann: a.1856 prof. Berlin, Univ.: a.1897 d.50 m.Berlin.
94, 105, 153, 229, 234-236, 238, 288

WESSEL Caspar, a.1745 d.159 n.Jonsrud (Vestby-Sogn): agrimensore geometra: a.1818 d.84 m.Kóbenhavn.
— *Essai sur la répresentation analytique de la direction*, Copenhague, a.1897. (traduct. de l'original de l'a.1797). 167, 185

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Abbreviationes in Bibliographia.

n. = nato. m. = mortuo.
Nomen proprio de Auctore Europæo es scripto secundo forma adoptato per auctore.
Nomen de urbe habe forma nationale.

CORRECTIONES

indicato per Doctores Chionio, Korselt, Pagliero, Pensa, Sannia.
+ indica linea ab summo ad imo; —indica linea ab imo ad summo.

pag. 3,	linea —9: in loco de	$(pq)r$	lege $(pq)\supset r$.
4	—3	$a\subset b$	$a\supset b$.
8	—3	$b\varepsilon N_1 + a$	$b\varepsilon N_1 \times a$
28	—12	0 et N_0	0 et +
44	+6	§ι P·1	§ι P2·2
45	Prop. 1·8	$a\times b/c$	$a\times c/b$
	linea —1	§—	§/
51	§ord P·4	$\geqq\ \leqq$	$\leqq\ \geqq$
82	P1·1	$x\varepsilon \bigcap u$	$x\varepsilon \bigcup u$
	P1·2	$\bigcap u \smile \bigcap v$	$\bigcup u \smile \bigcup v$
	P1·6, dele.		
95	linea 15, 16, 17	N_0	N_1
104	P2·1	$\smile$	$\frown$
113	P·3	$x=\ldots\ y=\ldots$	$2x=\ldots\ 2y=\ldots$
120	linea —15	$\Sigma(f, a\cdots b)$	$\Sigma(f, u)$

126 P16·7		1.o membro	$\Sigma[(-1)^r\, C(n,r)/(n+r+1)\|r,0\cdots n]$
130 P1·2		$a\times f$	$c\times f$
144 P·2		m	n
P·3 linea 3		NumCls'	Cls'
145 P4·0		$\varDelta$	∇
205 N. 381		(297)	(301)
N. 386		(298)	(302)
207 N. 421		(363)	(403)
217 P9·3, adde Hp $x\varepsilon$ Q f N_0			
221 linea	+16	P4·2	P4·1
	−4	$n>h/p$	$n>p/h$
223	+1	P·3	P·5
227	−10	P25·4	P24·4
246	+17	P3·1	P5·3
	+18	§lim 41·1	§lim 31·1
	+24	§lim 25·4	§lim 25·1
228	+5	§Σ 15·4	§Σ 14·4
	+21	§q 12·4	§q 13·4
237	+5	$\varDelta$	∇
248	+10	P4·1	P6·1
263 P6·2 Hp ut in P·3.			
291 linea	+4	$<h$	$<mh$
	+14	1+Q	q
295 P28 adde Hp : $\Sigma[f(x,n)\|n, N_0]\ \varepsilon$q. Vide pag. 364 P40.			
332 linea	−16	Q,0¦	Q,∞¦
358 P25·8, adde Hp $m+n\ \varepsilon\ N_0$			
383	+10	r^3	$(2r)^3$
386	−6	r^2h	rh^2
394 linea −4, −6, pag. 395 lin. −4, −5, pag. 397 lin. +6, +7, −6			
		μ	$\mu\theta$
412	+15	§30 P4·1	pag. 423
431	+17	e↾−S	e↾S
	−10	$-hx$	$+hx$
438	+2, adde Hp : $f\varepsilon$ cont.		
449	−11	$1-x$	$b-x$
	−9	$h0=h1=0$	$ha=hb=0$
		h‘q	g‘q
	−8	$x(1-x)$	$(x-a)(b-x)$

I

LOGICA-MATHEMATICA

I. LOGICA-MATHEMATICA.

§1 $= \supset \frown$

Signo $=$ vale « æqua », « es æquale ad ».

$a, b, \ldots x, y, z$ indica objecto arbitrario.

Nos pote scribe primo propositione symbolico:

·1 $x=x$

Lege: « x aequa x ». Numero ·1 es numero de propositione.

Signo $\supset$ vale « tunc », « seque » etc.

Puncto . : .˙. : : etc. et parenthesi ( ) [ ] } { divide formula in partes.

·2 $x=y \,.\supset.\, y=x$ « si $x=y$, tunc $y=x$ ».

Duo puncto divide propositione in tres parte; primo parte $x=y$ es hypothesi », abbreviato in Hp; secundo es signo de deductione; tertio parte $y=x$ es « thesi », abbreviato in Ths.

Signo $\frown$ vale « et ». Nos sæpe tace illo.

·3 $x=y \,.\, y=z \,.\supset.\, x=z$

« si $x=y$, et si $y=z$, tunc $x=z$ ».

Tres puncto divide propositione ·3 in quatuor parte; inter duo primo parte, que forma hypothesi, signo $\frown$ es tacito.

Nos conveni, in P·3 (lege: propositione indicato per numero ·3), que $pq\supset r$ vale $(pq)r$, et $p\supset qr$ vale $p\supset(qr)$.

Signo de aequalitate habe forma ∞, deformatione de initiale de « æquale » de Vieta ad Leibniz. Forma $=$, de Recorde, a. 1557, adoptato per Newton, es hodie commune in Mathematica.

Chuquet, Leibniz, Newton, indica decompositione de formula in partes per linea horizontale superiore aut inferiore, dicto « vinculo ». Mane hodie in plure tractato pro indica fractiones et radicale. Usu de parenthesi, cum valore actuale, es introducto verso anno 1700 (Bernoulli, Euler).

Usu de punctos, indicato per Leibniz, es commodo inter propositiones.

§2 Cls ε

Cls significa « classe ».

Si a es Cls, $x \varepsilon a$ vale « x es a », vel « x es individuo de classe a ».

Si a et b es classe, $a \supset b$ vale « omni a es b » vel « b contine a » vel « si aliquo objecto es a, tunc illo es b ».

Si a et b es Cls, $a \cap b$, vel ab, es classe commune ad a et ad b; id es, ab es classe de objecto, que es a et es b.

In lingua familiare, nomen commune et adjectivo indica classe; « plurale » saepe indica relatione $\supset$:

« homines sunt mortales » vale « homo $\supset$ mortale ».

Omni figura de Geometria es classe; in figura:

$\overline{\text{A} \quad \text{B} \quad \text{C} \quad \text{D}}$

B ε AC	vale:	« B es puncto de segmento AC ».
BC $\supset$ AD	»	« segmento BC continere in AD ».
AC $\cap$ BD = BC	»	« parte commune ad AC et BD es BC ».

In Arithmetica, nos indica classe plus importante per signos:

N_0 = numero (integro, positivo aut nullo).

N_1 = numero naturale (integro positivo) = N_0+1 = summa de aliquo numero N_0 cum 1.

R = numero rationale = N_1/N_1 = ratione de duo numero naturale.

$12 \varepsilon N_1 \times 4$	« 12 es multiplo de 4 ».
$25 \varepsilon N_1^2$	« 25 es numero quadrato ».
$N_1 \supset R$	« omni numero naturale es rationale ».
$6 \times N_1 \supset 2 \times N_1$	« omni multiplo de 6 es numero pari ».
$2 \times N_1 \cap 3 \times N_1 = 6 \times N_1$	« omni multiplo commune de 2 et de 3 es multiplo de 6, et vice-versa ».

Maximo numero de propositiones de Formulario es scripto ope solo signos de Logica $=, \varepsilon, \supset, \cap$, combinato cum signos de Arithmetica. Lege p. ex. pag. 30, 32, 34, 35, 48, 49, 55, 85-92, ...

Signo ε es litera initiale de vocabulo graeco ἐστί.

Nos pote indica relatione « b contine a » sub forma « $b \subset a$ », ubi signo $\subset$ es initiale de « contine », pauco deformato. Tunc si nos inverte duo membro, nos verte signo de relatione (ut in Arithmetica $a < b$ vale $b > a$), et nos indica idem relatione per « $a \subset b$ », lege « a continere in b ».

Signo $\supset$ occurre in Gergonne, a.1816, Abel t. 1 p.36, et in aliquo raro Auctore, et sub alio forma in Leibniz.

✻ 1·1 $a \varepsilon \mathrm{Cls} . \supset . a \supset a$

«Si a es classe, tunc omni a es a».

Nos saepe indica propositiones de Formulario per numero decimale 1·1, 2·3, etc. Mutatione de numero integro es indicato per signo ✻.

·2 $a \varepsilon \mathrm{Cls} . \supset : x,y \varepsilon a . = . x \varepsilon a . y \varepsilon a$ Df,

«Si a es classe, tunc $x,y \varepsilon a$ vale $x \varepsilon a . y \varepsilon a$». Definitione de «,».

Bipuncto «:» divide propositione in duo parte; primo parte consta de hypothesi et signo de deductione; secundo parte es aequalitate de logica. Primo membro es signo novo $x,y \varepsilon a$, que nos defini aequale ad secundo membro expresso per symbolos noto.

In modo simile, nos scribe $x,y,z \varepsilon a$ in loco de $x,y \varepsilon a . z \varepsilon a$, vel de $x \varepsilon a . y \varepsilon a . z \varepsilon a$.

Nos conveni, in P·2, que $p . \supset q$ et $p \supset . q$ vale $p . \supset . q$.

Exemplo: $5,7 \varepsilon \mathrm{Np}$ «5 et 7 es numero primo».

·3 $a,b \varepsilon \mathrm{Cls} . x \varepsilon a . a \supset b . \supset . x \varepsilon b$

·4 $a,b,c \varepsilon \mathrm{Cls} . a \supset b . b \supset c . \supset . a \supset c$ Syll

«Si a, b es classe, et si x es a, et si omni a es b, tunc x es b».

«Si a, b, c indica tres classe, et si omni a es b, et si omni b es c, tunc omni a es c».

Propositione ·3 et ·4 exprime forma de ratiocinio dicto «syllogismo».

Existe analogia intra signo ε et $\supset$; in vero si in propositione ·4 nos scribe ε in loco de primo $\supset$, nos habe P·3.

Existe differentia intra signo ε et $\supset$; non lice scribe ε in loco de secundo $\supset$ in P·4. In vero, ex hypothesi, in latino scholastico:

« Petrus et Paulus sunt apostoli; apostoli sunt duodecim » in symbolo:

« Petro, Paulo ε apostolo; apostolo ε classe de 12 individuo »,

seque nullo conclusione.

Relatione xay, intra duo objecto x et y, es dicto *transitivo* (de Morgan, CambridgeT. a. 1856 t. 6 p. 104), si de $xay . yaz$ seque xaz.

Relatione $x > y$ de arithmetica es transitivo.

§1 P·3 dice que relatione $x = y$ es transitivo.

Syllogismo, sub forma 2·4, dice que signo $\supset$ es transitivo.

Exemplo præcedente proba que signo ε non es transitivo.

Existe alio differentia inter signo $\supset$ et ε. Signo ε es commutabile cum signo $=$; signo $\supset$ non habe ce proprietate. Signo ε es distributivo ad $\cup$, quod non es vero pro signo $\supset$. Vide §4 P2·1 nota, et §5 P3·1 nota.

Aristotele (a.—383⁻—321) enuntia syllogismo, §2P1·4, sub forma:

« Εἰ τὸ Α κατὰ παντὸς τοῦ Β, καὶ τὸ Β κατὰ παντὸς τοῦ Γ, ἀνάγκη τὸ Α κατὰ παντὸς τοῦ Γ κατηγορεῖσθαι. »

Nota usu de literas variabile in Aristotele.

Alio forma de syllogismo, et de ratiocinio, considerato ab Aristotele, et ab logicos scholastico, non habe applicatione in Mathematica.

❋ 2·1 $a,b\varepsilon$ Cls $.\supset.$ $a \frown b$ ε Cls

·2 $a,b\varepsilon$ Cls $.\supset.$ $a\frown b \supset a$

·3 $a,b\varepsilon$ Cls $.\supset.$ $a\frown b \supset b$

❋ 3·1 $a\varepsilon$ Cls $.\supset.$ $a\frown a = a$

·2 $a,b\varepsilon$ Cls $.\supset.$ $a\frown b = b\frown a$ Comm ⌢

·3 $a,b,c\varepsilon$ Cls $.\supset.$ $(a\frown b)\frown c = a\frown(b\frown c)$ Assoc ⌢

·4 $a,bc\varepsilon$ Cls $.\supset.$ $a\frown b\frown c = (a\frown b)\frown c$ Df

Operatione x intra duo objecto a et b, es dicto «commutativo», si $axb = bxa$; et «associativo», si $(axb)xc = ax(bxc)$.

In Arithmetica, operationes + et × es commutativo et associativo.

P3·2·3 dice que operatione ⌢ es commutativo et associativo.

Valore commune de duo membro de aequalitate ·3 es indicato per $a\frown b\frown c$, sine parenthesi (P·4) per Df (definitione).

Aliquo theorema super minimo commune multiplo aut maximo commune divisore inter plure numero seque de solo P2·1·2·3 de Logica. Vide pag. 53 § mlt P1·2·21, 3·2·10·1·3.

❋ 4·1 $a,b\varepsilon$ Cls $.\supset:$ $a=b$ $.=.$ $a\supset b$. $b\supset a$

«Si a et b es Cls, aequalitate $a=b$ vale affirmatione simultaneo de: omni a es b, et omni b es a». Exprime signo = per ⊃.

·2 $a,b\varepsilon$ Cls $.\supset:$ $a\supset b$ $.=.$ $a = a\frown b$

Exprime ⊃ per = et ⌢.

·3 $a,b\varepsilon$ Cls $.\supset:$ $x\varepsilon\, a\frown b$ $.=.$ $x\varepsilon a$.⌢. $x\varepsilon b$ Distrib(ε, ⌢)

·4 $a,b,c\varepsilon$ Cls $.\supset:$ $a\supset b\frown c$ $.=.$ $a\supset b$. $a\supset c$ Distrib(⊃, ⌢)

Nos dice que operatione x es «distributivo» ad operatione y, si

$ax(byc) = (axb)y(axc)$ Distrib(x,y).

In Arithmetica, × se distribue ad + : $a\times(b+c) = a\times b + a\times c$.

P·3·4 affirma que ε et ⊃ es ambo distributivo ad ⌢.

❋ 5. ; *Indice ad signo* ⊃.

$x;y$ vel (x,y) indica systema composito éx duo objecto x et y.
x,y,z es systema de tres variabile:

·0 $x;y;z = (x;y);z$ Df

Notatione (x,y) es diffuso in Analysi. Me scribe $x;y$ quando existe periculo de ambiguitate cum conventione P1·2.

Si p_x et q_x es propositione que contine litera x, aut systema de literas x, scriptura:

$$p_x \supset_x q_x$$

significa « de p_x seque, pro omni valore de x, q_x ».

Nos tace indice ad $\supset$, quando non existe periculo de ambiguitate; id es in tres casu:

1°. Quando propositione contine uno solo signo $\supset$. Indice tacito ad $\supset$ es systema de variabiles in hypothesi.

Per ex. in P3·1, signo $\supset$ habe indice a, in P3·2 indice $a;b$, in P3·3 $a;b;c$.

2°. Quando illo es signo de deductione principale, que nos nosce per maximo numero de puncto apud illo. Indice tacito es systema de variabiles in hypothesi.

3°. In fine, nos tace indice ad secundo signo $\supset$ in theorema de forma $p\supset(q\supset r)$, id es ad signo de deductione, principale in thesi. Indice tacito es systema de variabiles, que figura in q, et non in p.

Si a es formula que contine litera x, nos dice que x es variabile reale in a, si valore de a depende ab valore de x. In casu contrario, nos dice que x es apparente in formula a.

Subsiste principio generale:

« Variabile que figura ut indice expresso aut tacito ad signo $\supset$ es apparente in toto deductione ».

·1 $a,b\varepsilon \mathrm{Cls} .\supset\therefore a\supset b .=: x\varepsilon a .\supset_x. x\varepsilon b$

Oper $x\varepsilon$ Oper $x\mathfrak{z}$

Si a et b es classe, tunc propositione universale affirmativo « omni a es b » vale deductione:

« Si x es a, seque pro omni valore de x, que x es b ».

Nos « opera per $x\varepsilon$ » quando nos scribe signo $x\varepsilon$ ante duo membro de $a\supset b$. Nos « opera per $x\mathfrak{z}$ », quando nos fac transformatione inverso.

Exemplo: $\mathrm{Np} \frown (4\mathrm{N}_1+1) \supset \mathrm{N}_1^2+\mathrm{N}_1^2$

« Omni numero primo multiplo de 4 plus 1, es summa de duo quadrato »

Nos opera per $x\varepsilon$: $x\varepsilon\, \mathrm{Np} \frown (4\mathrm{N}_1+1) .\supset. x\varepsilon\, \mathrm{N}_1^2+\mathrm{N}_1^2$

« Si x es numero primo de forma $4\mathrm{N}_1+1$, tunc x es summa de 2 quadrato ».

Nos distribue ε ad $\frown$ (P4·3): $x\varepsilon\, \mathrm{Np} . x\varepsilon\, 4\mathrm{N}_1+1 .\supset. x\varepsilon\, \mathrm{N}_1^2+\mathrm{N}_1^2$

« Si x es numero primo, et si x habe forma $4\mathrm{N}_1+1$, tunc... ».

·2 $x = y . = : a \varepsilon \text{Cls} . x \varepsilon a . \supset_a . y \varepsilon a$

«Duo objecto x et y es aequale, vel identico inter se, quando, de $x\varepsilon a$, seque $y\varepsilon a$, pro omni classe arbitrario a».

In loco de «classe» nos pote lege «proprietate». Ergo duo objecto es identico, si omni proprietate de primo es semper proprietate de secundo.

·3 $a,b,c \varepsilon \text{Cls} . \supset ::$

$x \varepsilon a . \supset_x : (x;y) \varepsilon b . \supset_y . (x;y) \varepsilon c .:. = : x \varepsilon a . (x;y) \varepsilon b . \supset_{x;y} . (x;y) \varepsilon c$

Import Export

Si p_x es propositione que contine variabile x, et si $q_{x;y}$ et $r_{x;y}$ es propositione cum duo variabile x et y, tunc propositione: «de p_x seque, pro omni valore de x, que de $q_{x;y}$ seque pro omni valore de y, $r_{x;y}$», in symbolo $p_x . \supset : q_{x;y} . \supset . r_{x;y}$

vale propositione:

«de p_x et $q_{x;y}$ seque, pro omni valore de x et de y, $r_{x;y}$», in symbolo

$$p_x \frown q_{x;y} \supset_{x;y} r_{x;y}$$

Nos «importa hypothesi p_x» vel «collige duo hypothesi p_x et $q_{x;y}$», quando nos transforma primo propositione in secundo; et nos «exporta hypothesi p_x» vel «separa hypothesi p_x et $q_{x;y}$» quando nos transforma secundo enuntiato in primo.

Pro scribe, per symbolos, toto interessante regula praecedente, nos reduce propositiones p_x, $q_{x;y}$, $r_{x,y}$ ad forma $x\varepsilon a$, $(x;y)\varepsilon b$, $(x;y)\varepsilon c$, ubi a, b et c es classe.

Exemplo. Regula P2·2, applicato ad propositiones, da:

$$a\varepsilon \text{Cls} . x\varepsilon a . \supset . x\varepsilon a$$

Signo $\supset$ habe indice tacito a et x (regula 1). Me exporta $a\varepsilon$Cls:

$$a\varepsilon \text{Cls} . \supset : x\varepsilon a . \supset . x\varepsilon a.$$

Primo signo $\supset$ porta indice tacito a (regula 2), et secundo signo $\supset$ habe indice tacito x (regula 3). Me supprime in thesi signo $x\varepsilon$, per P5·1, et me obtine:

$$a\varepsilon \text{Cls} . \supset . a \supset a$$

que es «principio de identitate» P1·1.

Alio exemplo. Propositione 5·1, ubi in loco de signo $=$ me scribe $\supset$, per P4·1, fi:

$$a,b\varepsilon \text{Cls} . \supset .\!\cdot. a \supset b . \supset : x\varepsilon a . \supset . x\varepsilon b$$

Me collige tres hypothesi, et habe:

$$a,b\varepsilon \text{Cls} . a \supset b . x\varepsilon a . \supset . x\varepsilon b$$

que es P1·3.

Exemplo ex Arithmetica:

$$a\varepsilon N_1 . b\varepsilon N_1 + a . c\varepsilon N_1 \times b . \supset . c\varepsilon N_1 \times a$$

«Si a es numero naturale, et si b es multiplo de a, et c es multiplo de b, tunc c es multiplo de a».

4 puncto divide propositione in 5 parte; tres primo es propositione inter que es tacito signo $\frown$; signo $\supset$ porta ut indice tacito systema a, b, c.
Ce propositione, pro lege de exportatione, es reductibile ad forma:

$$a \varepsilon N_1 . b \varepsilon N_1 \times a .\supset: c \varepsilon N_1 \times b .\supset. c \varepsilon N_1 \times a .$$

« Si a es numero naturale, et si b es multiplo de a, tunc, si numero c es multiplo de b, illo es multiplo de a ».

Bipuncto (:) divide propositione in 2 parte, primo consta de Hp et de signo de deductione. Secundo parte es thesi de toto propositione; ce thesi es deductione, que habe suo Hp et suo Ths. Primo signo $\supset$ habe indice tacito a et b, que figura in Hp de toto propositione. Secundo signo $\supset$ porta indice tacito c, que existe in Hp de Ths, et non in Hp de propositione. Litera c es reale in Hp de Ths $c \varepsilon N_1 \times b$, et in Ths de Ths $c \varepsilon N_1 \times a$, et apparente in Ths de toto propositione. Nos elimina litera apparente c in Ths, si nos « opera per $c \ni$ »; propositione fi:

$$a \varepsilon N_1 . b \varepsilon N_1 \times a .\supset. N_1 \times b \supset N_1 \times a$$

« Si a es numero naturale, et si b es multiplo de a, tunc omni multiplo de b es multiplo de a ».

Exemplo de signo $\supset$ cum indice explicito, in Arithmetica: pag.27 P1·3, pag.63 P2·1·2·3, P3·1.

§3 $\ni$ (que)

Si propositione p_x contine variabile x, tunc « $x \ni (p_x)$ » vale « x que p_x », vel « x que satisfac ad conditione p_x ».

·1 $a \varepsilon \text{Cls} .\supset. x \ni (x \varepsilon a) = a$

« Objectos x que satisfac ad conditione $x \varepsilon a$ forma classe a ».
Ergo operatione $x \ni$ et $x \varepsilon$ destrue se mutuo, et es inter se inverso.
Ita in Arithmetica, ubi operatione es scripto in ordine contrario,

$$a + x - x = a.$$

·2 $a,b \varepsilon \text{Cls} .\supset. x \ni (x \varepsilon a .\frown. x \varepsilon b) = a \frown b$ Distrib($\ni$,$\frown$)

« Classe commune ad duo classe a et b es classe de x que satisfac ad conditione $x \varepsilon a$ et $x \varepsilon b$ ». Exprime signo $\frown$ inter classe per idem signo inter propositione; et affirma que signo $\ni$ es distributivo ad $\frown$, ut suo inverso ε (§2 P5·1).

In scriptura $x \ni p_x$, litera x es apparente.

Exemplo:

$$a, b \varepsilon N_1 .\supset. \text{quot}(a, b) = \max[N_0 \frown x \ni (x \times b \leqq a)]$$

« dato duo numero a et b, quoto de a per b es maximo numero x tale que x multiplicato per b non supera a ». (pag. 48).

§4 - (non)

❋ 1. $a,b,c\varepsilon$ Cls $.\supset:$

·0 $x\varepsilon-a .=. x\varepsilon(-a) : a-b = a\frown-b : -a\supset b .=. (-a)\supset b : -a=b .=. (-a)=b$ Df

·1 $-a \ \varepsilon$ Cls

·2 $a\supset b .\supset. -b \supset -a$ Transp

« Si omni a es b, tunc omni non b es non a ».

·3 $ab\supset c .\supset. a-c \supset -b$ Transp

« Si de ab seque c, de a et non c seque non b ».

Nos « transporta », quando nos applica regula ·2 aut ·3.

·4 $-(-a) = a$

« Duo negatione forma affirmatione ».

·5 $a\supset b .=. -b \supset -a$

·6 $ab\supset c .=. a-c \supset -b$

❋ 2·0 $x -= y .=. -(x=y)$ Df

·1 $a\varepsilon$ Cls $.\supset. -(x\varepsilon a) = x\varepsilon -a = x -\varepsilon\ a$ Comm(ε,-)

« Negatione de propositioue $x\varepsilon a$ vale x es non a ».

Nos dice que operatione a es commutabile cum operatione b, si

$$abx = bax \qquad \text{Comm}(\varepsilon,-)$$

Ergo, operatione ε es commutabile cum -.

P. ex. si nos pone : (ignorante) = - (docto), seque :

-(Petro es docto) = Petro -ε docto = Petro ε- docto = Petro es ignorante.

Operatione ɔ non es commutabile cum -. P. ex. de -(homo ɔ docto), id es « non es vero que omni homo es docto », non seque (homo ɔ- docto), id es (homo ɔ ignorante), « omni homo es ignorante ».

Exemplo :

$$\text{Np} = (1+\text{N}_1)-[(1+\text{N}_1)\times(1+\text{N}_1)].$$

« Numero primo es numero superiore ad 1, non producto de duo factore superiore ad unitate »

$$a, b\varepsilon\text{q} . a-=b .\supset. a^2+b^2 > 2ab$$

« Si a et b es quantitate differente, tunc ... »

§5 ∪ (aut)

❋ 1. a,b,c,ε Cls $.\supset:$

·0 $a b\cup c = (ab)\cup c : a\cup bc = a\cup(bc) : a\cup b\supset c .=. (a\cup b)\supset c : a\supset b\cup c .=. a\supset(b\cup c) :$
$a\cup b\cup c = (a\cup b)\cup c : x\varepsilon\, a\cup b .=. x\varepsilon(a\cup b)$ Df

·1 $a\cup b\ \varepsilon$ Cls ·2 $a\supset a\cup b$ ·3 $b\supset a\cup b$

❋ 2. $a,b,c \varepsilon \text{Cls} . \supset .$

·1 $a \cup a = a$

·2 $a \cup b = b \cup a$ Comm $\cup$

·3 $(a \cup b) \cup c = a \cup (b \cup c) = a \cup b \cup c$ Assoc $\cup$

❋ 3. $a,b,c \varepsilon \text{Cls} . \supset .$

·1 $x \varepsilon a . \cup . x \varepsilon b .=. x \varepsilon a \cup b$ Distrib(ε,$\cup$)

·2 $a \supset c . b \supset c .=. a \cup b \supset c$

·3 $b \supset a .=. a \cup b = a$

Si nos muta $\cup$ in $\cap$, et $a \supset b$ in $b \supset a$, in de praesente §, nos habe propositiones de §2: P1·1·2·3, 2·1·2·3, 3·1·2·3 / P2·1·2·3, 3·1·2·3, 4·3·4·2

P3·1 dice que ε se distribue ad $\cup$.

Relatione $\supset$ non es distributivo ad $\cup$. Nam ex propositione

$N_0 \supset 2N_0 \cup (2N_0+1)$ « Omni numero es pari aut impari », non lice deduce: « Omni numero es pari » aut « omni numero es impari ».

❋ 4·1 $a,b,c \varepsilon \text{Cls} . \supset . a(b \cup c) = ab \cup ac$ Distrib($\cap$,$\cup$)

Exprime proprietate distributivo de $\cap$ ad $\cup$. Plure Auctore, per ce analogia, voca operatione $\cap$ et $\cup$ « multiplicatione et additione de logica ».

$\ni$ ❋ 5. $a,b \varepsilon \text{Cls} . \supset .$

·1 $x \ni (x \varepsilon a . \cup . x \varepsilon b) = a \cup b$ Distrib($\ni$,$\cup$)

·2 $a \cup b = x \ni (c \varepsilon \text{Cls} . a \supset c . b \supset c . \supset_c . x \varepsilon c)$ Dfp $\cup$

« Classe $a \cup b$ es systema de omni objecto x tale que, si nos sume classe arbitrario c, que contine a, et que contine b, seque, pro omni classe c, que x es c ».

Primo membro contine signo $\cup$, et secundo es composito per signos de §1-3. Ergo P5·2 es definitione possibile de signo $\cup$.

$-$ ❋ 6. $a,b,c,d \varepsilon \text{Cls} . \supset .$

·1 $-(a \cap b) = -a \cup -b$

« Negatione de producto logico es summa de negatione de factores ». Es analogo ad proprietate de logarithmo.

·2 $a \cup b = -[(-a) \cap (-b)]$ Dfp $\cup$

Es alio definitione possibile de signo $\cup$ per $\cap$ et $-$.

·3 $-(a \cup b) = -a \cap -b$ ·4 $a \cap b = -(-a \cup -b)$

·5 $a-b \supset c .=. a \supset b \cup c$ Transp

·6 $ab \supset c \cup d .=. a-c \supset d \cup -b$ Transp

·7 $a-b = x \ni (c \varepsilon \text{Cls} . a \supset b \cup c . \supset_c . x \varepsilon c)$ Dfp $-$

Exemplo de signo $\cup$, pag. 39, pag. 59.

§6 Λ (classe nullo) $\exists$ (existe)

❋ 1·0 $\Lambda = x\ni(a\varepsilon \text{Cls} .\supset_a. x\varepsilon a)$ Df Λ

Λ, lege «classe nullo», indica classe de objecto commune ad omni classe a. Responde ad 0 de Arithmetica.

Ergo $a\frown b=\Lambda$ repraesenta propositione universale negativo «nullo a es b».

$a,b\varepsilon \text{Cls} .\supset.$ ·1 $\Lambda\supset a$ ·2 $a\frown\Lambda = \Lambda$

·3 $a\smile\Lambda = a$ ·4 $a\smile b=\Lambda .=. a=\Lambda . b=\Lambda$

·5 $a=\Lambda .\smile. b=\Lambda .\supset. ab=\Lambda$

·6 $a-a=\Lambda$

·7 $a\supset b .=. a-b=\Lambda$ Transp

❋ 2. $a,b\varepsilon \text{Cls} .\supset:$ ·0 $\exists a .=. a -= \Lambda$ Df $\exists$

·01 $\exists ab .=. \exists(ab)$ Df

«Existe aliquo a» significa «classe a non es nullo».

·1 $x\varepsilon a .\supset. \exists a$ ·2 $a\supset b . \exists a .\supset. \exists b$

·3 $\exists a\frown b .\supset. \exists a . \exists b$

·4 $\exists(a\smile b) .=. \exists a .\smile. \exists b$ Distrib($\exists$,$\smile$)

❋ 3. $a,b,c\varepsilon \text{Cls} .\supset\therefore$

·1 $(x;y)\varepsilon a .\supset_{x,y}. y\varepsilon b :=: \exists x\ni[(x;y)\varepsilon a] .\supset_y. y\varepsilon b$ Elim x

Si in uno propositione, hypothesi habe duo variabile x et y, et thesi uno variabile solo y, tunc propositione:

«de Hp seque, pro omni valore de x et de y, Ths»,

es reductibile ad forma:

«Si existe x que satisfac ad hypothesi, seque pro omni valare de y, thesi».

Nos reduce hypothesi ad forma $(x,y)\varepsilon a$, et thesi ad forma $y\varepsilon b$; tunc regula nunc enuntiato sume forma praecedente.

Nos «elimina x» si nos transforma primo propositione in secundo; in hypothesi de secundo propositione litera x es apparente, nam praecede signo $\ni$.

·2 $\exists x\ni\exists y\ni[(x;y)\varepsilon a] .=. \exists y\ni\exists x\ni[(x;y)\varepsilon a]$

·3 $x\ni\{\exists y\ni[x\varepsilon a . (x;y)\varepsilon b]\} = x\ni\{x\varepsilon a . \exists y\ni[(x;y)\varepsilon b]\}$

·4 $\exists a\frown x\ni\{\exists b\frown y\ni[(x;y)\varepsilon c]\} .=. \exists b\frown y\ni\{\exists a\frown x\ni[(x;y)\varepsilon c]\}$

·5 $\exists (x;y)\ni(x\varepsilon a . y\varepsilon b) .=. \exists a . \exists b$

§7 ι ℩

❋ 1·0 $\iota x = y \ni (y = x)$ Df ι

·01 $y\varepsilon\, \iota x .=. y\varepsilon(\iota x) \;:\; a \supset \iota x .=. a \supset (\iota x) \;:\; a = \iota x .=. a = (\iota x)$ Df

·1 $y\varepsilon\, \iota x .=. y = x$ [Dfι . Oper $y\varepsilon$.$\supset$. P]

·2 $a\varepsilon\,\mathrm{Cls} .\supset: x\varepsilon a .=. \iota x \supset a$

·3 $a\varepsilon\,\mathrm{Cls} .\supset: x,y\varepsilon a .=. \iota x \cup \iota y \supset a$

·4 $a\varepsilon\,\mathrm{Cls} .\supset. -a = x \ni (\iota x \cap a = \Lambda)$ Dfp-

ιx, lege «aequale ad x», es classe composito de omni y, que satisfac ad conditione $y = x$. Id es, ιx indica classe continente solo objecto x. Signo ι es initiale de vocabulo ἴσος.

Si in Df de ι, P·0, nos opera per $y\varepsilon$, id es, si nos scribe $y\varepsilon$ ante duo membro de aequalitate, et si nos mene que $y\varepsilon$ et $y\ni$ destrue inter se, resulta (P·1) que $\varepsilon\iota$ vale signo $=$.

Vide exemplos de signo ι in pag. 37.

$$q \cap x \ni (x^2 - 3x + 2 = 0) = \iota 1 \cup \iota 2$$

«radices reale de aequatione scripto es 1 et 2». Nos opera per $x\varepsilon$, et distribue ε ad $\cap$ et distribue ε ad $\cup$·

$$x\varepsilon q \,.\, x^2 - 3x + 2 = 0 .=. x\varepsilon\iota 1 .\cup. x\varepsilon\iota 2 ,$$

unde, per Df ι:

$$x\varepsilon q \,.\, x^2 - 3x + 2 = 0 .=. x = 1 .\cup. x = 2.$$

❋ 2. $a\varepsilon\,\mathrm{Cls} \,.\, \exists a : x,y\varepsilon a .\supset_{x,y}. x = y :\supset::$

·0 $z = ℩a .=. a = \iota z$ Df℩

·1 $b\varepsilon\,\mathrm{Cls} .\supset\therefore ℩a\,\varepsilon b .=: a = \iota x .\supset_x. x\varepsilon b :=:$
$\exists x \ni (a = \iota x \,.\, x\varepsilon b) :=: \exists\, a \cap b :=: a \supset b$ Dfp

·2 $℩a\, \varepsilon a$ ·3 $\iota(℩a) = a$ ·4 $℩(\iota x) = x$

·5 $\Lambda = ℩\, x \ni [a\varepsilon\,\mathrm{Cls} .\supset_a. a - a = x]$ DfpΛ

Si classe a contine uno solo individuo z, id es si $a = \iota z$, nos indica per $℩a$ ce individuo z, $z = ℩a$. Nos exprime que classe a contine uno solo individuo, per phrasi: «classe a es existente vel non nullo; et si x et y pertine ambo ad classe a, tunc $x = y$».

Operatione ℩ es inverso de ι. In defectu de vocabulo aequivalente in lingua commune, nos pote lege signo ℩ per «illo», vel «to»; in vero, in aliquo casu, illo responde ad articulo de lingua commune. Vide definitione de subtractione in §— pag. 44, §/ P1·0, pag. 46 §max P1·0,...

§8 Df (definitione)
Dfp (definitione possibile)

Omni definitione, in mathematica, habe forma :

$$x = a \qquad \text{Df}$$

ubi x es signo novo, a es serie de signo noto. Df exprime conventione de scribe signo simplo x, dicto « definito », in loco de serie a, dicto « definiente ».

Dfp, lege « definitione possibile », es æqualitate que in primo membro habe signo non occurrente in altero membro. Charactere de Dfp hære ad propositione, et non depende de ullo conventione.

Si nos pone in ordine omni idea de Mathematica (vel de aliquo scientia), nos voca « definitione possibile de signo x, relativo ad ordine dato » æqualitate

$x =$ (expressione composito per signos præcedente x).

Inter definitiones possibile de signo x, nos elige forma magis conveniente; et definitione possibile fi reale per conventione.

Nos voca « idea primitivo, relativo ad ordine dato », idea que non habe definitione possibile, in ordine considerato.

Existe aliquo idea primitivo; in facto, nos non pote defini primo idea, que non habe praecedente; et nos non pote defini signo =, que figura intra duo membro de ommi definitione.

Idea primitivo, relativo ad uno ordine, pote es definito in ordine differente. Quare, si per idea a, b, c nos defini d, et si per idea a, b, d nos defini c, lice sume ut idea primitivo, vel a, b, c, vel a, b, d.

Nos pote elimina omni signo definito, si nos scribe in suo loco, valore definiente illo.

Ergo omni definitione exprime abbreviatione, in theoria non necessario, sed utile in practica. Si nos non pote elimina signo definito, suo definitione habe aliquo defectu.

Aristotele classifica de Df in reales et nominales :

Ὁ ὁριζόμενος δείκνυσιν ἢ τί ἐστιν ἢ τι σημαίνει τοὔνομα. (Anal. post. II 7).

In Mathematica, omni Df es nominale. Ce observatione occurre in Möbius (a.1815 t. 4, p.388)

« Definitionum divisi in verbales et reales omni caret sensu ».

et in Stuart Mill (a.1838) : :

« All definitions are of names, and names only ».

Vide Vailati, RdM. t. 8, p.57-63.

Df mathematico non satisfac ad regula que omni Df procede per genere proximo et per differentia specifico (Aristotele, Top. I, 8):
Ὁ ὁρισμὸς ἐκ γένους καὶ διαφορῶν ἐστίν.

Nam, supposito noto signo 1 et $+$, aequalitate:

$$2 = 1+1, \qquad 3 = 2+1, \qquad 4 = 3+1, \ldots$$

es definitione ds 2, 3, 4, ...; et non existe genere aut specie.

Exemplo: Df de classes: N_1, Np, n, R, ..., et de individuos ∞, e, i, ...
Df de operationes: $\uparrow$, $>$, $-$, $/$, max, quot, ... habe Hp.

Definitione per abstractione de functio φ habe forma

$$\varphi x = \varphi y \ .=. \text{(expressione composito per signos praecedente)}$$

Non defini symbolo simplo φx, sed solo aequalitate $\varphi x = \varphi y$.

Formul. contine Df per abstractione: §Num 1·0 §α 2·2 4·2 ...

Omni Df reale aut possibile debe es «homogeneo» pro literas variabile, id es ambo membro de aequalitate debe contine idem variabile reale. In vero si nos pone $f(x,y) =$ «expressione que depende de x, y, z», nos da idem nomen ad differente valores de expressione.

Dm (demonstratione)
Pp (propositio primitivo)
| (substitutione).

Demonstratione de uno propositione es suo deductione ex propositiones præcedente. Deductione de uno propositione ab præcedentes, es aliquo forma de ratiocinio. Collectione completo de forma de ratiocinio, que nos inveni in analysi de commune demonstrationes de Mathematica, constitue systema de formula de Logica-Mathematica.

Si nos suppone que propositiones de aliquo scientia es disposito in ordine, nos dice que uno propositione es primitivo, in relatione ad ordine dato, si nos non pote deduce illo ab præcedentes.

Si in aliquo scientia existe idea primitivo, et existe propositione primitivo, que fixa valore de idea non definito.

Nos proba que uno propositione es primitivo, in ordine dato, si nos inveni interpretatione de ideas primitivo, que satisfac ad omni propositione præcedente, et non ad propositione considerato. Nos proba que systema de propositione primitivo es mutuo ne-dependente, in modo absoluto, si, pro omni propositione, nos adduce interpretatione de systema de ideas primitivo, que satisfac ad omni Pp, excepto propositione considerato.

Si p es formula, que contine litera x,

$$(a \mid x)\, p$$

indica resultatu de substitutione de a ad x in formula p.

In Arithmetica, signo de substitutione figura solo in aliquo demonstratione. Illo es casu particulare de signo que occurre in theoria de functione (pag. 77).

In omni theorema, lice substitue omni litera pro valore arbitrario. Si hypothesi fi vero, nos supprime illo, et scribe solo thesi. Si hypothesi contine, ut factore logico, propositione vero, vel consequentia de alio factore, nos supprime illo.

Historia.

In praesente editione de Formulario, Logica-Mathematica es reducto ad minimo necessario pro intelligentia de partes sequente. Editione magis completo, es in Formularto t. 3 a.1901.

Ecce aliquo indicatione summario super historia.

Leibniz (a.1646–1716) es vero creatore de Logica-Mathematica. Vide: L. Couturat, *Opuscules et fragments inédits de Leibniz*. Paris a. 1903 pag. XVI+682

Me extrahe citationes sequente, cum indicatione de correspondente P de Formulario :

§2 P4·2 Omne A est B id est $AB \infty A$.

§2 P5·2 Eadem sunt quorum unum in alterius locum substitui potest, salva veritate ».

§4 P1·4 $A \infty \overline{non\ \overline{non\ A}}$ ».

§4 P1·5 A est B ergo non$B$ est non$A$.

§5 P1·2 N est in $A \oplus N$.

P2·1 Si idem secum ipso sumatur, nihil constituitur novum, seu $A+A \infty A$.

P3·2 Si A est in C et B est in C etiam $A+B$ erit in C.

P3·3 Si B est in A, erit $A+B \infty A$ ... Si $A+B \infty A$, tunc B erit in A.

P6·2 Sive A sive B, hoc est non neque A neque B.

§6 P1·7 Omne A est B, id est ... A non B est non Ens.

Lambert J. H., in a. 1781, exprime proprietate distributivo de $\cap$ ad $\cup$, §5P4·1 :

« Will man aber setzen $(m+n)A$, so ist dieses $= mA+nA$. »

De Morgan a. 1847, Boole a. 1854, Schröder a. 1890, etc. re-inveni theoremas de Leibniz, da ad illo forma symbolico semper magis completo, et evolve plure applicatione de Mathematica ad Logica, que non occurre in applicatione de Logica ad Mathematica.

Vide historia recente de Logica-mathematica in:

Rivista di Matematica, t.1 a.1901 - t.8 a.1905, et Formulario t.1-t.4.

C. Burali-Forti, *Logica Matematica*, Milano a. 1894.

L. Couturat, in Revue de Métaphysique et de Morale a. 1904-1905 (plure et interessante articulo super theoria, et applicationes ad Arithmetica et ad Geometria).

L. Couturat, *Manuel de Logistique*, Paris Alcan a. 1905.

» *Les définitions mathématiques*, Enseignement mathématique, a.1905 p. 27.

» Congrès international de Philosophie, tenu à Genève; Rapports p.706.

B. Russell, *The principles of mathematics*, Cambridge, University Press, a. 1903.

E. B. Wilson, *The foundations of Mathematics*, AmericanB. a. 1904, p. 74.

Plure alio articulo de Prof. Whitehead et Huntington in American J. of Mathem., et in Bull. of the American Society.

VOCABULARIO I.

Me collige aliquo vocabulo, frequente in Mathematica, commune in generale ad A (lege; Anglo, English), D (lege: Deutsch, vel Germano in sensu stricto), F (lege: Franco), H (lege: Hispano), I (lege: Italo) et R (lege: Russo).

Maximo numero de vocabulo internationale es L (lege: Latino). Omni vocabulo, que non habe indicatione contrario, es latino.

Plure vocabulo habe origine commune in G (lege: Graeco) trans L.

In aliquo casu, vocabulo L vel G habe vocabulo || (lege: parallelo) in ADR; et origine commune in E (lege: Europaeo antiquo), id es, ramo de Indo-Europaeo, que resulta ex comparatione de GLADR. Tunc es utile comparatione de vocabulo cum S (lege: Sanscrito).

Me trahe omni elemento linguistico, quando occurre, ex grammatica et vocabulario, commune et etymologico, de singulo lingua ADEFGHILRS. Resulta que studio linguistico, utile, quasi necessario, ad Mathematico, non es multo; sed hodie es sparso in plure et voluminoso libro.

Pro citatione de Auctores, transcriptione de GRS, etc. vide in fine de libro.

In praesente libro, me adopta vocabulo magis diffuso. Pro grammatica, me seque Leibniz «Grammatica rationalis» (Vide RdM. t.8, p.74), que duce ad suppressione de omni flexione. Principio de internationalitate duce ad idem resultatu; in vero, raro elemento grammaticale es commune ad linguas moderno.

In orthographia, me seque Latino.

Abbreviationes.

⊃ vale « genera ».

⊂ ,, « deriva ».

+ es signo de unione de duo elemento de lingua. Valore de signo —, et de 0, es consequentia de regulas de Arithmetica.

a⌢b indica elemento commune ad vocabulo *a*, *b*; es idea commune expresso per literas commune; exprime quod linguistas voca thema, radice, praefixo, suffixo, etc.

-, tractu de unione; indica fragmento de vocabulo de lingua considerato.

§1.

1. ***es***, H es, F es-t, es, I è, es-sere.
|| G es-ti, A is, D is-t, R es-tĭ, S as. ⊂ E es.

Lege: *es* es vocabulo latino (imperativo et thema de *sum*, *esse;* in generale thema de verbo vale suo imperativo); mane cum pauco variatione in Hispano, in Franco et in Italo. Illo es parallelo ad vocabulo Graeco, Anglo, Deutsch, Russo et Sanscrito.

Nos pone, secundo conventione adoptato per Linguistas,

E *e* = L *e* = G *e* = A *e* (*i*) = D *e* (*i*) = R *e* = S *a*

E *s* = LADRS *s* = G *s* (*h*)

Tunc forma de vocabulo Europaeo antiquo, ex LGRS, resulta *es*.

E *e* produce in Germanico, id es in A et D, per regula, *e*, et per exceptione *i*. Si secundo syllaba de aliquo vocabulo E contine *i*, tunc vocale *e*, *a*, *o*, *u* in primo syllaba fi *i*, *ä*, *ö*, *ü*. Exemplo:

D *e*rde, *i*rd*i*sch, g*o*tt, g*ö*ttl*i*ch,... A *a*ngle, engl*i*sh;...

De vocabulo G esti = R esti = S asti nos induce E esti, que genera Gotico ist, D ist, A is.

Linguas moderno habe comune vocabulo Indo-Europaeo «es». Latino «es» mane in vocabulo «es-sentia» ADFHIR, et Graeco «es» mane celato in «ontologia» ADFHIR. Plure elemento es commune ad linguas moderno sub triplice forma E L G.

2. ***aequo***, I equo, H ecuo. ⊃ aequ-atore ADFHIR. || S aica, eca.

Lege: *aequo* es vocabulo latino (ablativo et thema de nominativo *aequus*; in generale, thema de nomen es suo ablativo). Illo mane, pauco alterato, in I et H. Es elemento internationale. Parallelo ad vocabulo Sanscrito.

L ae ⊃ AFHIR e, D ae,

id es diphthongo L *ae* mane in D, et fi *e*, in omni alio lingua.

3. ***aequa*** = fac aequo, es aequo. ⊂ aequo (2) — -o + -a (4)

Resulta de vocabulo praecedente, in quo nos supprime desinentia -o et adde desinentia -a.

4. -a — -o = libera — libero = sana — sano = firma — firmo ...
⊃ HI -a — -o, F 0, A 0.

Substitutione de desinentia -a ad desinentia -o, nunc considerato, es frequente in L. Illo mane in Hispano et in Italo; in F et A ce differentia de forma evanesce, F ferme = A firm = L firmo (adjectivo), firma (verbo).

5. ***aequale***, aequali, A equal, F égal, H egual, I eguale. = aequo.
⊂ aequa (3) + -le (6) = aequo — -o + -ale (239).

« aequali » es vocabulo latino, ablativo, et thema de « aequali-s (que occurre in vocabulario), aequali-um ».

« aequale » es vocabulo latino, nominativo neutro, thema de « aequale-m, aequale-s »; generante I « eguale ».

In tardo latino, ablativo in -i es mutato in -e, et mane sub ce forma in linguas moderno. Ita me cita vocabulos latino.

6. -le, -li = (5) ∩ differentia-le ∩ fide-le ∩ simi-le ∩ -i-le (199) ∩ -a-le .
AF -l, -le, DH -l, I -le, R -l' || G -lo (224).

⁂

Scriptura de forma: $x = y$
es dicto « aequalitate », « aequatione ». x et y es « membro » de aequatione.

7. ***aequalitate***, A equality, F égalité, H igualdad.
⊂ aequale (5) — -e + -itate (8) = aequali + -tate.

8. -itate = bon-itate ∩ facil-itate ∩ nov-itate ∩ univers-itate ...
A -ity, D -ität, F -ité, H -idad, I -ità, R -itat'. ⊂ -i- + -tate.

9. **-i-** = voc-i-fera ∩ aequ-i-voco ∩ nov-i-tate ∩ ..,
Es litera de unione de duo elemento. Deriva de finale -i de plure vocabulo: facili-tate, classi-fica, ...

10. **-tate** = (8) ∩ liber-tate ∩ hones-tate ∩ ... || G -τητι, S -tati.

11. ***aequatione***, A equation, F équation, H ecuacion, I equazione, D aequation (in Astronomia). ⊂ aequa (3) + -tione (12).

12. -tio, -tione = (11) ∩ lec-tione ∩ defini-tione ∩ no-tione ∩ ...
ADF -tion, H -cion, I -zione, R -tsija.

13. ***membro*** I, A member, F membre, H miembro.

14. ***ergo*** A.

15. ***seque*** (L de anno + 200, thema de L classico), HI segue, F sui-t.
⊃ seque-nte A, con-seque-ntia D, secu-ndo (minuto) DR, soc-io, ...
|| G hep-, S sac'e.

16. ***si*** FH, I se.

17. ***tunc***, F donc, I dunque.

⁂

Scriptura de forma $a \supset b$
ubi a et b es propositione, es dicto « deductione ».

18. ***deductione***, AD deduction, F déduction, H deduccion, I deduzione, R deductsija. ⊂ deduce (19) — -e + -tione (12).
L -ctione ⊃ I -zione.

19. ***deduce*** AHI, F dédui-re. ⊂ de (21) + duce (20).

20. ***duc***, ***duce*** ⊃ de-duce, pro-duce, intro-duc-tione, ...
|| A tow, D ziehe.
Ad latino *d* responde GSR *d*, A *t*, D *z*. Exemplo:
L duo = G dyo = A two = D zwei = R dva = S dva.
L decem = G deca = A then = D zehn = R desja-tị = S daça.

21. ***de*** FH, I di, ADFHIR de-. (A to, D zu, R do, E do = L ad, habe solo primo litera commune cum L de).

22. ***thesi*** θέσι-ς G, AD thesis, F thèse, H tesis, I tesi, R tesis.
⊃ (25), syn-thesi, par-en-thesi,... = L positione.
⊂ the- (23) + -si (24).

23. the-, θε- ⊃ (22), the-ma, homo-the-tia. = L pone, fac (137).

24. -si ⊃ the-si, analysi, basi mathesi, genesi, phasi, phrasi, physi, ellipsi.
|| G.dorico -ti, L -ti, -ti-one (12).

25. ***hypothesi*** ὑπόθεσις GADFHIR. ⊂ hypo + thesi (22).
= L sup-posi-tione.

26. hypo G ⊃ hypo-thesi, hypo-crita, hypo-teinusa. || L sub (95).
Ad E *s* initiale et ante vocale responde G h:
L sex, septem, serpe, sede, semi-, super
|| G hex, hepta, herpe, hed-, hemi, hyper.

27. ***et***, F et, I e, ed, H y. ⊃ LAF et-cetera.
|| G eti, S ati, Gotico ith, R i, ot'.

§2.

28. ***classe*** FHI, A class, classis, D klasse, R class'.
⊂ G.dorico clasi, G.classico clêsi ⊂ cla- (= cla-ma) + -si (24).

29. ***omni***, I ogni. ⊃ omni-potente AFHI, omni-voro, omni-bus, ...

30. ***-s***, suffixo de plurale. = matre-s ∩ rosa-s ∩ anno-s ∩ sensu-s ∩ die-s.
F -s = rose-s — rose, H I(sardo) et Port. -s = rosa-s — rosa.
|| A -s = rose-s — rose = day-s — day. Gotico: dag-os.
G -s, -es, mêter-es ∩ sphaira-s ∩ ...
S açva-s — açva, matar-as — matar. ⊂ E -s, -es.
Nos scribe suffixo *-s* de plurale quando es utile; vale «omni, classe».

31. ***syllogismo*** συλλογισμός GADFHIR. ⊂ syl- + logo — -o + -ismo.

32. syn G, syl- (ante l), sym- (ante m, b, p), sy- (ante s) = L cum.
⊃ syn-chrono, syn-taxi, syl-laba, sym-bolo, sy-stema, ...
|| R s', so, su-, S sam.

33. logo G. ⊃ dia-logo, astro-logo, log-ica, log-arithmo. = ratiocinio.

34. ***logico*** GHI, A logical, D logisch, F logique, R logic'.
⊂ logo (33) — -o + -ico.

35. -ico LGHI, A -ic, -ical, D -isch, F -ique, R -ic'.
= G log-ico, con-ico, geometr-íco, ...
= L am-ico, pud-ico, ... un-ico, med-ico ...
|| D -ig = stein-ig ∩ einz-ig, R in-oc', S -ica. ⊂ -i- (9) + -co (201).

36. ***logica*** λογική G, A logic, D logik, F logique, HIR logica.
⊂ logico (34) — -o + -a (37).

37. -a (transcriptione L de desinentia G classico -ή, G dorico -ά).
= logic-a ⌒ arithmetic-a ⌒ mathematic-a ⌒ parabol-a ⌒ hyperbol-a ...
AD 0, F -e, HIR -a.
Indica femminile naturale aut artificiale. Vide (126).

38. -ismo GHI, F -isme, A -ism, D -ismus, R -ism'.
⊃ syllog-ismo, barbar-ismo, social-ismo,... ⊂ iz- (276) + -mo (125).

39. ***associativo***; introducto ab Hamilton, a. 1841, in sensu de § 2 P 3·1.
⊂ associa + -tivo.

40. ***associa*** I, A associa-te, F associe, A asocia.
⊂ ad- (41) + socio (42) — -o + -a (4).

41. ***ad***, ad-, as- (ante s), ac- (ante c). I ad, a, F à, H a.
ADFHIR ad-. || A at.

42. ***socio*** HI. ⊃ soci-ale R, soci-etate D. ⊂ soc- + -io (43)
soc- = seque (15).

Nota. Transformatione de *e* in *o* existe in L.intn.:
t*e*ge, t*o*ga; p*e*nde, p*o*ndo; m*e*ne (thema de com-men-to), m*o*ne; n*e*ce, n*o*ce; s*e*que, s*o*cio; f*e*r, f*o*rtuna; v*e*rtice, v*o*rtice.

In G: ph*e*re, ph*o*ro; l*e*ge, l*o*go;...

Responde ad D s*e*he, s*a*h; g*e*be, g*a*b; nehme, n*a*hm.

Et A s*ee*, s*aw*; g*i*ve, g*a*ve; s*i*t, s*a*t; sp*ea*k, sp*o*ken; wr*i*te, wr*o*te,...

43. -io = soc-io ⌒ exim-io ⌒ gen-io ⌒ patr-io ⌒ egreg-io ⌒ rad-io...
|| G -io = ax-io-ma ⌒ patr-io ⌒ ...
|| S pitri-ja, R -yj, -ij = matern-ij ⌒ dobr-yj ⌒ ...

44. -tivo = ac-tivo ⌒ posi-tivo ⌒ accusa-tivo ⌒ associa-tivo ⌒... Vide (213).

45. ***commutativo***; vocabulo introducto ab Servois a. 1815, in sensu de § 4 P 2·1. In sensu de § 2 P 3·2, illo occurre in theoria de quaternione de Hamilton. ⊂ commuta (46) + -tivo (44).

46. ***commuta*** I, AF commute, H conmuta. ⊂ com- (47) + muta (49).

47. ***cum,*** con-, co- (ante vocale, g, h), col- (ante l), com- (ante m, b, p); HI con. ⊃ co-efficiente ADFHIR.

48. ***muta*** I, F mue, H muda. ⊃ per-muta-tione AD. ⊂ mov- + -ta.

49. ***distributivo*** ⊂ distribue (50) — -e + -tivo (44).
Ce vocabulo es introducto ab Servois, a. 1815.

50. ***distribue*** FHI, A distribu-te. ⊂ dis- (51) + tribue (52).

51. dis- (ante c, p, q, s, t, præcedente vocale), di- (in omnia alio casu).
= dis-junge⌒ dis-puta ⌒ dis-tribue ⌒ di-vide ⌒ di-gere ...
|| D zwi- = zwi-schen ⌒ zwie-fach ⌒ ..., G διά.

52. ***tribue*** ⊃ dis-tribue ⌒ at-tribue ⌒ con-tribue ⌒ ...

53. ***opera-***, A opera-te, D operire, F opère, H obra-r, I opera.
⊃ opera-tione (191), opera-tore. ⊂ opere (54) — -e + -a.

54. ***opere***, ***opus***, ***opera***, I opera, F oeuvre, H obra. || S apasi.
⊃ oper-ario, oper-oso, opus-culo. opere ⊂ op- (55) + -ere (56).

55. op- = opera, fac. || S ap-, D übe, L ape, ap-to, op-ta.

56. -ere, -es, -us, -ore ⊃ gen-us, gen-ere; temp-us, temp-or-ale, temp-es-tate, temp-us-culo; hon-ore, hon-es-to; am-ore, dol-ore; ama-re, dole-re, es-se. || G gen-os = S g'an-as = L genus. ⊂ E -es.

Nota. In L, litera *s* inter duo vocale, muta se in *r*: plu*s*, plu*r*e; ma*s*culo, ma*r*ito; ge*s*tu, ge*r*ente; o*s*culo, o*r*ale. ...

57. ***per*** I, F par, H por. ADFHR per-. || L prae, por-, pro (134), A for, far, D für, ver-, vor, G peri, parà, pro, R pere-, pri, pro, S pari, pra, pura. Es differente casu de idem vocabulo.

Nota. Alio exemplo de correspondentia:
L *p* || A *f*, D *f*, *v*, G *p*, R *p*, S *p*.
L *p*ater || A *f*ather, D *v*ater, G *p*ater, S *p*itar.
L *p*leno || A *f*ull, D *v*oll, G *p*leio, R *p*olno, S *p*rana.

58. ***systema***, systemate GADFHIR. ⊂ sy- (32) + ste- (59) + -ma (60).

59. ste-, sta- ⊃ sta-tica ADFHIR, ec-sta-si, sta-dio, ... || L sta (77).

60. -ma, -mate, -$\mu\alpha$ = syste-ma ⌒ theore-ma ⌒ axio-ma ⌒ lem-mat-ico... || L -men, -mento (251).

§3.

61. ***que*** (thema L de que-m), qui, quo, quod, quam.
F que, qui, H que, quien, I che, chi.
|| S ca; G po-; Gotico hvas, hvo, hva; A who, what; D wer, was; R co, c-to, c'-to.

§4.

62. ***ne, non***, in-, F ne, non; H no, ni; I nè, non.
⊃ ne-sci, ne-utro ADFHIR, in-variante D, in-ertia R.
|| G ne-, an-, a- ⊃ an-hydro, an-archia, a-symmetrico ...
|| A no, none, un- = un-even ⌒ un-just, ...
D nein, nicht, un- = un-sicher ⌒ un-ähnlich ⌒ ... R ne, S na.

63. ***transporta***, A transport, D transport-ire, F transporte, HI trasporta.
⊂ trans (64) + porta (67).

64. ***trans*** H tras, I tra, ADFHIR trans-. ⊂ tra- + -ns.

65. tra- ⊃ in-tra, pene-tra, tra-mite.

66. -ns = (64) ⌒ stude-ns ⌒ .... = -ente (142).

Id es, *trans* es participio praesente neutro de verbo hypothetico *tra*, existente in plure composito.

67. ***porta*** (verbo) HI, F porte. ⊃ porta (nomen) FHI, D pforte, im-porta, ex-porta,... ⊂ por- (68) + -ta (69).

68. por- = porta, fer, i. ⊃ L por-ta, por-tu ADFHR || A ford, D furt.
|| G peir-e, por-ɔ; A fare, ferry, D fahre, führe; R por-ytį, S par.

69. -ta, suffixo L, saepe cum valore = 0. Exemplo :
canta LHI = cane L; consulta LHI = consule L; ...
A abbreviate = L abbrevia, A create = L crea, ...
⊂ -to (135) — -o + -a (4).

§5.

70. ***aut***, F ou, H o, u, I o, od. ⊂ au- (|| G ay, D au-ch) + -t (⊃ e-t, u-t).
71. ***vel*** = aut; non indica oppositione. ⊂ vol, I vuoi.

§6.

72. ***nullo***, AD null, F nul, nulle, H nulo, I nullo, R nulį.
⊂ ne (62) — -e + ullo (73).

73. ***ullo*** = aliquo. ⊂ un- (114) + -lo (224).

74. ***existe*** FH, A exist, D existire, I esiste. ⊂ ex + siste.

75. ***ex*** = ne in. ⊃ ex-ponente ADHIR, ex-tra-.
|| G ex, ec ⊃ ec-centrico ADFHIR, ec-clesia, ec-lipsi,... || R iz'.

76. siste ⊃ as-siste-nte AD, con-siste, per-siste. || G histê-, S tistha.
⊂ sta (77), per « reduplicatione », frequente in L, in G di-dac-tica ADR, in S, et in Gotico.

77. ***sta*** I, H esta. || A stand, D stehe, G sta- (59), R sta-tį, S stha.
⊃ con-sta, di-sta, re-sta, sta-bile AFHI, sta-tione. ADFHI.

§7.

78. ***illo***, ille, illa (demonstrativo in L). ⊃ F il, le, elle, la, ce-lui, c-elle; H él, el, ello, la; Port. o, a; I il, lo, la, egli, ella, qu-ello.
⊂ ollo (L antiquo) ⊂ on- (|| R ono, S ana) + -lo (224), secundo Vanic'ek et Fick.

79. ***to*** (thema L) ⊃ is-to, t-ale, t-anto, to-t, t-am, t-um.
F ce-t, ce-tte, H es-to, I ques-to.
|| G to (demonstrativo in Homero, articulo in Græco classico).
A the, D der, das, S ta, R to (demonstrativo). ⊂ E to.

Leibniz, et suo contemporaneos, adopta articulo « to », in scripto latino, quando es utile.

Alio exemplo de correspondentia : ELGRS t || A th, D d:
L te, tu, A the, D du, de-in, G (dorico) te, R tebja, ty, S tu, te.

§8.

80. ***definitione***, AD definition, F définition, H definicion, I definizione, R definitsija. ⊂ defini + -tione (12).

81. ***defini***, A define, F défini-r, H defini-r, DI defini-re.
⊂ de (21) + fini.

82. ***fini***, A fini-sh, FH fini-r, I fini-sce. ⊂ fine — -e + -i. = da fine.

83. ***fine*** I, FH fin, AL finis. ⊃ fin-ale D.

84. -i = fin-i ∩ un-i ∩ im-ped-i ∩ vest-i ...

85. ***demonstratione***. A demonstration, F démonstration, H demonstracion, I dimostrazione, R (in politica) demonstratsija.
⊂ demonstra + -tione (12).

86. ***demonstra***. A demonstra-te, F démontre, H demostra, I dimostra.
⊂ de (21) + monstra.

87. ***monstra***. F montre, HI mostra. ⊂ monstro — -o + -a (92).

88. ***monstro***, A monster, F monstre, I mostro (in sensu malo).
= que mone, re notabile. ⊂ mone + -es (57) + -tro.

89. ***mone***, A mon-ish. ⊃ mon-itore AFHI.
= fac mene, causativo de mene (90). Vide (42).

90. ***mene*** (thema L) = memora, recorda, cogita.
⊃ men-te, com-men-to, men-tione...
|| A mean, D meine, G men-, R mni-tı, S man-je.
⊂ E mene ⊃ A mind = G menos = S manas = L mente.

91. -tro ⊃ ara-tro ⌢ mons-tro ⌢ ras-tro ⌢ ...
|| G -tro = me-tro ⌢ cen-tro ⌢ ... || S -tra, A -der, D -ter.

92. -a — -o = dona — dono = loca — loco = numera — numero
= da, fac; numera = da numero. Vide (4).

93. ***substitutione*** ADFHI. ⊂ substitue (94) — -e + -tione (12).

94. ***substitue*** F, A substitute, D substituire, H substitu-ir, I sostitu-ire.
⊂ sub (95) + -stitue (96).

95. ***sub, subto***, F sous; H so, soto; I sotto; Port. sob; ADFHIR sub-.
|| G hypo (26). Vide (124) et (257).

96. -stitue ⊃ sub-stitue, con-stitue, in-stitue. = statue (97).
Transformatione de *a* in *i*, mane in L.intn.: c*a*de, ac-c*i*de-nte; f*a*c, coef-f*i*c-iente; h*a*be, ex-h*i*be;....
Illo es constante in latino classico, si litera *a* es ultimo litera de syllaba.

97. ***statue*** = fac sta, D statu-ire. ⊃ statu-to ADFHIR.
⊂ statu (99) + -e (98).

98. -e = acu-e ⌢ statu-e ⌢ tribu-e.

99. ***statu***. A state, D staat, F état, H estado, I stato. || R statı, G stasi, D stadt. ⊂ sta (77) + -tu.

100. -tu = fruc-tu ⌢ can-tu ⌢ sta-tu ⌢ gus-tu ⌢ adven-tu ⌢ por-tu.
|| G bro-ty, as-ty; S vas-tu, gan-tu; D fur-t, lus-t.
⊃ HI -to, AF -t, -te, et se confunde cum L -to:
L cantu = AHI canto, AF chant; L canto = HI cantato.

Continua post Arithmetica, pag. 65.

II

ARITHMETICA

II. ARITHMETICA.

§1 +

N_0 vale « numero », et es nomen commune de 0,1,2, etc.
0 » « zero ».
$+$ » « plus ». Si a es numero, $a+$ indica « numero sequente a ».

Quæstione, si nos pote defini N_0, significa si nos pote scribe æqualitate de forma

$N_0 =$ expressione composito per signos noto $\cup \cap - \dots \iota$,

quod non es facile.

Ergo nos sume tres idea N_0, 0, $+$ ut idea primitivo, per que nos defini omni symbolo de Arithmetica.

Nos determina valore de symbolo non definito N_0, 0, $+$ per systema de propositio primitivo sequente.

✻ 1. Pp

·0 $N_0 \; \varepsilon \; \mathrm{Cls}$

·1 $0 \, \varepsilon \, N_0$

·2 $a \varepsilon N_0 \,.\supset.\, a+ \, \varepsilon \, N_0$

·3 $s \varepsilon \mathrm{Cls} \,.\, 0 \varepsilon s : a \varepsilon s \,.\supset_a.\, a+ \, \varepsilon s :\supset.\, N_0 \supset s$ Induct

·4 $a,b \varepsilon N_0 \,.\, a+ = b+ \,.\supset.\, a = b$

·5 $a \varepsilon N_0 \,.\supset.\, a+ \,-\!= 0$

Lege :

·0 N_0 es classe, vel « numero » es nomen commune.
·1 Zero es numero.
·2 Si a es numero, tunc suo successivo es numero.
·3 N_0 es classe minimo, que satisfac ad conditione ·0·1·2;

id es, si s es classe, que contine 0; e si a pertine ad classe s, seque pro omni valore de a, que et $a+$ pertine ad s; tunc omni numero es s.

Ce propositione es dicto « principio de inductione », et nos indica illo per abbreviatione « Induct ».

Omni conditione determina uno classe; ergo nos pote lege principio de inductione sub forma:

Si s es conditione, satisfacto ab numero 0, et si omni vice que illo es vero pro numero a, et es vero pro suo successivo, tunc conditione s es vero pro omni numero.

·4 Duo numero, que habe successivo æquale, es æquale inter se.

·5 0 non seque ullo numero.

Systema praecedente de Pp suffice pro deduce omni propositione de Arithmetica, de Algebra et de Calculo infinitesimale.

Illo es necessario, vel singulo Pp non depende, in modo explicito aut implicito, de systema de cetero propositione. Quod nos proba ut seque:

1. Nos considera serie periodico, per exemplo, serie de hora astronomico de die, 0, 1, 2,... 23, 0, 1,... ubi hora que seque 23 es de novo 0.

Ce serie es classe (·0), que contine 0 (·1), que contine successivo de omni suo elemento (·2); illo satisfac ad principio de inductione (·3); et ad conditione ·4. Sed, in serie considerato, 0 seque numero 23; exemplo considerato satisfac ad omni Pp, excepto ·5. Ergo Pp ·5 exprime que serie de numero non es periodico, proprietate non implicito in conditione ·0 ·1 ·2 ·3 ·4.

2. Serie periodico, sequente antiperiodo, ut serie

$$0, 1, 1, 1, \ldots$$

satisfac ad conditione ·0 ·1 ·2 ·3 ·5, et non ad conditione ·4; nam $0+ = 1$, et $1+ = 1$, sed 0 non $= 1$.

3. Si in loco de N_0 nos lege « numero rationale positivo aut nullo », et si conserva ad 0 et N_0 valore commune, tunc conditione ·0 ·1 ·2 ·4 ·5 es saticfacto, sed non principio de inductione ·3. Nam non suffice de nosce que si uno proprietate es vero pro aliquo numero x es quoque vero pro $x+1$, ut nos deduce que omni numero rationale habe ce proprietate.

4. Serie finito, ut 0, 1, 2, ubi $0+=1$, $1+=2$, $2+=3$ (que non pertine ad serie), satisfac ad conditione ·0 ·1 ·3 ·4 ·5, et non ad ·2; nam ultimo numero non habe successivo in serie dato.

5. Si nos considera serie 1, 2, 3,... composito ex N_0, post suppressione de 0, nos habe exemplo que satisfac ad omni Hp, excepto ·1.

6. In fine, ut nos satisfac ad omni Pp, excepto ·0, nos muta valore de signo Cls; nos tribue ad Cls valore de « omni classe, excepto N_0 », et ad $x \varepsilon N_0$ valore « es numero », ut si N_0 es classe.

✻ 2. *Definitione de cifras.*

$1 = 0+ \ . \ 2 = 1+ \ . \ 3 = 2+ \ . \ 4 = 3+ \ . \ 5 = 4+ \ .$
$6 = 5+ \ . \ 7 = 6+ \ . \ 8 = 7+ \ . \ 9 = 8+ \ . \ X = 9+$ Df

Nos voca 1 successivo de 0; 2 es successivo de 1,....
Signo Romano X es necessario usque ad numeratione in §↑.
De P1·1·2 resulta que

$0+ \quad 0++ \quad 0+++ \quad 0++++$ etc.

es numero. Si nos subintellige 0, et in loco de + nos scribe tractu, nos habe signo de numero

| || ||| |||| etc.

expresso ut reunione de unitate. Illo es systema primitivo de numeratione et hodie adoptato in joco de chartas, sub forma

. : ∴ :: etc.

In hieroglyphos de antiquo Ægypto, numero 1, 2,... 9 es indicato per 1, 2,... 9 tractu. Signo ∩ vale decem; existe alio signo cum valore 100, 1000. In transformatione de scriptura hyerogliphico super saxo, in scriptura hieratico et demotico super papyro, in anno —2000 circa, scribas de Ægypto collega ce tractu, et forma signo simplice, vel cifra; illos transforma

= in 2, ≡ in 3, = ≡ in 23, et post in 5, etc.

Tale es origine de nostro cifras (Lindemann, *Zur Geschichte der ... Zahlzeichen*, MünchenA. a. 1896 t. 26 p. 625).

Usu de cifra ab Ægypto transi in India, post in Arabia, et in fine in Europa verso anno + 1200.

Indos in anno + 400 circa, introduce signo 0 sub formas • , ₀ , 0, pro indica loco vacuo in numeratione decimale. 0 es considerato ut numero per mathematicos de a. 1600.

Signo + habe forma actuale post anno 1500; es deformatione de anteriore signo *p*.

✻ 3. *Definitione de additione.*

·1 $a \varepsilon N_0 \ .\supset. \ a+0 = a$
·2 $a,b \varepsilon N_0 \ .\supset. \ a+(b+) = (a+b)+$) Df+

Si a es numero, tunc $a+0$ vale a.

Et si a et b indica numero, tunc summa de a cum successivo de b es successivo de summa de a cum b.

·3 $a \varepsilon N_0 \ .\supset. \ a+1 = a+$ [$(0 \mid b)$P·2 . Pp·1 . P2 . P3·1 $.\supset.$ P]
» $a+2 = (a+)+$ [$(1 \mid b)$P·2 $.\supset.$ P]

Si in P·2, in loco de b nos scribe 0, nos habe

$a,0 \varepsilon N_0 \ .\supset. \ a+(0+) = (a+0)+.$

Hp vale $a\varepsilon N_0$. $0\varepsilon N_0$. Secundo propositione es vero, per Pp·1, ergo nos supprime illo. Signo 0+, per definitione, (P2) vale 1 ; $a+0$ vale a, per P3·1 ; ergo : $a\varepsilon N_0 .\supset. a+1 = a+$,
id es, numero successivo de a, jam indicato per $a+$, vale $a+1$; notatione commune, de que nos vol fac usu constante.

* 4.

·0 $a,b,c\varepsilon N_0 . a=b .\supset. a+c = b+c$

[I§1 P·1 .⊃: $a,c\varepsilon N_0 .\supset. a\varepsilon\ x\ni(a+c = x+c)$
I§2 P5·2 .⊃: $a,b,c\varepsilon N_0 . a=b .\supset. b\varepsilon\ x\ni(a+c = x+c)$ (1)
(1) ⊃ P]

·1 $a,b\varepsilon N_0 .\supset. a+b\ \varepsilon\ N_0$

[$a\varepsilon N_0$. Df+ .⊃. $a+0\ \varepsilon N_0$ (1)
$a,b\varepsilon N_0 . a+b\ \varepsilon N_0$. Df+ . P1·2 .⊃. $a+(b+1) = (a+b)+1$.
$(a+b)+1\ \varepsilon N_0 .\supset. a+(b+1)\ \varepsilon N_0$ (2)
(1) . (2) . Induct .⊃. P]

·2 $a,b,c\varepsilon N_0 .\supset. (a+b)+c = a+(b+c)$ Assoc+

[$a,b\varepsilon N_0 .\supset. (a+b)+0 = a+b . a+(b+0) = a+b .\supset.$
$(a+b)+0 = a+(b+0)$ (1)
$a,b,c\varepsilon N_0 . (a+b)+c = a+(b+c)$. Oper+1 .⊃. $[(a+b)+c]+1 = [a+(b+c)]+1$
» » » Df+ .⊃. $(a+b)+(c+1) =$ »
» » » » .⊃. » $= a+[(b+c)+1]$
» » » » .⊃. » $= a+[b+(c+1)]$
(2)
(1) . (2) . Induct .⊃. P]

·3 $a,b,c\varepsilon N_0 .\supset. a+b+c = (a+b)+c$ Df

·4 $a,b\varepsilon N_0 .\supset. a+b = b+a$ Comm+

[Df+ .⊃. $0+0=0$ (1)
$a\varepsilon N_0 . 0+a=a$. Df+ .⊃. $0+(a+1) = (0+a)+1 = a+1$ (2)
(1) . (2) . Induct .⊃: $a\varepsilon N_0 .\supset. 0+a=a$ (3)
Df+ .⊃. $1+0 = 0+1 = 1$ (4)
$a\varepsilon N_0 . 1+a = a+1 .\supset. 1+(a+1) = (1+a)+1 = (a+1)+1$ (5)
(4) . (5) . Induct .⊃: $a\varepsilon N_0 .\supset. 1+a = a+1$ (6)
$a,b\varepsilon N_0 . a+b = b+a$. Df+ .⊃. $a+(b+1) = (a+b)+1 = (b+a)+1$
Df+ .⊃. » $= b+(a+1)$
(6) .⊃. » $= b+(1+a)$
Assoc+ .⊃. » $= (b+1)+a$ (7)
(3) .(7) . Induct .⊃. P]

·5 $a,b,c \varepsilon N_0 \,.\, a+c = b+c \,.\supset.\, a=b$

$[\; a,b \varepsilon N_0 \,.\, a+0 = b+0 \,.\supset.\, a=b$ (1)
$a,b,c \varepsilon N_0 : a+c=b+c \,.\supset.\, a=b : P1{\cdot}4 :\supset:$
$a+(c+1) = b+(c+1) \,.\supset.\, a+c = b+c \,.\supset.\, a=b$ (2)
$(1) \,.\, (2) \,.\, \text{Induct} \,.\supset.\, P \;]$

·6 $a,b,c \varepsilon N_0 \,.\supset:\, a=b \,.=.\, a+c = b+c$

$[\; P4{\cdot}0 \,.\, \text{Export} \,.\supset.\!\cdot.\; a,b,c \varepsilon N_0 \,.\supset:\, a=b \,.\supset.\, a+c=b+c$ (1)
$P4{\cdot}5 \,.\, \text{Export} \,.\supset.\!\cdot.\;$ » $\,.\supset:\, a+c=b+c \,.\supset.\, a=b$ (2)
$(1) \,.\, (2) \,.\, I\S2P4{\cdot}1 \,.\supset.\, P \;]$

·7 $a,b \varepsilon N_0 \,.\, a \mathrel{-=} 0 \,.\supset.\, a+b \mathrel{-=} 0$

$[\; a \varepsilon N_0 \,.\, a \mathrel{-=} 0 \,.\, Df+ \,.\supset.\, a+0 \mathrel{-=} 0$ (1)
$a,b \varepsilon N_0 \,.\, P1{\cdot}5 \,.\supset.\, a+(b+1)=(a+b)+1 \,.\, (a+b)+1 \mathrel{-=} 0 \,.\supset.$
$a+(b+1) \mathrel{-=} 0$ (2)
$(1) \,.\, (2) \,.\, \text{Induct} \,.\supset.\, P \;]$

·8 $a,b \varepsilon N_0 \,.\supset:\, a+b=0 \,.=.\, a=0 \,.\, b=0$

$[\; P{\cdot}7 \,.\, \text{Transp} \,.\supset:\, a,b \varepsilon N_0 \,.\, a+b=0 \,.\supset.\, a=0$ (1)
$(1) \,.\, \text{Comm}+ \,.\supset:$ » » $\,.\supset.\, b=0$ (2)
$(1) \,.\, (2) \,.\supset:$ » » $\,.\supset.\, a=0 \,.\, b=0$ (3)
$Df+ \,.\supset.\, 0+0=0$ (4)
$(3) \,.\, (4) \,.\supset.\, P \;]$

Super analysi de idea de numero, et suo historia, vide Formulario anteriore, et novo publicationes:

Huntington, *A complete set of postulates for the theorie of absolute continuous magnitude.* AmericanT. a.1902 p.264.

Dickson, *Definitions of a field by independent postulates,* AmericanJ. a.1903 p.13.

Dickson, *Definitions of a linear associative algebra by independent postulates,* Id. p.21.

Huntington, *Two definitions of an Abelian group by sets of independent postulates,* Id. p.27.

Huntington, *Definitions of a field by sets of independent postulates,* Id. p.31.

Huntington, *On a new edition of* Stolz's *Allgemeine Arithmetik, with an account of* Peano's *definition of number.* AmericanB. a.1902, t.9 p.40.

C. Burali-Forti. *Sulla teoria generale delle grandezze e dei numeri.* Atti Acc. Sc. Torino, 1904.

Encyclopédie des sciences mathématiques, t. 1, a. 1904.

§2 ×

❋ 1. $a,b,c \varepsilon N_0 .\supset.$ ·0 $a\times 0 = 0$] Df×
·01 $a\times(b+1) = (a\times b)+a$]

Definitione de signo × per inductione.

Lege $a\times b$ « a multiplicato per b », vel « a per b », vel « producto de a per b ». a et b es dicto « factore ».

Signo × se inveni in Oughtred, *Clavis mathematica*, a. 1631.

Illo es signo hieroglyphico N. 87 de antiquo Ægyptios.

·02 $a\times 1 = a$ [$(0 \mid b)$ P·01 $\supset$ P]
$a\times 2 = a+a$ [$(1 \mid b)$ » »]

·03 $ab = a\times b$. $a\times b\times c = (a\times b)\times c$.
$a\times b+c = (a\times b)+c$. $a+b\times c = a+(b\times c)$ Df

·1 $a\times b \ \varepsilon\ N_0$

[$a\varepsilon N_0$. Df× $.\supset.$ $a\times 0 \ \varepsilon\ N_0$ (1)
$a,b\varepsilon\ N_0$. $a\times b \ \varepsilon\ N_0$. Df× . §+ 4·1 $.\supset.$ $a\times(b+1) \ \varepsilon\ N_0$ (2)
(1) . (2) . Induct $.\supset.$ P]

·2 $a(b+c) = ab+ac$ Distrib(×,+)

[$a,b\varepsilon N_0$ $.\supset.$ $a\times(b+0) = a\times b + a\times 0$ (1)
$a,b,c\varepsilon N_0$. $a(b+c) = ab+ac$. Assoc+ . Df× $.\supset.$ $a[b+(c+1)] =$
$a[(b+c)+1] = a(b+c)+a = ab+ac+a = ab+a(c+1)$ (2)
(1) . (2) . Induct $.\supset.$ P]

·3 $(a+b)c = ac + bc$ Distrib(×,+)

[$a,b\varepsilon N_0$ $.\supset.$ $(a+b)\times 0 = a\times 0+b\times 0$ (1)
$a,b,c\varepsilon N_0$. $(a+b)c = ac+bc$. Df × $.\supset.$ $(a+b)(c+1) = (a+b)c+(a+b) =$
$ac+bc+a+b = (ac+a)+(bc+b) = a(c+1)+b(c+1)$ 2)
(1) . (2) . Induct $.\supset.$ P]

·4 $ab = ba$ Comm×

[Df× $.\supset.$ $0\times 0 = 0$ (1)
$a\varepsilon N_0$. $0\times a = 0$. Df× $.\supset.$ $0\times(a+1) = 0\times a+0 = 0$ (2)
(1) . (2) . Induct $.\supset.$ $0\times a = 0$ (3)
Df× $.\supset.$ $1\times 0 = 0$ (4)
$a\varepsilon N_0$. $1\times a = a$ $.\supset.$ $1\times(a+1) = 1\times a+1 = a+1$ (5)
(4) . (5) . Induct $.\supset.$ $1\times a = a$ (6)

$a\varepsilon N_0 \,.\, P{\cdot}0 \,.\, (3) \,.\supset.\, a\times 0 = 0\times a$ (7)

$a,b\varepsilon N_0 \,.\, ab = ba \,.\, Df\times \,.\supset.\, a(b+1) = ab+a = ba+a$

$(6) \,.\, P{\cdot}3 \,.\supset.$ » $= ba+1a = (b+1)a$ (8)

$(7) \,.\, (8) \,.\,$ Induct $.\supset.\, P$]

·5 $(a\times b)\times c = a\times(b\times c)$ Assoc×

[$(a\times b)\times 0 = a\times(b\times 0)$ (1)

$a,b,c\varepsilon N_0 \,.\, (ab)c = a(bc) \,.\, Df\times \,.\,$ Distrib$(\times,+)$ $.\supset.$

$(ab)(c+1) = (ab)c+ab = a(bc)+ab = a(bc+b) = a[b(c+1)]$ (2)

$(1) \,.\, (2) \,.\,$ Induct $.\supset.\, P$]

·6 $a\times b = 0 \,.\, a \mathrel{-=} 0 \,.\supset.\, b=0$

·7 $a\times b = 0 \,.=.\, a=0 \,.\cup.\, b=0$

·8 $a\times c = b\times c \,.\, c \mathrel{-=} 0 \,.\supset.\, a=b$

❋ 2. $a,b,c,d\varepsilon N_0 \,.\supset.$

·1 $(a+b)(c+d) = ac+bc+ad+bd$

·2 $(a+b)(b+c)(c+a)+abc = (ab+bc+ca)(a+b+c)$

·3 $ab(a+b)+bc(b+c)+ca(c+a)+2abc = (a+b)(b+c)(c+a)$

§3 $\upharpoonright$

* 1. $a,b,c \varepsilon N_0 . \supset . \cdot 0 \quad a\upharpoonright 0 = 1$
$\cdot 01 \quad a\upharpoonright(b+1) = (a\upharpoonright b) \times a$ | Df $\upharpoonright$

Lege $a\upharpoonright b$, « a ad potentia b » « a ad potestate b », « a ad b ». Es definito per inductione. a es « basi », b es « exponente ».

Notatione de P·5 es de Descartes a. 1637. De Morgan a. 1845 propone signo $\upharpoonright$ (sub forma pauco differente), que es signo de radice verso. Illo es commodo quando exponente es expressione complexo, et necessario in citatione de P1·0, 1·3, etc.

·02 $a\upharpoonright 1 = a$ [Df $\upharpoonright$. $b=0 . \supset .$ P]
$a\upharpoonright 2 = a \times a$ [» . $b=1$. »]

·03 $1\upharpoonright b = 1$

[$1\upharpoonright 0 = 1$ (1)
$b \varepsilon N_0 . 1\upharpoonright b = 1 . \supset . 1\upharpoonright(b+1) = (1\upharpoonright b) \times 1 = 1 \times 1 = 1$ (2)
(1) . (2) . Induct $. \supset .$ P]

·04 $0\upharpoonright(a+1) = 0$

·05 $a\upharpoonright b+c = (a\upharpoonright b)+c \quad . \quad a+b\upharpoonright c = a+(b\upharpoonright c)$
$a\upharpoonright b \times c = (a\upharpoonright b) \times c \quad . \quad a \times b\upharpoonright c = a \times (b\upharpoonright c)$ Df

·1 $a\upharpoonright b \,\varepsilon N_0$

[$a \varepsilon N_0$. Df $\upharpoonright . \supset . a\upharpoonright 0 \,\varepsilon N_0$ (1)
$a,b \varepsilon N_0 . a\upharpoonright b \varepsilon N_0 . \supset . (a\upharpoonright b) \times a \,\varepsilon N_0$. Df $\upharpoonright . \supset . a\upharpoonright(b+1) \,\varepsilon N_0$ (2)
(1) . (2) . Induct $. \supset .$ P]

·2 $a\upharpoonright(b+c) = a\upharpoonright b \times a\upharpoonright c$

[$a,b \varepsilon N_0 . \supset . a\upharpoonright(b+0) = a\upharpoonright b = (a\upharpoonright b) \times 1 = a\upharpoonright b \times a\upharpoonright 0$ (1)
$a,b,c \varepsilon N_0 . a\upharpoonright(b+c) = a\upharpoonright b \times a\upharpoonright c . \supset .$
$a\upharpoonright[b+(c+1)] = a\upharpoonright[(b+c)+1] = a\upharpoonright(b+c) \times a =$
$a\upharpoonright b \times a\upharpoonright c \times a = a\upharpoonright b \times (a\upharpoonright c \times a) = a\upharpoonright b \times a\upharpoonright(c+1)$ (2)
(1) . (2) . Induct $. \supset .$ P]

·3 $(a \times b)\upharpoonright c = (a\upharpoonright c)(b\upharpoonright c)$ Distrib($\upharpoonright, \times$)

[$a,b \varepsilon N_0 . \supset . (a \times b)\upharpoonright 0 = a\upharpoonright 0 \times b\upharpoonright 0$ (1)
$a,b,c \varepsilon N_0 . (a \times b)\upharpoonright c = a\upharpoonright c \times b\upharpoonright c . \supset . (a \times b)\upharpoonright(c+1) = (a \times b)\upharpoonright c \times (a \times b) =$
$(a\upharpoonright c) \times (b\upharpoonright c) \times a \times b = [(a\upharpoonright c) \times a] \times [(b\upharpoonright c) \times b] = a\upharpoonright(c+1) \times b\upharpoonright(c+1)$ (2)
(1) . (2) . Induct $. \supset .$ P]

·4 $(a \upharpoonright b) \upharpoonright c = a \upharpoonright (b \times c)$

[$(a \upharpoonright b) \upharpoonright 0 = a \upharpoonright (b \times 0) = 1$ (1)

. $(a \upharpoonright b) \upharpoonright c = a \upharpoonright (b \times c)$.⊃. $(a \upharpoonright b) \upharpoonright (c+1) = (a \upharpoonright b) \upharpoonright c \times (a \upharpoonright b) =$

$a \upharpoonright (b \times c) \times a \upharpoonright b = a \upharpoonright (b \times c + b) = a \upharpoonright [b(c+1)]$ (2)

(1) . (2) . Induct .⊃. P]

·5 $a^b = a \upharpoonright b$ Df

✻ 2. $a, b \varepsilon N_0$.⊃.

·1 $(a+b)^2 = a^2+2ab+b^2$

[» $= (a+b)(a+b) = aa+ab+ba+bb =$ »]

·2 $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

[» $= (a+b)^2(a+b) = (a^2+2ab+b^2)(a+b) = a^3+2a^2b+ab^2+$

$a^2b+2ab^2+b^3 =$ »]

·3 $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

·4 $(a+b)^3 = a^3+b^3+3ab(a+b)$

·5 $(a+b)^4+a^4+b^4 = 2(a^2+ab+b^2)^2$

$(a+b)^6+a^6+b^6 = 2(a^2+ab+b^2)^3+3[ab(a+b)]^2$

·6 $a^2(a+b)^2+a^2b^2+(a+b)^2b^2 = (a^2+ab+b^2)^2$

·7 $(a+b)^5 = a^5+b^5+5ab(a+b)(a^2+ab+b^2)$

·8 $(a+b)^7 = a^7+b^7+7ab(a+b)(a^2+ab+b^2)^2$

·9 $(a+b)^8+a^8+b^8 = 2(a^2+ab+b^2)[(a^2+ab+b^2)^3+4a^2b^2(a+b)^2]$

✻ 3. $a, b, c \varepsilon N_0$.⊃.

·1 $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$

·2 $(a+b+c)^3 = a^3+b^3+c^3+3(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)+6abc$

·3 » $= a^3+b^3+c^3+3(a+b)(a+c)(b+c)$

·4 » $+3abc = a^3+b^3+c^3+3(a+b+c)(ab+ac+bc)$

·5 » $+a^3+b^3+c^3 = (b+c)^3+(c+a)^3+(a+b)^3+6abc$

·6 $(a+b+c)^4+a^4+b^4+c^4 = 2(a^2+b^2+c^2)(a+b+c)^2+$

$8abc(a+b+c)+2(a^2b^2+a^2c^2+b^2c^2)$

·7 $(a+b+c)^4+a^4+b^4+c^4 =$

$(a+b)^4+(a+c)^4+(b+c)^4+3 \times 4abc(a+b+c)$

·8 $(a+b+c)^5 = a^5+b^5+c^5+5(a+b)(a+c)(b+c)(a^2+b^2+c^2+ab$

$+ac+bc)$

$(a+b+c)^4+(a^4+b^4+c^4)+4abc(a+b+c) =$

$2\{(a+b)^2(b+c)^2+(b+c)^2(c+a)^2+(c+a)^2(a+b)^2\}$

·9 $a(b+c)^2+b(c+a)^2+c(a+b)^2 = (a+b)(b+c)(c+a)+4abc$

§4 Cls'

✻ 1·0 $k\varepsilon\, \mathrm{Cls} \,.\supset.\, \mathrm{Cls}'k = \mathrm{Cls} \cap x\ni(x\supset k)$ Df

Si k es classe, Cls'k, lege « classe de k » indica classe que continere in k, vel omni classe x, que satisfac ad conditione $x\supset k$.

Si u es classe de numero, et a es numero, tunc $u+a$ indica classe de numero que nos obtine, si nos adde ad singulo numero u, numero a; vel es classe composito de omni objecto x, reductibile ad forma $x=y+a$, ubi y es aliquo individuo de classe u; in symbolo:

$u,v,w\varepsilon\, \mathrm{Cls}'\mathrm{N}_0 \,.\, a\varepsilon \mathrm{N}_0 \,.\supset.$

·1 $u+a = x\ni[\exists\, u \cap y\ni(x=y+a)]$ Df

·2 $a+u =$ » » $(x=a+y)]$ Df

·3 $u+v = x\ni[\exists(y;z)\ni(y\varepsilon u \,.\, z\varepsilon v \,.\, x=y+z)]$ Df

·4 $a+u=u+a$ ·5 $u+v=v+u$ { Comm + }

·6 $u+(v+w)=(u+v)+w=u+v+w$ { Assoc + }

·7 $a+(u\cup v)=(a+u)\cup(a+v)$ { Distrib $(+,\cup)$ }

$u+(v\cup w)=(u+v)\cup(u+w)$ »

✻ 2. $u,v,w\varepsilon\, \mathrm{Cls}'\mathrm{N}_0 \,.\, a,b\varepsilon \mathrm{N}_0 \,.\supset:$

·1-·3 $(\times \mid +)$ P1·1-·3 Df

·4 $au=ua$ ·5 $uv=vu$ ·6 $u(vw)=(uv)w=uvw$

·7 $a(u+v)=au+av$

·8 $u(a+b)\supset ua+ub$ ·9 $u(v+w)\supset uv+uw$

✻ 3. $u,v\varepsilon\, \mathrm{Cls}'\mathrm{N}_0 \,.\, a,b\varepsilon \mathrm{N}_0 \,.\supset.$

·1-·3 $(\uparrow \mid +)$ P1·1-·3 Df

·4 $a\uparrow(u+v)=(a\uparrow u)\times(a\uparrow v)$ ·5 $(u\times v)^a = u^a\times v^a$

·6 $(u\uparrow a)\uparrow b = u\uparrow(ab)$

·7 $(a\times b)\uparrow u \supset a\uparrow u \times b\uparrow u$ ·8 $u^{a+b}\supset u^a u^b$

§5 N_1 $>$

✱ 1·0 $N_1 = N_0+1$ Df

N_1, lege « numero naturale », indica successivo de numero.

·1 $N_1 \supset N_0$

[$a\varepsilon N_0 . b=a+1$. §+1·2 .⊃. $b\varepsilon N_0$ (1)
(1) . Elim$a$.⊃: $\exists\, N_0 \frown a\ni(b=a+1)$.⊃. $b\varepsilon N_0$
(2) . DfN_1 .⊃: $b\varepsilon N_1$.⊃. $b\varepsilon N_0$ (3)
(3) . Oper $b\ni$.⊃. P]

·2 $N_0 = \iota 0 \cup N_1$

[§+1·1 . §ι1·2 . P·1 .⊃. $\iota 0 \supset N_0 . N_1 \supset N_0$. I §∪P3·2 .⊃. $\iota 0 \cup N_1 \supset N_0$ (1)
§= ·1 .⊃. $0\,\varepsilon\,\iota 0$ (2)
Df N_1 .⊃: $a\varepsilon N_0$.⊃. $a+1\,\varepsilon N_1$ (3)
(2) . (3) . Induct .⊃. $N_0 \supset \iota 0 \cup N_1$ (4) (1) . (4) .⊃. P]

·3 $0 -\varepsilon N_1$

[§+1·5 .⊃: $a\varepsilon N_0 . b=a+1$.⊃. $b -= 0$
Elim a . DfN_1 .⊃: $b\varepsilon N_1$.⊃. $b -\varepsilon \iota 0$
Comm(−,ε) .⊃: $b\varepsilon N_1$.⊃. $b\varepsilon -\iota 0$
Oper $b\ni$.⊃. $N_1 \supset -\iota 0$
Transp .⊃. $\iota 0 \supset -N_1$
§ι1·2 .⊃. P]

·4 $N_1 = N_0 - \iota 0$ Dfp

[P·3 . Oper$-\iota 0$.⊃. $N_0 - \iota 0 . = N_1 - \iota 0 . N_1 - \iota 0 = N_1$.⊃. P]

·5 $N_0 - N_1 = \iota 0$ ·6 $0 = \overline{\iota}\, N_0 - N_1$

✱ 2. $a,b,c,d\varepsilon N_0$.⊃:

·0 $b>a$.=. $b\varepsilon\, a+N_1$ Df>

·01 $a<b$.=. $b>a$ Df<

Nos dice que b es majore de a, et que a es minore de b, et nos scribe $b>a$, $a<b$, quando b es summa de a cum aliquo numero naturale.

·02 $a>b>c$.=. $a>b . b>c$ Df

·1 $c>b . b>a$.⊃. $c>a$

[$x,y\varepsilon N_1 . c=b+y . b=a+x$.⊃. $c=a+x+y=a+(x+y)$.⊃. $c>a$ (1)
(1) . Elim$(x;y)$.⊃. P]

·2 $b>a .=. b+c>a+c$

[$x\varepsilon N_1 . b=a+x .\supset. b+c=a+x+c=a+c+x .\supset. b+c>a+c$ (1)

$x\varepsilon N_1 . b+c=a+c+x . \S+ \mathrm{P}4{\cdot}5 .\supset. b=a+x .\supset. b>a$ (2)

(1).(2) . Elim $x .\supset.$ P]

·3 $b>a . d>c .\supset. b+d>a+c$

[Hp . P·2 $.\supset. b+d>a+d . a+d>a+c$. P·1 $.\supset.$ Ths]

·4 $a=0 .\cup. a>0$ [P1·2 $.\supset. a\varepsilon \iota 0 \cup N_1 .\supset.$ P]

·5 $a=b .\cup. a<b .\cup. a>b$

[$a\varepsilon N_0 . b=0$. P·4 $.\supset.$ P (1)

$a,b\varepsilon N_0 . a=b .\supset. a<b+1$ (2)

» $. a<b .\supset.$ » (3)

» $. c\varepsilon N_0 . a=b+(c+1)$. P·4 $.\supset. a=b+1 .\cup. a>b+1$ (4)

» $. a>b$. (4) . Elim$c .\supset.$ » (5)

» $: a=b .\cup. a<b .\cup. a>b$: (2)(3)(5) $:\supset:$

$a<b+1 .\cup. a=b+1 .\cup. a>b+1$ (6)

(1) . (6) . Induct $.\supset.$ P]

·6 $-(a>a)$ [$0 -\varepsilon N_1 .\supset. a+0 -\varepsilon a+N_1$]

* 3. $a,b\varepsilon N_0 .\supset:$ ·0 $b\geqq a .=. a\leqq b .=. b\varepsilon a+N_0$ Df$\geqq$

·1 » » $.=. -(b<a)$ Dfp

·2 » » $.=. b>a .\cup. b=a$ Dfp

* 4·0 $a\varepsilon N_0 . b\varepsilon a+N_0 .\supset. a\cdots b=(a+N_0)-(b+N_1)$ Df...

·1 » $= N_0 \cap x\ni(a\leqq x\leqq b)$ Dfp

$a\cdots b$ indica classe de numero inter a et b.

·2 $c,a\varepsilon N_0 . b\varepsilon a+N_0 .\supset. c+(a\cdots b)=(c+a)\cdots(c+b)$

Distrib$(+,\cdots)$

·3 $a\varepsilon N_0 .\supset. a\cdots a=\iota a$

$a,b\varepsilon N_0 .\supset.$

·4 $(0\cdots a)+(0\cdots b)=0\cdots(a+b)$ Distrib$(0\cdots,+)$

·5 $0\cdots a=0\cdots b .=. a=b$

·6 $a\leqq b .=. 0\cdots a\supset 0\cdots b .=. a\varepsilon 0\cdots b$ Dfp

·7 $N_0=N_0\times(a+1)+0\cdots a$

* 5. $a,b,c\varepsilon N_0 .\supset:$

·1 $b\varepsilon N_0\times a . c\varepsilon N_0\times b .\supset. c\varepsilon N_0\times a$ [$(\times|+)$P2·1 $\supset$ P]

·2 $b\varepsilon N_0\times a .\supset. N_0\times b\supset N_0\times a$

[P·1 . Export $.\supset.\because. a\varepsilon N_0 . b\varepsilon N_0\times a .\supset: c\varepsilon N_0\times b .\supset. c\varepsilon N_0\times a$

Oper $c\ni .\supset.$ P]

·3 $b,c \varepsilon N_0 \times a . \supset . b+c \varepsilon N_0 \times a$

[$x,y \varepsilon N_0 . b = xa . c = ya . \supset . b+c = (x+y)a . \supset . b+c \varepsilon N_0 \times a$ (1)

(1) . Elim$(x;y)$ $. \supset .$ P]

·4 $N_0 \times a + N_0 \times a = N_0 \times a$ [= P·3]

·5 $b, b+c \varepsilon N_0 \times a . \supset . c \varepsilon N_0 \times a$

·6 $b \varepsilon N_0 \times a . \supset . bc \varepsilon N_0 \times ac$

·7 $m,n \varepsilon N_0 \times c . \supset . am+bn \varepsilon N_0 \times c$

·8 $a(a+1) \varepsilon 2N_0 . a(a+1)(a+2) \varepsilon 6N_0 . a(a+1)(2a+1) \varepsilon 6N_0$

·9 $a+b \varepsilon 2N_0 .=. a,b \varepsilon 2N_0 .\cup. a,b \varepsilon 2N_0+1$

❋ 6. $a,b,c,d \varepsilon N_1 . \supset :$

·0 $a>b . \supset . ac > bc$

[$x \varepsilon N_1 . a = b+x . \supset . ac = bc+xc . \supset . ac > bc$ (1)

(1) . Elim$x$ $. \supset .$ P]

·1 $ac > bc . \supset . a>b$

[P·0 . $a \leqq b . \supset . ac \leqq bc$. Transp $. \supset .$ P]

·2 $a>b .=. ac > bc$ [=P·0·1]

·3 $a>b . c>d . \supset . ac > bd$

[Hp . P·0 $. \supset . ac>bc . bc>bd . \supset .$ Ths]

·4 $a>b . c>d . \supset . ac+bd > ad+bc$

[Hp . $x,y \varepsilon N_1 . a = b+x . c = d+y . \supset . ac+bd = 2bd+by+dx+xy .$

$ad+bc = 2bd+by+dx . \supset .$ Ths]

❋ 7. $a,b,m,n \varepsilon N_1 . \supset :$

·1 $a>b . \supset . a^m > b^m$

[$m = 1 . \supset .$ P (1)

$m \varepsilon N_1 . a>b . a^m > b^m . \supset . a^{m+1} > b^{m+1}$ (2)

(1) . (2) . Induct $. \supset .$ P]

·2 $a>1 . m>n . \supset . a^m > a^n$

[Hp . $p \varepsilon N_1 . m = n+p . \supset . a \uparrow (n+p) = a \uparrow n \times a \uparrow p . a \uparrow p > 1 . \supset .$ P]

❋ 8·1 $N_0^2 \supset 4N_0 \cup (4N_0+1)$

[$N_0 = 2N_0 \cup (2N_0+1) . (2N_0)^2 \supset 4N_0 . (2N_0+1)^2 \supset 4N_0+1 . \supset .$ P]

$N_0^2 \supset 3N_0 \cup (3N_0+1)$

$N_0^2 \supset 4N_0 \cup (8N_0+1)$

$N_0^2 \supset 5N_0 \cup (5N_0+1) \cup (5N_0+4)$

$N_0^3 \supset 7N_0 \cup (7N_0+1) \cup (7N_0+6)$

$N_0^3 \supset 9N_0 \cup (9N_0+1) \cup (9N_0+8)$

$N_0^4 \supset 5N_0 \cup (5N_0+1)$

❋ 9.

·1 $N_1 \supset N_1^2+N_0^2+N_0^2+N_0^2$ } Bachet a.1621 p.241 :
« Omnem autem numerum vel quadratum esse vel ex duobus aut tribus aut etiam quatuor quadratis componi, satis experiendo deprehendis. » {

·2 $N_1^3+N_1^3 \supset N_1 - N_1^3$ } Alchodschandî a.992 {

·3 $N_1^4+N_1^4 \supset N_1 - N_1^2$ } Fermat t.1 p.327 {

·4 $n \varepsilon N_0+3 \,.\supset.\, N_1^n+N_1^n \supset N_1 - N_1^n$ } Fermat t.1 p.291 :
« Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duas eiusdem nominis fas est dividere: cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet ». {

❋ 11. $a,b \varepsilon N_1 \,.\, a -= b \,.\supset.$

·1 $a^2+b^2 > 2ab$ [P6·4 ⊃ P]

·2 $2(a^2+b^2) > (a+b)^2$ [P·1 ⊃ P]

·3 $(a+b)^2 > 4ab$ } Euclide vi P27 { [P·1 ⊃ P]

·4 $a^3+b^3 > ab(a+b)$ } Harriot p.79 {
[P·1 $.\supset.$ $(a^2+b^2)(a+b) > 2ab(a+b)$ $.\supset.$ P]

·5 $2(a^3+b^3) > (a+b)(a^2+b^2)$
[P·4 $.\supset.$ $2(a^3+b^3) > ab(a+b)+a^3+b^3$ $.\supset.$ P]

·6 $4(a^3+b^3) > (a+b)^3$
[P·5 $.\supset.$ $4(a^3+b^3) > (a+b) \times 2(a^2+b^2)$. P·2 $.\supset.$ P]

·7 $3(a^4+a^2b^2+b^4) > (a^2+ab+b^2)^2$ } Bertrand a.1855 p.142 {

·8 $2(a^4+a^2b^2+b^4) > 3ab(a^2+b^2)$

·9 $4(a^2+ab+b^2)^3 > 3^3(a^2b+ab^2)^2$ } Harriot a.1631 p.85 {

·91 $3^3(a^3+a^2b+ab^2+b^3)^4 > 4^4a^3b^3(a^2+ab+b^2)^3$

❋ 12. $a,b,c \varepsilon N_1 \,.\, -(a=b=c) \,.\supset.$

·11 $a^2+b^2+c^2 > ab+ac+bc$ [P11·1 ⊃ P]

·12 $3(a^2+b^2+c^2) > (a+b+c)^2 > 3(ab+ac+bc)$ [P·11 ⊃ P]

·13 $a \leqq b \leqq c \,.\, a+b > c \,.\supset.\, 2(ab+ac+bc) > a^2+b^2+c^2$
[Hp $.\supset.$ $(b+c)a > a^2 \,.\, (c+a)b > b^2 \,.\, (a+b)c > c^2$ $.\supset.$ P]

·15 $2(a^3+b^3+c^3) > a^2(b+c)+b^2(c+a)+c^2(a+b)$ [P11·4 ⊃ P]

·16 $3(a^3+b^3+c^3) > (a+b+c)(a^2+b^2+c^2)$ [P·15 ⊃ P]

·17 $3(a^3+b^3+c^3) > (a+b+c)(ab+ac+bc)$ [P·11 . P·16 .⊃ P]

·18 $9(a^3+b^3+c^3)>(a+b+c)^3$ [P·16 . P.12 .⊃ P]

·19 $a^2(b+c)+b^2(c+a)+c^2(a+b)>6abc$ [P11·1 .⊃ P]

·20 $a^3+b^3+c^3>3abc$ [P·15 . P·19 .⊃P]

{ ·11·12·19·20 HARRIOT a.1631 p.84 {

·21 $8(a^3+b^3+c^3)>3(a+b)(a+c)(b+c)$ [P·15 . P·20 .⊃P]

·22 $(a+b+c)(a^2+b^2+c^2)>9abc$ [P·19 . P·20 .⊃P]

·23 $2(a+b+c)(a^2+b^2+c^2)>3[a^2(b+c)+b^2(c+a)+c^2(a+b)]$ [P·15⊃P]

·24 $(a+b+c)^3>3^3abc$ [P·19 . P·20 .⊃. P]

{HARRIOT a.1631 p.85 : « Si quantitas secetur in tres partes inæquales Cubus è tertia parte totius major est solido è tribus partibus inæqualibus. Si sint quantitatis tres partes inæquales p, q, r, est...

$$\left.\begin{array}{l} p+q+r \\ p+q+r \\ \underline{p+q+r} \end{array}\right| > 27pqr$$

» {

·25 $(a+b)(b+c)(c+a)>8abc$ [P·19 ⊃ P]

·26 $a^3+b^3+c^3+3abc>a^2(b+c)+b^2(c+a)+c^2(a+b)$

·27 $a>b>c$.⊃. $a^2b+b^2c+c^2a>a^2c+b^2a+c^2b$

·30 $(ab+ac+bc)^2>3abc(a+b+c)$ [$(bc,ca,ab) \mid (a,b,c)$ P·12 ⊃ P]

·31 $a^4+b^4+c^4>abc(a+b+c)$

[$(a^2,b^2,c^2) \mid (a,b,c)$P·11 .⊃. $a^4+b^4+c^4>a^2b^2+a^2c^2+b^2c^2$ (1)
$(ab,ac,bc) \mid (a,b,c)$P·11 .⊃. $a^2b^2+a^2c^2+b^2c^2>abc(a+b+c)$ (2)
(1) . (2) .⊃. P]

·32 $(a+b+c)(a^3+b^3+c^3)>(a^2+b^2+c^2)^2$

❋ 13. $a,b,c,d \varepsilon N_1$. -$(a=b=c=d)$.⊃.

·1 $4(a^2+b^2+c^2+d^2)>(a+b+c+d)^2$

·2 $3(a^2+b^2+c^2+d^2)>2(ab+ac+ad+bc+bd+cd)$ [P·1⊃P]

·3 $3(a+b+c+d)^2 > 8($ » » $)$ [P·2⊃P]

{ ·2·3 MACLAURIN a.1726 LondonT. t.34 p.109, 104, 112 {

·4 $a,b,c,d \varepsilon N_1$. -$(ad=bc)$.⊃. $(a^2+b^2)(c^2+d^2)>(ac+bd)^2$

·5 $a,b,c,d \varepsilon N_1$. -$(ad=bc)$.⊃. $(a^3d^2+b^3c^2)(c+d)^2>(a+b)^3c^2d^2$

·6 $a,b,c,d,e,f \varepsilon N_1$. -$(aef=bdf=cde)$.⊃.
$(a^2+b^2+c^2)\times(d^2+e^2+f^2)>(ad+be+cf)^2$

✻ 14. $a,b,m \varepsilon N_1 . a -= b .\supset.$

·1 $a^{m+2}+b^{m+2} > ab(a^m+b^m)$

[Hp . $a>b$ $.\supset.$ $a^{m+1}>b^{m+1}$. P6·4 $.\supset.$ P]

·2 $2^m(a^{m+1}+b^{m+1}) > (a+b)^{m+1}$

✻ 15. $a,b,c \varepsilon N_1 .\supset:$

·1 $a^2 \varepsilon N_1 b^2 .=. a \varepsilon N_1 b$ ·2 $a^3 \varepsilon N_1 b^3 .=. a \varepsilon N_1 b$

} EUCLIDE VIII P14-17:

Ἐὰν τετράγωνος τετράγωνον μετρῇ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσει καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρῇ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσει.

Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μετρῇ, ——————————————————————————————, *καὶ ὁ κύβος τὸν κύβον* ———. {

·3 $a \varepsilon N_1 \times b .=. a^m \varepsilon N_1 \times b^m$ } EUCLIDE VIII P6 {

·4 $a^2+b^2 \varepsilon 3N_1 .\supset. a,b \varepsilon 3N_1$: $a^2+b^2 \varepsilon 7N_1 .\supset. a,b \varepsilon 7N_1$

·5 $a^2 = b^2+c^2 .\supset. b \varepsilon 3N_1 \cup c \varepsilon 3N_1 . b \varepsilon 4N_1 \cup c \varepsilon 4N_1 .$
$a \varepsilon 5N_1 \cup b \varepsilon 5N_1 \cup c \varepsilon 5N_1 . abc \varepsilon 3\times4\times5\times N_1$

} FRÉNICLE a.1676 p.76-79 {

·6 $m \varepsilon 2N_1+1 .\supset. a^m+b^m \varepsilon N_1\times(a+b)$

·7 $n \varepsilon N_1 - 3N_1 .\supset. a^{2n}+a^n+1 \varepsilon N_1 \times (a^2+a+1)$

} EULER Op. post. t.1 p.186 {

·8 $a^2 \varepsilon N_1^3 .\supset. a \varepsilon N_1^3$ } EUCLIDE IX P6:

Ἐὰν ἀριθμὸς ἑαυτὸν πολλαπλασιάσας κύβον ποιῇ, καὶ αὐτὸς κύβος ἔσται. {

[$a,b \varepsilon N_1 . a^2=b^3 .\supset. a^2=b\times b^2 .\supset. a^2 \varepsilon N_1\times b^2$. P15·1 $.\supset. a \varepsilon N_1\times b$
$a,b,c \varepsilon N_1 . a=bc . a^2=b^3 .\supset. b^2c^2=b^3 .\supset. c^2=b .\supset. a=c^3 .\supset. a \varepsilon N_1^3$]

·9 $(N_1\times a^2) \frown N_1^2 = N_1^2 \times a^2$ [P15·1 $.\supset.$ P]

✻ 16. $a \varepsilon N_1 .\supset.$

·1 $a+1, a^2+1 \varepsilon 2N_1^2 .=. a=1 \cup a=7$ } FERMAT t.2 p.434 {

·2 $a^2+2 \varepsilon N_1^3 .=. a=5$ ·3 $a^2+4 \varepsilon N_1^3 .=. a=2 \cup a=11$

} FERMAT a.1657 t.2 p.345:

«... il n'y a qu'un seul nombre quarré en entiers qui, joint au binaire, fasse un cube, et le dit carré est 25... »

«... si on cherche un quarré qui, ajouté à 4 fasse un cube, on n'en trouvera jamais que deux en nombres entiers, savoir 4 et 121. » {

·4 $a^2+a \varepsilon 2N_1^4 .\supset. a=1$ } FERMAT t.1 p.341 {

·5 $a(a+1)(a+2)(a+3)+1 \ \varepsilon\ N_1^2$ $[= (a^2+3a+1)^2]$

·6 $a(a+1), a(a+1)(a+2) -\varepsilon\ N_1^2 \cup N_1^3$

·7 $2\times(N_0^2+N_0^2) \supset N_0^2+N_0^2 \ . \ (N_1^2+N_1^2)\times(N_1^2+N_1^2) \supset N_0^2+N_1^2$

$(2N_0+1)-(8N_0+7) \supset N_0^2+N_0^2+N_1^2$ } LEGENDRE a.1797 p.398 {

$2N_0+1 \supset N_0^2+N_0^2+2N_0^2$ » »

$n\varepsilon N_1 .\supset. (N_0^2+N_0^2+N_0^2)^n \supset N_0^2+N_0^2+N_0^2$

·8 $(N_1+1)^2 \supset N_1^2 + N_1^2 + N_0^2 + N_0^2$
} P. TANNERY IdM. a.1898 t.5 p.281 {

·9 $(8N_0+7)\times N_1^2 \supset N_1^2+N_1^2+N_1^2+N_1^2$
} FERMAT a.1636 t.2 p.66:
« Octuplum cuiuslibet numeri unitate deminutum componitur ex quatuor quadratis tantum, non solum in integris sed etiam in fractis ».{

✻ 17.

·1 $a\varepsilon N_1 . a^2 \varepsilon N_1^2+2N_1^2 .\supset. a\varepsilon N_1^2+2N_1^2$
} FERMAT t.1 p.340; Dm. EULER *Algèbre* t.2 c.13 {

·3 $N_1^4+N_1^2 \supset N_1-N_1^4$ } FERMAT t.1 p.340 {

·6 $N_1^2\times(4N_0+3) \supset N_1-(N_0^2+N_0^2)$ } FERMAT a.1640 t.2 p.203:
« Un nombre moindre de l'unité qu'un multiple du quaternaire n'est ni quarré, ni composé de deux quarrés, ni en entiers, ni en fractions ».{

·7 $N_1^2\times(8N_0+7) \supset N_1-(N_0^2+N_0^2+N_0^2)$ } FERMAT voir 9·22 {

§6 —

* 1·0 $a\varepsilon N_0 . b\varepsilon\, a+N_0 .\supset. b-a = \imath[N_0 \frown x\ni(x+a=b)]$ Df —

·1 » » $b-a\, \varepsilon N_0 \quad . \quad (b-a)+a = b$

[$Hp .\supset. \exists\, N_0 \frown x\ni(b=x+a)$ (1)

$Hp . x,y\varepsilon N_0 . b=x+a . b=y+a . \S+ 4\cdot5 .\supset. x=y$ (2)

$(1) . (2) . \S\imath P\cdot1 .\supset. b-a\, \varepsilon\, N_0 \frown x\ni(x+a=b)$]

Si a es numero, et b es numero superiore aut æquale ad a, tunc $b-a$, lege « b minus a » indica illo numero x tale que $x+a$ vale b. Definitione de signo —.

Si nos opera per signo $x=$ super ambo membro de definitione, id es, si nos scribe $x=$ ante ambo membro, nos obtine

$$x=b-a .=. x=\imath[N_0 \frown x\ni(x+a=b)]$$

Si, in loco de $x=$, in secundo membro, nos scribe $x\varepsilon\iota$ (per $\S\iota$), et si nos mene que ι et $\imath$ se destrue, nos habe

$$x=b-a .=. x\varepsilon\, [N_0 \frown x\ni(x+a=b)]$$

Nos distribue signo $x\varepsilon$ ad duo factore logico de secundo membro, et supprime $x\varepsilon x\ni$, que mutuo se destrue, et habe

$$x=b-a .=. x\varepsilon N_0 . x+a=b.$$

Viceversa, de ultimo propositione, si nos extrahe signo $b-a$ in primo membro, nos deduce definitione de —.

·2 $a,b\varepsilon N_0 .\supset. (b+a)-a=b$

[$(b+a)|b\, P\cdot1 .\supset. [(b+a)-a]+a=b+a . \S+4\cdot5 .\supset. P$]

·3 $a\varepsilon N_0 .\supset. a-0=a \quad . \quad a-a=0$

[$a+0=a$. Oper—0 . Oper—a $.\supset. P$]

·4 $a,b,c\varepsilon N_0 .\supset.$

$a+b-c = (a+b)-c . a-b+c = (a-b)+c . a-b-c = (a-b)-c$ Df

·5 $b,c\varepsilon N_0 . a\varepsilon\, b+c+N_0 .\supset. a-(b+c) = a-b-c$

[$a-b-c+(b+c) = a-b-c+(c+b) = a-b-c+c+b =$
$a-b+b = a$. Oper $-(b+c)$ $.\supset. P$]

·6 $a,c\varepsilon N_0 . b\varepsilon\, c+N_0 .\supset. a+(b-c) = a+b-c$

[$a+(b-c)+c = a+[(b-c)+c] = a+b$. Oper—c $.\supset. P$]

·7 $c\varepsilon N_0 . b\varepsilon\, c+N_0 . a\varepsilon\, b+N_0 .\supset. a-(b-c)=a-b+c$

[$a-b+c+(b-c) = a-b+c+b-c = a-b+b+c-c = a-b+b=a$.
Oper—$(b-c)$ $.\supset. P$]

·8 $b,c\varepsilon N_0 . a\varepsilon\, b+N_0 .\supset. a-b+c = a+c-b$

[$a-b+c = c+(a-b) = c+a-b = a+c-b$]

·9 $b,c\varepsilon N_0 . a\varepsilon\, b+c+N_0 .\supset. a-b-c = a-c-b$ [$P\cdot5 \supset P$]

* 2·1 $a,b \varepsilon N_0 . a > b . \supset . a - b \; \varepsilon N_1$

·2 $a,b \varepsilon N_0 . a > b . c \varepsilon a + N_0 . \supset . c - a < c - b$
[$Hp . \supset . a - b \; \varepsilon N_1 . (c-a)+(a-b) = c - b . \supset . Ths$]

·3 $a,b,c \varepsilon N_0 . a > b . b \varepsilon c + N_0 . \supset . a - c > b - c$
[$Hp . \supset . a - b \; \varepsilon N_1 . a - c = (b-c)+(a-b) . \supset . P$]

·4 $a,b,c,d \varepsilon N_0 . a > b . c < d . b \geqq d . \supset . a - c > b - d$
[$Hp . \supset . a - c > b - c . b - c > b - d . \supset . Ths$]

* 3·1 $b,c \varepsilon N_0 . a \varepsilon b + N_0 . \supset . (a-b)c = ac - bc$
[$Df- . Distrib(\times,+) . \supset . ac = [(a-b)+b]c = (a-b)c + bc . \supset . P$]

·2 $b,d \varepsilon N_0 . a \varepsilon b + N_0 . c \varepsilon d + N_0 . \supset . (a-b)(c-d) = ac + bd - bc - ad$

§7 /

* 1·0 $a \varepsilon N_1 . b \varepsilon N_1 \times a . \supset . b/a = \imath N_1 \cap x \ni (x \times a = b)$ Df/

·1 » » $b/a \; \varepsilon N_1 . (b/a) \times a = b$

·2 $a,b \varepsilon N_1 . \supset . (b \times a)/a = b$

·3 $a \varepsilon N_1 . \supset . a/1 = a . a/a = 1$

·4 $a,b,c \varepsilon N_1 . \supset . a \times b/c = (a \times b)/c .$
$a/b \times c = (a/b) \times c .$
$a/b/c = (a/b)/c$ Df

·5 $b,c \varepsilon N_1 . a \varepsilon b \times c \times N_1 . \supset . a/(b \times c) = a/b/c$

·6 $a,c \varepsilon N_1 . b \varepsilon c \times N_1 . \supset . a \times (b/c) = a \times b/c$

·7 $c \varepsilon N_1 . b \varepsilon c \times N_1 . a \varepsilon b \times N_1 . \supset . a/(b/c) = a/b \times c$

·8 $b,c \varepsilon N_1 . a \varepsilon b \times N_1 . \supset . a/b \times c = a \times b/c$

·9 $b,c \varepsilon N_1 . a \varepsilon b \times c \times N_1 . \supset . a/b/c = a/c/b$

* 2·1 $c \varepsilon N_1 . a,b \varepsilon N_1 \times c . \supset . (a+b)/c = a/c + b/c$
[$a/c+b/c) \times c = a/c \times c + b/c \times c = a + b . Oper /c . \supset . P$]

·2 $c \varepsilon N_1 . a,b \varepsilon N_1 \times c . a > b . \supset . (a-b)/c = a/c - b/c$
[$(a-b)/c + b/c = a/c . Oper -b/c . \supset . P$]

$(N_1, \times, /, 1) \mid (N_0, +, -, 0)$ §— P1·0-·9 $. \supset .$ §/ P1·0-·9

Lege b/a « b diviso per a », vel « b in a ». Ab plure P de §— nos deduce P de §—, si nos lege N_1, $\times$, /, 1 in loco de N_0, +, —, 0.

§8 num

·0 $u \varepsilon \mathrm{Cls} . \supset: \mathrm{num} u = 0 .=. u = \Lambda$ Df

·1 $u \varepsilon \mathrm{Cls} . m \varepsilon N_0 . \supset \therefore$
$\mathrm{num} u = m+1 .=: \exists u : x \varepsilon u . \supset_x . \mathrm{num}(u - \iota x) = m$ Df

·2 $u \varepsilon \mathrm{Cls} . \supset \therefore \mathrm{num} u = 1 .=: \exists u : x,y \varepsilon u . \supset_{x,y} . x = y$

[Df ·1 .⊃. $\mathrm{num} u = 1 .=: \exists u : x \varepsilon u . \supset_x . \mathrm{num}(u - \iota x) = 0$
Df ·0 » » » $. u - \iota x = \Lambda$
Transp » » » $. u \supset \iota x$
Oper $y\varepsilon$ » » » $. (y \varepsilon u . \supset_y . y = x)$
Import⊃ P]

·3 $\mathrm{num}\, \iota x = 1$ ·4 $m \varepsilon N_1 . \supset . \mathrm{num}\, 1 \cdots m = m$

·5 $u,v \varepsilon \mathrm{Cls} . u \supset v . \mathrm{num} v \,\varepsilon N_0 . \supset . \mathrm{num} u \,\varepsilon N_0 . \mathrm{num} u \leqq \mathrm{num} v$

·6 $u,v \varepsilon \mathrm{Cls} . \mathrm{num} u, \mathrm{num} v \,\varepsilon N_0 . u \frown v = \Lambda . \supset . \mathrm{num}(u \smile v) =$
$\mathrm{num} u + \mathrm{num} v$

Si u es classe, nos defini « num$u$ », lege « numero de u », per inductione, ut seque :

·0 Numero de u vale zero, quando u es classe nullo.

·1 Et numero de u vale $m+1$, quando numero de u differente ab uno suo elemento arbitrario x, vale m.

§9 max (maximo)

* 1. $u \varepsilon \mathrm{Cls}'N_0 . a,b \varepsilon N_0 . \supset:$

·0 $\max u = \imath\, u \frown x \ni (y \varepsilon u - \iota x . \supset_y . y < x)$ Df max

·1 » » $(y \varepsilon u . \supset_y . y \leqq x)$ Dfp

[P·0 . Transp .⊃. P]

Si u es classe de numero, per « maximo de u » nos intellige illo u et x tale que, si y es numero arbitrario de classe u, et differente de x, seque $y < x$

·2 $x = \max u .=: x \varepsilon u : y \varepsilon u . \supset_y . x \geqq y$

[P·1 . Oper $x \varepsilon \iota$.⊃. P]

·3 $\max \iota a = a$ ·4 $a > b . \supset . \max(\iota a \smile \iota b) = a$

·5 $\max u = \imath\, u \frown x \ni (u \supset 0 \cdots x)$ Dfp

·6 $b \geqq a \;.\supset.\; \max(a \cdots b) = b$

·7 $(0 \cdots a) \cup (0 \cdots b) = 0 \cdots \max(\iota a \cup \iota b)$

·8 $\exists u \,.\, m \varepsilon N_0 \,.\, -\exists\, u \cap (m + N_1) \;.\supset.\; \max u \;\varepsilon u$

[$Hp \,.\, m = 0 \;.\supset.\; 0 = \max u \;.\supset.\; Ths$ (1)
$m \varepsilon N_0 : u \varepsilon Cls'N_0 \,.\, \exists u \,.\, -\exists\, u \cap (m + N_0) \;.\supset_u.\; \max u \;\varepsilon\; N_0 : u \varepsilon Cls'N_0 \,.\, \exists u \,.$
$-\exists\, u \cap (m + 1 + N_0) :\supset: m \varepsilon u \;.\cup.\; m - \varepsilon u$ (2)
$Hp(2) \,.\, m \varepsilon u \;.\supset.\; m = \max u \;.\supset.\; \max u \;\varepsilon\; N_0$ (3)
$Hp(2) \,.\, m - \varepsilon u \;.\supset.\; -\exists\, u \cap (m + N_0) \,.\, \max u \;\varepsilon\; N_0$ (4)
$Hp(2) \,.\, (3) \,.\, (4) \;.\supset.\; \max u \;\varepsilon\; N_0$ (5)
$(1) \,.\, (5) \,.\, Induct \;.\supset.\; P$]

❋ 2. $u, v \varepsilon Cls'N_0 \,.\, \max u, \max v \;\varepsilon\; N_0 \;.\supset.$

·1 $\max(u \cup v) = \max(\iota \max u \cup \iota \max v)$

·2 $\max(u + v) = (\max u) + (\max v)$ Distrib(max, +)

[$Hp \;.\supset.\; \max u \;\varepsilon u \,.\, \max v \;\varepsilon v \;.\supset.\; \max u + \max v \;\varepsilon\; u + v$ (1)
$x \varepsilon u \,.\, y \varepsilon v \,.\, z = x + y \;.\supset.\; x \leqq \max u \,.\, y \leqq \max v \;.\supset.\; z \leqq \max u + \max v$ (2)
$z \varepsilon\; u + v \,.\, (2) \,.\, Elim(x;y) \;.\supset.\; z \leqq \max u + \max v$ (3)
$(1) \,.\, (3) \,.\, Df \max \;.\supset.\; P$]

·3 $\max(u \times v) = (\max u) \times (\max v)$ Distrib(max, ×)

·4 $u \varepsilon Cls'N_1 \;.\supset.\; \max(u \uparrow v) = (\max u) \uparrow (\max v)$ » » $\uparrow$

·5 $a \varepsilon N_1 \;.\supset.\; \max N_1 \cap x \ni (a \varepsilon N_1 \times x) = a$

·6 $\max u \;\varepsilon u \;.=.\; \text{num}\, u \;\varepsilon N_1$

§10 min (minimo)

❋ 1·0-·4 $(\min, <) \mid (\max, >)$ §max P·0-·4

·5 $u \varepsilon Cls'N_0 \;.\supset.\; \min u = \text{℩}\, u \cap x \ni (u \supset x + N_0)$

·7 $a, b \varepsilon N_0 \;.\supset.\; (0 \cdots a) \cap (0 \cdots b) = 0 \cdots \min(\iota a \cup \iota b)$

·8 $u \varepsilon Cls'N_0 \,.\, \exists u \;.\supset.\; \min u \;\varepsilon u$

·9 $\min N_0 = 0 \;.\; \min N_1 = 1$

❋ 2. min | max §max 2·1-·4

·5 $u \varepsilon Cls'N_0 \;.\supset.\; \max u = \min N_0 \cap x \ni (u \supset 0 \cdots x)$

·6 » $\min u = \max N_0 \cap x \ni (u \supset x + N_0)$

§11 quot (quoto) rest (resto)

* 1. $a,b \varepsilon \mathrm{N}_0 . c,d \varepsilon \mathrm{N}_1 .\supset.$

·0 $\mathrm{quot}(a,c) = \max[\mathrm{N}_0 \cap x \ni (x \times c \leqq a)]$ Df quot

·1 $\mathrm{quot}(a,c)\ \varepsilon\ \mathrm{N}_0$

$[\ 0\ \varepsilon\ \mathrm{N}_0 \cap x \ni (xc \leqq a) . -\exists \mathrm{N}_0 \cap x \ni (xc \leqq a) \cap (a + \mathrm{N}_1) .$ §max 1·8 $.\supset.$ P]

·2 $\mathrm{quot}(0,c) = 0$ $[\ \mathrm{quot}(0,c) = \max[\mathrm{N}_0 \cap x \ni (x \times c \leqq 0)] = \max \iota 0 = 0\ ]$

·3 $\mathrm{quot}(a,1) = a$ $[\mathrm{quot}(a,1) = \max[\mathrm{N}_0 \cap x \ni (x \leqq a)] = \max 0 \cdots a = a\ ]$

·4 $\mathrm{quot}(ad, cd) = \mathrm{quot}(a,c)$

$[\ \mathrm{quot}(ad, cd) = \max \mathrm{N}_0 \cap x \ni (cdx \leqq ad) = \max \mathrm{N}_0 \cap x \ni (cx \leqq a) = \mathrm{quot}(a,c)\ ]$

·5 $c \times \mathrm{quot}(a,c) \leqq a < c \times [\mathrm{quot}(a,c) + 1]$

·6 $\mathrm{quot}(a,c) = \iota\, \mathrm{N}_0 \cap x \ni [x \times c \leqq a . (x+1) \times c > a]$ Dfp

·7 $\mathrm{quot}(ac+b, c) = a + \mathrm{quot}(b,c)$

[P·5 $.\supset.$ $c \times \mathrm{quot}(b,c) \leqq b < c[\mathrm{quot}(a,c)+1]$

Oper$+ac$ $.\supset.$ $c[a + \mathrm{quot}(b,c)] \leqq ac+b < c[a + \mathrm{quot}(a,c)+1]$

P·6 $.\supset.$ P]

·8 $\mathrm{quot}(a, cd) = \mathrm{quot}[\mathrm{quot}(a,c),d]$

[$d \times \mathrm{quot}[\mathrm{quot}(a,c),d] \leqq \mathrm{quot}(a,c)$ (1)

$\mathrm{quot}(a,c) < d \times \{\mathrm{quot}[\mathrm{quot}(a,c),d]+1\}$ (2)

(2) $.\supset:$ $\mathrm{quot}(a,c)+1 < d \times \{\mathrm{quot}[\mathrm{quot}(a,c),d]+1\}$ (3)

$. \ c \times \mathrm{quot}(a,c) \leqq a$ (4)

$a < c \times [\mathrm{quot}(a,c)+1]$ (5)

(1) . Oper$\times c$. (4) $.\supset.$ $cd \times \mathrm{quot}[\mathrm{quot}(a,c),d] \leqq a$ (6)

(3) . Oper$\times c$. (5) $.\supset.$ $a < cd \times \{\mathrm{quot}[\mathrm{quot}(a,c),d]+1\}$ (7)

(6) . (7) $.\supset.$ P]

* 2. Hp P1 $.\supset:$

·0 $a < c .=. \mathrm{quot}(a,c) = 0 \ : \ a \geqq c .=. \mathrm{quot}(a,c)\ \varepsilon \mathrm{N}_1$

·1 $a > b .\supset. \mathrm{quot}(a,c) \geqq \mathrm{quot}(b,c)$

·2 $c > d .\supset. \mathrm{quot}(a,c) \leqq \mathrm{quot}(a,d)$

·3 $a > b . c < d .\supset. \mathrm{quot}(a,c) \geqq \mathrm{quot}(b,d)$

·4 $\mathrm{quot}(a+b, c) \geqq \mathrm{quot}(a,c) + \mathrm{quot}(b,c)$

·5 $\mathrm{quot}(a+b, c) \leqq \mathrm{quot}(a,c) + \mathrm{quot}(b,c) + 1$

·6 $a > \mathrm{quot}(b,c) .=. ac > b$

·7 $\text{quot}(a,c) > \text{quot}(b,d) \;.=.\; ad > bc$

·8 $c \times \text{quot}(a,d) \leqq \text{quot}(ac,\, d) < [\text{quot}(a,d)+1]c$

·9 $\text{quot}(a,c) \times \text{quot}(b,d) \leqq \text{quot}(ac,\, bd) <$
$[\text{quot}(a,c)+1] \times [\text{quot}(b,d)+1]$

✻ 3·0 $a \varepsilon N_1 \,.\, b \varepsilon N_1 \times a \;.\supset.\; \text{quot}(b,a) = b/a$

✻ 4. $a,b \varepsilon N_0 \,.\, c,d \varepsilon N_1 \;.\supset:$

·0 $\text{rest}(a,c) = a - c \times \text{quot}(a,c)$ Df rest

·1 $a = c \times \text{quot}(a,c) + \text{rest}(a,c)$ [$=$P·0]

·2 $\text{rest}(a,c) < c$

[§quot1·5 $.\supset.\; a < c \times [\text{quot}(a,c)+1] \;.\supset.\; a - c \times \text{quot}(a,c) < c$. Df rest $.\supset.$ P]

·3 $q,r \varepsilon N_0 \,.\, a = cq + r \,.\, r < c \;.\supset.\; q = \text{quot}(a,c) \,.\, r = \text{rest}(a,c)$

[$\text{quot}(a,c) = \text{quot}(cq+r,c) = c + \text{quot}(r,c) \,.\, \text{quot}(r,c) = 0 \;.\supset.$ P]

·4 $\text{rest}(a+bc,\, c) = \text{rest}(a,c)$

[$\text{rest}(a+bc,c) = a + bc - c \times \text{quot}(a+bc,c) = a + bc - c \times [b + \text{quot}(a,c)]$
$= a - c \times \text{quot}(a,c) = \text{rest}(a,c)$]

·5 $\text{rest}(ad,\, cd) = d \times \text{rest}(a,c)$

[$\text{rest}(ad,cd) = ad - cd \times \text{quot}(ad,cd) = ad - cd \times \text{quot}(a,c)$
$= d \times [a - \text{quot}(a,c)] = d \times \text{rest}(a,c)$]

·6 $\text{rest}(a+b,\, c) = \text{rest}[\text{rest}(a,c) + \text{rest}(b,c)\,,\, c]$

[$a = c \times \text{quot}(a,c) + \text{rest}(a,c) \,.\, b = c \times \text{quot}(b,c) + \text{rest}(b,c) \;.\supset.\; a+b =$
$c \times [\text{quot}(a,c) + \text{quot}(b,c)] + \text{rest}(a,c) + \text{rest}(b,c) \;.\supset.\; \text{rest}(a+b,c) =$
$\text{rest}[\text{rest}(a,c) + \text{rest}(b,c), c]$]

·7 $\text{rest}(ab,\, c) = \text{rest}[\text{rest}(a,c) \times \text{rest}(b,c)\,,\, c]$

·8 $m \varepsilon N_1 \;.\supset.\; \text{rest}(a^m,c) = \text{rest}\{[\text{rest}(a,c)]^m,\, c\}$

✻ 5. Hp P1 $.\supset$

·0 $\text{rest}(0,c) = 0 \,.\, \text{rest}(c,c) = 0$ ·1 $\text{rest}[\text{rest}(a,c),\, c] = \text{rest}(a,c)$

·2 $\text{rest}(a,c) = \text{rest}(b,c) \;.=.\; \text{rest}(a+d,\, c) = \text{rest}(b+d,\, c)$

·3 $a < c \;.\supset.\; \text{rest}(a,c) = a$ ·4 $a > c \;.\supset.\; a > 2\,\text{rest}(a,c)$

·5 $a \varepsilon N_0 \times c \;.=.\; \text{rest}(a,c) = 0$

·6 $\text{rest}(a,c) = \text{rest}(b,c) \;.\supset.\; \text{rest}(ad,c) = \text{rest}(bd,c)$

·7 $\text{rest}(a+b,\, c) = \text{rest}(b,c) \;.\supset.\; a \varepsilon N_0 \times c$

·8 $\text{rest}(a,c) + \text{rest}(b,c) \,\varepsilon\, N_0 \times c \;.\supset.\; a + b \,\varepsilon\, N_0 \times c$

·9 $\text{rest}(a^3,6) = \text{rest}(a,6)$. $\text{rest}(a^2,4) \,\varepsilon\, \iota 0 \cup \iota 1$

✻ 6. Hp P1 $.\supset.$

·1 $\text{quot}[\text{rest}(a,c), c] = 0 \,.\, \text{quot}[\text{rest}(a, cd), c] = \text{rest}[\text{quot}(a,c), d]$

·2 $\text{rest}(a,c) + c \times \text{rest}[\text{quot}(a,c), d] = \text{rest}(a, cd)$

·3 $a \geqq c \,.\, \text{quot}(a,c) -\varepsilon\, d \times N_1 \,.\supset.$
$\text{rest}(a, cd) > \text{rest}(a,c) \,.\, \text{rest}(a, cd) - \text{rest}(a,c) \,\varepsilon\, c \times N_1$

·4 $\text{quot}(a+b, c) = \text{quot}(a,c) + \text{quot}\,[b + \text{rest}(a,c), c]$

·5 $\text{quot}(a,c) = \text{quot}(a, c+d) \,.=.\, \text{rest}(a,c) \geqq [\text{quot}(a,c)] \times d$

·6 $c > d \,.\supset:\, \text{quot}(a,c) = \text{quot}(a, c-d) \,.=.$
$c - \text{rest}(a,c) > [\text{quot}(a,c) + 1] \times d$

·7 $\text{quot}(a,c) = \text{quot}(a+b, c+b) \,.=.\, \text{rest}(a,c) \geqq [\text{quot}(a,c) - 1] \times b$

·8 $\max N_1 \frown x \ni \{\text{quot}(a+x,c) = \text{quot}(a,c)\} = c - \text{rest}(a,c) - 1$

·9 $ac \geqq \,.\supset.\, \text{quot}[a, \text{quot}(a,c)] = c + \text{quot}[\text{rest}(a,c), \text{quot}(ac,)]$
$----- \text{rest}[a, \text{quot}(a,c)] = \text{rest}\,[\text{rest}(a,c), \text{quot}(a,c)]$

§12 Cfr

$a \varepsilon N_0 \,.\, n \varepsilon N_1 \,.\supset.$

·0 $Cfr_0 a = rest(a,X)$, $Cfr_n a = Cfr_0\, quot(a,X^n)$ Df

·1 $rest(a,2) = rest(Cfr_0 a, 2) \,.\, rest(a,5) = rest(Cfr_0 a, 5)$
$[\ a = X quot(a,X) + Cfr_0 a \,.\, X = 2 \times 5 \,.\supset.\, P\ ]$

·2 $rest(a,4) = rest(Cfr_0 a + 2 Cfr_1 a, 4)$
$[\ rest(a,4) = rest[X^2 \times quot(a,X^2) + X \times Cfr_1 a + Cfr_0 a \,.\, 4\ ] \,.$
$rest(X^2,4) = 0 \,.\, rest(X,4) = 2 \,.\supset.\, P\ ]$

·3 $rest(a,8) = rest(Cfr_0 a + 2 Cfr_1 a + 4 Cfr_2 a, 8)$

·4 $a \varepsilon\, 2N_0 \,.=.\, Cfr_0 a\, \varepsilon\, 2N_0 \ :\ a \varepsilon\, 5N_0 \,.=.\, Cfr_0 a\, \varepsilon\, 5N_0$

·5 $Cfr_0\, a^2\ \varepsilon\ \iota 0 \cup \iota 1 \cup \iota 4 \cup \iota 5 \cup \iota 6 \cup \iota 9$
$Cfr_0\, a^2 = 6 \,.=.\, Cfr_1\, a^2\, \varepsilon\, 2N_0 + 1$
$a \varepsilon\, (2N_1) - (5N_1) \,.\supset.\, Cfr_0\, a^4 = 6$
$a \varepsilon\, N_1 - (2N_1) - (5N_1) \,.\supset.\, Cfr_0\, a^4 = 1$
$a,n \varepsilon N_1 \,.\supset:\, Cfr_0\, a^5 = Cfr_0\, a \ .\ Cfr_0\, a^{n+4} = Cfr_0\, a^n$
$m \varepsilon\, (4N_0) \cup (4N_0+1) \,.\supset.\, Cfr_1\, 7^m = 0$
$m \varepsilon\, (4N_0+2) \cup (4N_0+3) \,.\supset.$ » 4

Si a es numero, $Cfr_0 a$ indica « cifra de unitate », et $Cfr_n\, a$ indica « cifra de loco n, ante unitate ».

§13 ord

$a,b \varepsilon N_1 \,.\supset.$ ·0 $ord a = \max N_0 \cap n \ni (X^n \leqq a)$ Df

·1 $X \uparrow ord a \leqq a < X \uparrow (ord a + 1)$

·2 $ord(a+b) \geqq \max(\iota ord a \cup \iota ord b)$
$\leqq$ » $+1$

·3 $a > b \,.\supset.\, ord a \leqq ord b$

·4 $ord(a \times b) \leqq ord a + ord b$
» $\geqq ord a + ord b + 1$

·5 $b \times ord a \eqslantless ord(a \uparrow b) < b \times ord a + b$

·6 $a > b \,.\supset.\, ord(a-b) \eqslantless ord a$

·7 $a > b \,.\supset.\, ord\ quot(a,b) \leqq ord a - ord b - 1$
» $\geqq ord a - ord b$

ord$a$, lege : ordine de a. Numero de cifras de a vale ord$a$ + 1.

§14 ! C

✻ 1·0 $0! = 1 \quad : \quad a \varepsilon N_0 \,.\supset.\, (a+1)! = a! \times (a+1)$ Df

·01 $a,b \varepsilon N_0 \,.\supset.\, a+b! = a+(b!) \quad . \quad a-b! = a-(b!)$
$a \times b! = a \times (b!) \quad . \quad a/b! = a/(b!)$ Df

·1 $a,b \varepsilon N_0 \,.\supset.\, (a+b)! \,\varepsilon\, N_1 \times (a!)(b!)$ } B. PASCAL t.3 p.274:
« Omnis productus a quotlibet numeris continuis est multiplex producti a totidem numeris continuis quorum primus est unitas.» {

Notatione $a!$ « factoriale de a », introducto per Kramp a. 1808, es plus diffuso que notatione Πa, $\underline{|a}$.

✻ 2. $mn \varepsilon N_0 \,.\supset.$

·0 $C(m,0) = 1 \quad . \quad C(m, n+1) = C(m,n) \times (m-n)/(n+1)$ Df

Numeros $C(m+1,2)$, $C(m+2,3)$ es vocato ab Pythagoricos (Jamblicho, Diophanto, Boetio ...) « numeros triangulare, pyramidale, figurato ».

Functione $C(m,n)$, aut $C_{m,n}$, se inveni et sub formas
$\left[\frac{m}{n}\right]$ (Euler), $(m)_n$ (Cauchy), $\binom{m}{n}$ (Raabe), m_n, etc.
et es vocato « numero de combinationes de m objecto ad n » (Pascal).

·1 $C(m,1) = m \quad . \quad C(m,m) = 1$

·2 $C(m+n, m) = C(m+n, n)$ } PASCAL t.3 p.289:
« Duo quilibet numeri æque combinantur in eo quod amborum aggregatum est. » {

·3 $C(m+1, n+1) = C(m, n+1) + C(m,n)$

·4 $m \varepsilon\, n+N_0 \,.\supset.\, C(m+1, n) = C(m,n) \times (m+1)/(m-n+1)$

·5 $C(m+1, n+1) = C(m, n) \times (m+1)/(n+1)$
} PASCAL t.3 p.289 {

·6 $p \varepsilon N_0 \,.\, n \varepsilon\, p+N_0 \,.\, m \varepsilon\, n+N_0 \,.\supset.\, C(m,n) \times C(n,p) =$
$C(m,p) \times C(m-p, n-p)$

·7 $m \varepsilon N_1 \,.\supset.\, C(m,n) = \text{num}[\text{Cls}'1\cdots m \frown x \ni (\text{num}\, x = n)]$ Dfp

·8 $n \varepsilon N_0 \,.\, m \varepsilon\, n+N_0 \,.\supset.\, C(m,n) = m!\,/\,[n!\,(m-n)!]$ Dfp

§15 mlt Dvr

* 1. $a,b,c \varepsilon N_1 . \supset$:

·0 $\text{mlt}(a,b) = \text{m}(a,b) = \min [(a \times N_1) \frown (b \times N_1)]$ Df m

= minimo commune multiplo de a et b.

In praesente § nos scribe $\text{m}(a,b)$ et $\text{D}(a,b)$; in § successivos nos ute notatione plus longo $\text{mlt}(a,b)$ et $\text{Dvr}(a,b)$.

Notationes $\text{m}(a,b)$, $\text{D}(a,b)$, introducto per Lebesgue a. 1859, adoptato per Lucas, es hodie de usu commune.

·01 » $= a \times \min[N_1 \frown y \ni (ay \,\varepsilon\, N_1 \times b)]$

·1 $\text{m}(a,b) \,\varepsilon N_1 \;.\; \text{m}(a,b) \,\varepsilon\, N_1 \times a \frown N_1 \times b \;.\; \text{m}(a,b) \leqq a \times b$

[$a \times b \,\varepsilon\, (N_1 \times a) \frown (N_1 \times b) . \supset . \exists (N_1 \times a) \frown (N_1 \times b)$. §min 1·8 $. \supset$ P]

·2 $\text{m}(a,a) = a \;.\; \text{m}(1,a) = a$

[$\text{m}(a,a) = \min(a \times N_1 \frown a \times N_1) = \min(a \times N_1) = a \times \min N_1 = a \times 1 = a$]

[$\text{m}(1,a) = \min(N_1 \frown a \times N_1) = \min(a \times N_1) = a$]

·21 $\text{m}(a,b) = \text{m}(b,a)$ [Comm$\frown \supset$ P]

·3 $a \varepsilon\, N_1 \times b . \supset . \text{m}(a,b) = a$

[Hp $. \supset . N_1 \times a \supset N_1 \times b . \supset . (N_1 \times a) \frown (N_1 \times b) = N_1 \times a . \supset . \text{m}(a,b) = \min(N_1 \times a) = (\min N_1) \times a = 1 \times a = a$]

·4 $a \varepsilon\, N_1 \times b . = . \text{m}(a,b) = a$ [P·3 $\supset$ P]

·5 $\text{m}(ac, bc) = c \times \text{m}(a,b)$ Distrib($\times$,m)

[$\text{m}(ac, bc) = a \times c \times \min[N_1 \frown y \ni (a \times c \times y \,\varepsilon\, N_1 \times b \times c)]$

$= a \times \min[(N_1 \frown y \ni (ay \,\varepsilon\, N_1 \times b)] \times c = \text{m}(a,b) \times c$]

·6 $N_1 \times a \frown N_1 \times b \supset N_1 \times \text{m}(a,b)$ ¦ Euclide vii P35 ¦

[$x = \text{m}(a,b) . c \varepsilon (N_1 \times a) \frown (N_1 \times b) . \supset . \text{rest}(c,x) \,\varepsilon\, 0 \cdots (x-1) \frown (N_0 \times a) \frown (N_0 \times b) . -\exists\, 1 \cdots (x-1) \frown (N_0 \times a) \frown (N_0 \times b) . \supset . \text{rest}(c,x) = 0 . \supset . c \varepsilon N_1 \times \text{m}(a,b)$]

·7 $N_1 \times a \frown N_1 \times b = N_1 \times \text{m}(a,b)$ [P·1·6 $\supset$ P]

·8 $\text{m}(a,b) = \iota\, N_1 \frown x \ni [N_1 \times x = (N_1 \times a) \frown (N_1 \times b)]$ Dfp

* 2. $a,b,c \varepsilon N_1 . \supset$:

·1 $a \times b \,\varepsilon\, N_1 \times \text{m}(a,b)$ [$a \times b \,\varepsilon\, N_1 \times a \frown N_1 \times b$. P1·6 $. \supset$ P]

·2 $a,b \varepsilon\, N_1 \times c . \supset . a \times b / \text{m}(a,b) \,\varepsilon\, N_1 \times c$

[Hp $. \supset . ab/c \,\varepsilon\, N_1 \times a \frown N_1 \times b . \supset . ab/c \,\varepsilon\, N_0 \times \text{m}(a,b) . \supset$ P]

·3 $a \times b \; \varepsilon \; N_1 \times c \; .\supset. \; a \times b / \mathrm{m}(a,b) \; \varepsilon \; N_1 \times c$
[$a = [ab/\mathrm{m}(a,b)] \times \mathrm{m}(a,b)/b$]

·4 $N_1 \frown x \ni (a, b \varepsilon N_1 x) = N_1 \frown x \ni (a \times b / \mathrm{m}(a,b) \; \varepsilon \; N_1 x)$ [= P·2·3]

* 3. $a,b,c \, \varepsilon \, N_1 \; .\supset:$

·0 $\mathrm{Dvr}(a,b) = \mathrm{D}(a,b) = \max N_1 \frown x \ni (a,b \varepsilon N_1 \times x)$ DfD
 = maximo commune divisore

·01 » $= \max[N_1 \frown a/N_1 \frown b/N_1]$ Dfp

·02 $\mathrm{D}(a,b) = ab/\mathrm{m}(a,b)$
[DfD $.\supset.$ $\mathrm{D}(a,b) = \max N_1 \frown x \ni (a,b \varepsilon N_1 \times x)$
P1·7 $.\supset.$ » $= \max N_1 \frown x \ni [ab/\mathrm{m}(a,b) \; \varepsilon \; N_1 \times x] = ab/\mathrm{m}(a,b)$]

·1 $\mathrm{D}(a,b) \; \varepsilon N_1 \; . \; a,b \varepsilon \; N_1 \times \mathrm{D}(a,b)$
[$1 \varepsilon \; N_1 \frown a/N_1 \frown b/N_1 \; . \; -\exists \; N_1 \frown a/N_1 \frown b/N_1 \frown (a + N_1)$. §max1·8 $.\supset.$ P]

·2 $\mathrm{D}(a,a) = a \; . \; \mathrm{D}(1,a) = 1 \; . \; \mathrm{D}(a,b) = \mathrm{D}(b,a)$

·3 $a \varepsilon \, N_1 \times b \; .\supset. \; \mathrm{D}(a,b) = b$
[Hp $.\supset.$ $a,b \varepsilon \; N_1 \times b \; . \; -\exists \, (b + N_1) \frown (b/N_1)$ $.\supset.$ P]

·4 $\mathrm{D}(a,b) = b \; .=. \; a \varepsilon \; N_1 \times b$

·5 $\mathrm{D}(ac, bc) = c \times \mathrm{D}(a,b)$ Distrib(×,D)
[P·02 . P1·5 $.\supset.$ $\mathrm{D}(ac,bc) = ac \times bc \, / \, \mathrm{m}(ac,bc) = [ab/\mathrm{m}(a,b)] \times c = \mathrm{D}(a,b) \times c$]

·6 $a,b \varepsilon \; N_1 \times c \; .\supset. \; \mathrm{D}(a,b) \; \varepsilon \; N_1 \times c$ [P·5⊃P]

}EUCLIDE VII P2: *« ἐὰν ἀριθμὸς δύο ἀριθμοὺς μετρῇ, καὶ τό μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει »* {

·7 $N_1 \frown a/N_1 \frown b/N_1 = N_1 \frown \mathrm{D}(a,b)/N_1$

·8 $\mathrm{D}(a,b) = \imath \, N_1 \frown x \ni (N_1 \frown x/N_1 = N_1 \frown a/N_1 \frown b/N_1)$

* 4. $a,b \varepsilon N_1 \, .\supset:$

·1 $a \, -\varepsilon \, N_1 b \; .\supset. \; \mathrm{D}(a,b) = \mathrm{D}[b, \mathrm{rest}(a,b)]$
[Hp . $x \varepsilon N_1$. $a,b \varepsilon \; N_1 \times x$ $.\supset.$ $a - b \times \mathrm{quot}(a,b) \; \varepsilon \; N_1 \times x$ $.\supset.$ $\mathrm{rest}(a,b) \; \varepsilon \; N_1 \times x$ (1)
Hp . b, $\mathrm{rest}(a,b) \; \varepsilon \; N_1 \times x$ $.\supset.$ $b\,\mathrm{quot}(a,b) + \mathrm{rest}(a,b) \; \varepsilon \; N_1 \times x$ $.\supset.$ $a \varepsilon \; N_1 \times x$ (2)
Hp . (1) . (2) $.\supset.$ $N_1 \frown x \ni (a,b \varepsilon \; N_1 \times x) = N_1 \frown x \ni [b, \mathrm{rest}(a,b) \; \varepsilon \; N_1 \times x]$.
Oper max $.\supset.$ P]

4·2 $\mathrm{D}(a,b) = \mathrm{D}[b, b - \mathrm{rest}(a,b)]$

·3 $a > b \; .\supset. \; \mathrm{D}(a,b) = \mathrm{D}(a - b, b)$ }EUCLIDE VII P1{

·4 $\mathrm{D}(a+1, a) = 1$ ·5 $\mathrm{D}(2a+1, 2a-1) = 1$

·6 $a \varepsilon \; 2N_1 \; . \; b \varepsilon \; 2N_1 + 1 \; . \; a > b \; .\supset. \; \mathrm{D}(a+b, a-b) = \mathrm{D}(a,b)$

·7 $a \, , \, b \; \varepsilon \; 2N_1 + 1 \; . \; \text{——} \; .\supset. \; \text{————} = 2\mathrm{D}(a,b)$

·8 $a > b \; . \; \mathrm{D}(a,b) = 1 \; .\supset. \; \mathrm{D}(a+b, a-b) \; \varepsilon \; \iota 1 \cup \iota 2$

✻ 5. $a,b\varepsilon N_1 .\supset:$

·1 $D(a,b)=1 .\supset. m(a,b)=a\times b$ [P2·02 ⊃ P]

·2 $m(a,b) = a\times b/D(a,b)$ { Euclide vii P34 }

✻ 6. $a,b,c,d\varepsilon N_1 .\supset:$

·1 $ab\ \varepsilon\ N_1\times c . D(a,c)=1 .\supset. b\varepsilon\ N_1\times c$

[Hp $.\supset.\ ab\ \varepsilon\ N_1\times c\ .\ cb\ \varepsilon\ N_1\times c\ .\supset.\ D(ab, cb)\ \varepsilon\ N_1\times c\ .\ D(ab, cb) = b\times D(a,c) = b\ .\supset.$ P]

[Hp $.\supset.\ ab\ \varepsilon N_1\times a \frown N_1\times c\ .\supset.\ ab\ \varepsilon\ N_1\times m(a,c)\ .\supset.\ ab\ \varepsilon\ N_1\times ac\ .\supset$ P]

·2 $D(a,b)=1 . c\varepsilon\ N_1\times a \frown N_1\times b .\supset. c\varepsilon\ N_1\times ab$

·3 $D(a, b\times c) = D[a, b\times D(a,c)]$

·4 $D(a,c)=1 .\supset. D(ab,c) = D(b,c)$

·5 $D(a,b)=1 . a\varepsilon\ N_1\times c .\supset. D(b,c)=1$ {Euclide vii P23}

·6 $D(a,c)=1 . D(b,c)=1 .=. D(ab,c)=1$ {Euclide vii P24}

·7 $D(a,c) = D(b,c) = D(a,d) = D(b,d) = 1 .\supset. D(ab, cd)=1$ { Euclide vii P26 }

·8 $D(b,c)=1 .\supset. D(a, b\times c) = D(a,b)\times D(a,c)$

·9 $D[a/D(a,b) , b/D(a,b)]=1$

[Distrib$(\times,D)$ $.\supset.\ D(a,b)\times D[a/D(a,b), b/D(a,b)] = D(a,b)\ .\supset.$ P]

✻ 7. $a,b,c,d\varepsilon N_0 .\supset:$

·1 $a/b=c/d . D(a,b)=1 .\supset. c/a = d/b = D(c,d)$

[Hp $.\supset.\ ad = bc\ .\supset.\ bc\ \varepsilon\ N_1\times a\ .\ D(a,b)=1\ .\supset.\ c\varepsilon\ N_1\times a$]

·2 $D(a,b)=D(c,d)=1 . a/b=c/d .\supset. a=c . b=d$

{ Euclide vii P20, 21 }

✻ 8. $a,b,m,n\varepsilon N_1 .\supset.$

·1 $D(a,b)=1 .\supset. D(a^m,b)=1$

[$m=1\ .\supset.$ P (1)

$m\varepsilon N_1\ .\ D(a,b)=1\ .\ D(a^m,b)=1\ .$ P6·6 $.\supset.\ D(a^{m+1}, b)=1$ (2)

(1) . (2) . Induct .⊃ P]

·2 $D(a,b)=1 .\supset. D(a^m,b^n)=1$ { Euclide viii P2,3 }

[Hp . P·1 $.\supset.\ D(a^m,b)=1$. P·1 .⊃ P]

·3 $a,b,m\varepsilon N_1 . D(a,b)=1 . ab\ \varepsilon\ N_1{}^m .\supset. a,b\varepsilon\ N_1{}^m$

❋ 9·1 $a,b \varepsilon N_1+1 . D(a,b)=1 .\supset.$
$\text{rest}(ax,b) \mid x` 1\cdots(b-1) = 1\cdots(b-1)$

[$x\varepsilon 1\cdots(b-1) .\supset. \text{rest}(ax,b) \varepsilon 1\cdots(b-1)$
$x,y\varepsilon N_1 . x<y . \text{rest}(ax,b)=\text{rest}(ay,b) .\supset. \text{rest}[(y-x)a, b]=0 .\supset.$
$(y-x)a \varepsilon N_1\times b .\supset. y-x \varepsilon N_1\times b$
$x,y\varepsilon 1\cdots(b-1) . x<y .\supset. y-x -\varepsilon N_1\times b .\supset. \text{rest}(ax,b) -= \text{rest}(ay,b)$
$\text{num rest}(ax,b) \mid x`1\cdots(b-1)=b-1 .\supset. P$]

·2 $a,b\varepsilon N_1+1 . D(a,b)=1 .\supset. \exists N_1 \frown n\ni(a^n \varepsilon N_1 b+1)$

❋ 10. $a,b,c\varepsilon N_1 .\supset.$

·0 $m(a,b,c) = \min(N_1\times a \frown N_1\times b \frown N_1\times c)$ Df

·1 $m(a,b,c) = m[m(a,b), c]$ } EUCLIDE VII P36 {

[$m(a,b,c) = \min[(N_1\times a)\frown(N_1\times b)\frown(N_1\times c)]$
$= \min\{[N_1\times m(a,b)]\frown(N_1\times c)\} = m[m(a,b), c]$]

·2 $D(a,b,c) = \max N_1 \frown x\ni(a,b,c\varepsilon N_1\times x)$ Df

·3 $D(a,b,c) = D[D(a,b), c]$ } EUCLIDE VII P3 {

[P·3 . Oper $x\ni$ $.\supset.$ $N_1 \frown x\ni(a,b\varepsilon N_1\times x) = N_1 \frown x\ni[D(a,b) \varepsilon N_1\times x]$.
Oper $[\frown x\ni(c\varepsilon N_1\times x)]$. Distrib$(\ni,\frown)$. Oper max . Df D $.\supset.$ P]

·4 $a,b,c\varepsilon 2N_0+1 .\supset. D(a,b,c)=D[(a+b)/2, (a+c)/2, (b+c)/2]$

·5 $D(a,b,c)\times m(ab,ac,bc) = abc$
m ———D————————

·6 $m(a,b,c)\, D(a,b)\, D(a,c)\, D(b,c) = abc\, D(a,b,c)$
} ·5-·6 LEBESGUE a.1859 p.31,34 {

·7 $[N_1,\ b\varepsilon N_1\times a,\ D(a,b),\ m(a,b),\ 1] \mid [\text{Cls},\ a\supset b,\ a\frown b,\ a\smile b,\ \Lambda]$
I §ε 1·1·4 2·1·2·3 3·1·2·3 4·1·2·4
§∪ 1·1·2·3 2·1·2·3 3·2·3 4·1 §Λ 1·1·2·3·4

❋ 11. $a,b,m,n,\varepsilon N_1 .\supset.$

·0 $D(a,b)=1 . ab \varepsilon N_1^2 .\supset. a,b\varepsilon N_1^2$
} LEIBNIZ a.1678 *Math. Schr.* t.7 p.122 {

·1 $D(a,b)=1 .\supset. N_1^a \frown N_1^b = N_1^{ab}$

·2 $D(a,b)=1 .\supset. (N_1+1)\frown (a^2+b^2)/N_1 \supset N_1^2+N_1^2$

·3 $D(a,b)=1 .\supset. N_1\frown (a^2+b^2)/(N_1+1) \supset (4N_0+1) \smile \iota 2$
» $N_1\frown (a^4+b^4)/(N_1+1) \supset (8N_0+1) \smile \iota 2$
» $N_1\frown [a\uparrow(2^m)+b\uparrow(2^m)]/(N_1+1) \supset (N_0\times 2^{m+1}+1)\smile\iota 2$
} EULER PetrNC. t.1 a.1747-48 p.32 {

·4 $N_1 \frown (2^{4n}+1)/N_1 \supset 16nN_0+1$

·5 $D(a,b)=1 . ab\varepsilon 4N_1+1 .\supset. N_1 \frown (a^{2abn}+b^{2abn})/N_1 \supset (8abnN_0+1) \cup \iota 2$
{ LUCAS TorinoA. a.1878 t.13 p.281 {

·6 $D(a,b) =$ num $1 \cdots b \frown x \ni (ax \,\varepsilon\, N_1 \times b)$ Dfp

·7 $D(a,b,c) =$ num $1 \cdots c \frown x \ni (ax, bx \,\varepsilon\, N_1 \times c)$ Dfp

·8 $n \varepsilon N_1+1 .\supset. \mathrm{m}(1 \cdots 2n) = \mathrm{m}[(n+1) \cdots 2n]$

·9 $a,b,m,n,x \varepsilon N_1 . x^m-1 \,\varepsilon\, N_1 \times a . x^n-1 \,\varepsilon\, N_1 \times b . D(a,b)=1 .\supset.$
$x \uparrow \mathrm{m}(m,n) -1 \,\varepsilon\, N_1 \times a \times b$

✻ 12. $u,v \varepsilon \mathrm{Cls}'N_1 . a,b \varepsilon N_1 .\supset:$

·0 $\mathrm{m}u = \min[N_1 \frown x \ni (u \supset N_1 \frown x/N_1)]$ Df

·1 $\mathrm{m}\iota a = a$ ·2 $\mathrm{m}(\iota a \cup \iota b) = \mathrm{m}(a,b)$

num$u \,\varepsilon N_1 .\supset:$ ·3 $\mathrm{m}u \,\varepsilon N_1$ ·4 $\mathrm{m}(u \times a) = a \times \mathrm{m}u$

num$u$, num$v \,\varepsilon N_1 .\supset:$ ·5 $\mathrm{m}(u \cup v) = \mathrm{m}(\mathrm{m}u, \mathrm{m}v)$

·6 $\mathrm{m}(u \times v) = \mathrm{m}u \times \mathrm{m}v$ Distrib(×,m)
{ STIELTJES a.1895 p.4 {

✻ 13. $u,v \varepsilon \mathrm{Cls}'N_1 . a,b \varepsilon N_1 .\supset:$

·0 $\mathrm{D}u = \max[N_1 \frown x \ni (u \supset N_1 \times x)]$ Df

·1 $\mathrm{D}\iota a = a$ ·2 $D(a,b) = D(\iota a \cup \iota b)$

$\exists u .\supset.$ ·3 $\mathrm{D}u \,\varepsilon\, N_1$

[$u \varepsilon \mathrm{Cls}'N_1 . v = N_1 \frown x \ni (u \supset N_1 \times x) .\supset. 1 \varepsilon v . \S\exists\, 2{\cdot}1 .\supset. \exists v$ (1)
Hp(1) . $m \varepsilon u .\supset. -\exists\, v \frown (m+N_1)$. (1) . §max ·2 $.\supset.$ max$v \,\varepsilon\, N_1$ (2)
(1) . (2) . Elim m . Elim$v .\supset.$ P]

·4 $D(u \times a) = a \times \mathrm{D}u$

$\exists u . \exists v .\supset.$ ·5 $D(u \cup v) = D(\mathrm{D}u, \mathrm{D}v)$

·6 $D(u \times v) = (\mathrm{D}u) \times (\mathrm{D}v)$ Distrib(×,D)
{STIELTJES a.1895 p.4{

§16 Np (numero primo)

✻ 1·0 $\mathrm{Np} = (1+\mathrm{N}_1) - [(1+\mathrm{N}_1)\times(1+\mathrm{N}_1)]$ Df Np

·1 $b\varepsilon \mathrm{N}_1 \,.\, a\varepsilon \mathrm{Np}\cap(\mathrm{N}_1\times b) \,.\supset.\, b\varepsilon \iota 1 \cup \iota a$

[$b,c\varepsilon \mathrm{N}_1 \,.\, a\varepsilon \mathrm{Np} \,.\, a=bc \,.\, \mathrm{Df\,Np} \,.\supset.\, -(b,c\varepsilon \mathrm{N}_1+1) \,.\supset.\, b=1 \,.\cup.\, c=1 \,.\supset.\, b\varepsilon \iota 1\cup\iota a$]

·11 $a\varepsilon \mathrm{Np} \,.\supset.\, \mathrm{N}_1 \cap a/\mathrm{N}_1 = \iota 1 \cup \iota a$ [$=$ P·1]

·2 $a\varepsilon \mathrm{Np} \,.\, b\varepsilon \mathrm{N}_1 - (\mathrm{N}_1\times a) \,.\supset.\, \mathrm{D}(a,b)=1$

[$\mathrm{N}_1 \cap a/\mathrm{N}_1 = \iota 1 \cup \iota a \,.\, a -\varepsilon\, b/\mathrm{N}_1 \,.\supset.\, \mathrm{N}_1\cap a/\mathrm{N}_1 \cap b/\mathrm{N}_1 = \iota 1$]

{ EUCLIDE VII P29:

Ἅπας πρῶτος ἀριθμὸς πρὸς ἅπαντα ἀριθμόν, ὃν μὴ μετρεῖ, πρῶτός ἐστιν.}

·3 $a\varepsilon \mathrm{Np} \,.\, b,c\varepsilon \mathrm{N}_1 \,.\, bc\, \varepsilon\, \mathrm{N}_1\times a \,.\supset.\, b\varepsilon \mathrm{N}_1\times a \,.\cup.\, c\varepsilon \mathrm{N}_1\times a$

{ EUCLIDE VII P30:

Ἐὰν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, τὸν δὲ γενόμενον ἐξ αὐτῶν μετρῇ τις πρῶτος ἀριθμός, καὶ ἕνα τῶν ἐξ ἀρχῆς μετρήσει.}

[$\mathrm{Hp} \,.\, \mathrm{P}{\cdot}1 \,.\supset.\, \mathrm{D}(b,a)\, \varepsilon\, \iota 1\cup\iota a$ (1)
$\mathrm{Hp} \,.\, \mathrm{D}(b,a)=1 \,.\, \S\mathrm{D}\, 6{\cdot}1 \,.\supset.\, c\varepsilon \mathrm{N}_1\times a$ (2)
$\mathrm{Hp} \,.\, \mathrm{D}(b,a)=a \,.\supset.\, b\varepsilon \mathrm{N}_1\times a$ (3)
(2) . (3) . Oper $\cup$. (1) $.\supset.$ P]

·4 $a\varepsilon \mathrm{N}_1+1 \,.\supset.\, \min[(\mathrm{N}_1+1)\cap(a/\mathrm{N}_1)]\, \varepsilon\, \mathrm{Np}$

[$a\varepsilon (\mathrm{N}_1+1)\cap(a/\mathrm{N}_1) \,.\supset.\, \min(\mathrm{N}_1+1)\cap(a/\mathrm{N}_1)\, \varepsilon\, \mathrm{N}_1+1$ (1)
$x\varepsilon (\mathrm{N}_1+1)\cap(a/\mathrm{N}_1) \,.\, y\varepsilon \mathrm{N}_1+1 \,.\, x\varepsilon (\mathrm{N}_1+1)\times y \,.\supset.\, y\varepsilon (\mathrm{N}_1+1)\cap(a/\mathrm{N}_1) \,.$
$y<x \,.\supset.\, x -= \min(\mathrm{N}_1+1)\cap(a/\mathrm{N}_1)$ (2)
$x\varepsilon (\mathrm{N}_1+1)\cap(a/\mathrm{N}_1) \,.\, x\varepsilon (\mathrm{N}_1+1)\times(\mathrm{N}_1+1) \,.\, (2) \,.\, \mathrm{Elim}\,y \,.\supset.$
$x -= \min(\mathrm{N}_1+1)\cap(a/\mathrm{N}_1)$ (3)
$x = \min(\mathrm{N}_1+1)\cap(a/\mathrm{N}_1) \,.\, (3) \,.\, \mathrm{Transp} \,.\supset.\, x\varepsilon (\mathrm{N}_1+1)-[(\mathrm{N}_1+1)\times(\mathrm{N}_1+1)] \,.$
$\supset.\, x\varepsilon \mathrm{Np}$]

·5 $\mathrm{N}_1+1 \supset \mathrm{N}_1\times\mathrm{Np}$ { EUCLIDE VII P31:

Ἅπας σύνθετος ἀριθμὸς ὑπὸ πρώτου τινὸς ἀριθμοῦ μετρεῖται. }

[$a\varepsilon \mathrm{N}_1+1 \,.\, x=\min(\mathrm{N}_1+1)\cap(a/\mathrm{N}_1) \,.\supset.\, x\varepsilon \mathrm{Np}\cap(a/\mathrm{N}_1) \,.\supset.\, a\varepsilon \mathrm{N}_1\times\mathrm{Np}$]

·6 $m\varepsilon \mathrm{N}_1 \,.\supset.\, \exists\, \mathrm{Np}\cap(m+\mathrm{N}_1)$

{ EUCLIDE IX P20: *Οἱ πρῶτοι ἀριθμοὶ πλείους εἰσὶν παντὸς τοῦ προτεθέντος πλήθους πρώτων ἀριθμῶν.* }

[$\mathrm{Hp} \,.\supset.\, m!+1\, \varepsilon\, \mathrm{N}_1+1 \,.\, \mathrm{P}{\cdot}3 \,.\supset.\, m!+1\, \varepsilon\, \mathrm{N}_1\times\mathrm{Np}$ (1)
$x\varepsilon \mathrm{Np} \,.\, x\leqq m \,.\supset.\, m!\, \varepsilon\, \mathrm{N}_1\times x \,.\, 1 -\varepsilon\, \mathrm{N}_1\times x \,.\supset.\, m!+1 -\varepsilon\, \mathrm{N}_1\times x$ (2)
$x\varepsilon \mathrm{Np} \,.\, m!+1\, \varepsilon\, \mathrm{N}_1\times x \,.\, (2) \,.\, \mathrm{Transp} \,.\supset.\, x>m \,.\supset.\, \exists\, \mathrm{Np}\cap(m+\mathrm{N}_1)$ (3)
(1) . (3) . Elim$x$ $.\supset$ P]

$*$ 2·0 $a\varepsilon (N_1+1)\times(N_1+1) .\supset. \exists\, Np \cap x\ni(x^2 \leqq a . a\varepsilon N_1\times x)$

[$x = \min (1+N_1)\cap(a/N_1) . y = a/x .\supset. x\varepsilon Np . y\varepsilon N_1+1 .$
$\min Np\cap(y/N_1) \geqq x .\supset. y \geqq x .\supset. a = xy \geqq x^2 .\supset.$
$x\varepsilon Np\cap(a/N_1) . x^2 \leqq a$]

2·1 $a\varepsilon N_1+1 . -\exists\, Np\cap x\ni(x^2\leqq a . a\varepsilon N_1\times x) .\supset. a\varepsilon Np$
[P·0 . Transp $.\supset.$ P]

{ LEONARDO PISANO a.1202 p.38:

«Qui primum numerum… cognoscere voluerit… semper eat dividendo ipsum per primos numeros ordinate, donec aliquem primum numerum invenerit, per quem propositum numerum absque alia superatione possit dividere, vel donec ad eiusdem pervenerit radicem: si per nullum ipsorum dividi potuerit, tunc primum ipsum esse indicabit.» }

·2 $a\varepsilon Np . b,n\varepsilon N_1 . b^n \varepsilon N_1\times a .\supset. b\varepsilon N_1\times a$ {EUCLIDE IX P12}
[$b -\varepsilon N_1\times a .\supset. D(a,b)=1 .\supset. D(b^n,a)=1 .\supset. b^n -\varepsilon N_1\times a$ (1)
(1) . Transp $.\supset.$ P]

·3 ———————— . $a^n \varepsilon N_1\times b .\supset. b\varepsilon a \upharpoonright N_1$ { » » P13}

$*$ 3. $a\varepsilon Np . b\varepsilon (N_1+1) - (N_1\times a) .\supset.$ ·1 $b^{a-1}-1 \varepsilon N_1\times a$

[Hp . $c = a-1 .\supset. D(a,b)=1 .\supset. \text{rest}(bx,a) \mid x`1\cdots c = 1\cdots c .\supset.$
$\Pi[\text{rest}(bx,a)|x, 1\cdots c] = c! .\supset. \Pi(bx|x, 1\cdots c)\,\varepsilon\, c! + N_1\times a .\supset.$
$c!b^c \,\varepsilon\, c! + N_1\times a .\supset. (b^c-1)c! \,\varepsilon\, N_1\times a . D(a,c!)=1 .\supset. b^c-1 \,\varepsilon\, N_1\times a$]

·2 $\min[N_1\cap x\ni(b^x-1 \,\varepsilon\, a\times N_1)] \,\varepsilon\, N_1\cap(a-1)/N_1$

·3 $N_1\cap x\ni(b^x-1 \,\varepsilon\, a\times N_1) = N_1\times\min[N_1\cap x\ni(b^x-1\varepsilon\, a\times N_1)]$

{ FERMAT a.1640 t.2 p.209:

« Tout nombre premier mesure infailliblement une des puissances —1 de quelque progression que ce soit, et l'exposant de la dite puissance est sous multiple du nombre premier donné —1; et après qu'on a trouvé la première puissance qui satisfait à la question, toutes celles dont les exposants sont multiples de l'exposant de la première satisfont tout de même à la question. » }

$*$ 4·1 $Np\cap(N_1+2) \supset 2N_1+1$

·2 $Np\cap(N_1+3) \supset (6N_1+1)\cup(6N_1-1)$
[$(N_1+3)-(2N_1)-(3N_1) \supset (6N_1+1)\cup(6N_1-1)$]

{ BUNGUS a.1599 p.399 : «…semper … numeri primi post binarium et ternarium, in senariorum multiplicium vicinia collocati comperientur, aut uno minores, aut uno majores. » }

·3 $Np\cap(N_1+5) \supset 30N_1 \pm (\iota 1\cup\iota 7\cup\iota 11\cup\iota 13)$

❋ 5·1 $x \varepsilon 0\cdots 9 .\supset. x^2+x+11 \varepsilon \text{Np}$

·2 $x \varepsilon 0\cdots 15 .\supset. x^2+x+17 \varepsilon \text{Np}$ ⎨ EULER *Op. post.* t.1 p.185 ⎬

·3 $x \varepsilon 0\cdots 39 .\supset. x^2+x+41 \varepsilon \text{Np}$ ⎨ EULER BerlinM. a.1772 p.36 ⎬

·4 $x \varepsilon 0\cdots 28 .\supset. 2x^2+29 \varepsilon \text{Np}$ ⎨ LEGENDRE a.1797 p.10 ⎬

·5 $p \varepsilon \iota 3 \cup \iota 5 \cup \iota 11 \cup \iota 17 \cup \iota 41 . x \varepsilon 0\cdots(p-2) .\supset. x^2+x+p \varepsilon \text{Np}$

❋ 6·1 $\text{Np} \frown (4\text{N}_1+1) \supset \text{N}_1^2+\text{N}_1^2$ ⎨ GIRARD a.1634 p.156: « Tout nombre premier qui excede un nombre quaternaire de l'unité se peut diviser en deux quarrez entiers. » ⎬

·2 $a,b,c,d \varepsilon \text{N}_1 . a^2+b^2=c^2+d^2 . a^2+b^2 \varepsilon \text{Np} .\supset. \iota a \cup \iota b = \iota c \cup \iota d$

⎨ FERMAT t.1 p.294 : « Numerus primus qui superat unitate quaternarii multiplicem, semel tantum est hypotenusa trianguli rectanguli » ⎬

·3 $a \varepsilon \text{Np} \frown (4\text{N}_0+3) . b,c \varepsilon \text{N}_1 . b^2+c^2 \varepsilon \text{N}_1 \times a .\supset. b,c \varepsilon \text{N}_1 \times a$

⎨ FERMAT a.1640 t.2 p.204 :

« Si un nombre est composé de deux quarrés premiers entre eux, je dis qu'il ne peut être divisé pas aucun nombre premier moindre de l'unité qu'un multiple du quaternaire ». ⎬

·4 $\text{Np} \frown (3\text{N}_1+1) \supset \text{N}_1^2+3\text{N}_1^2$ ⎨FERMAT a.1654 t.2 p.313⎬

$\text{Np} \frown [(8\text{N}_1+1) \cup (8\text{N}_0+3)] \supset \text{N}_1^2+2\text{N}_1^2$ ⎨ » » ⎬

·5 $2^1+1, 2^2+1, 2^4+1, 2^8+1, 2^{16}+1 \varepsilon \text{Np}$ ⎨FERMAT a.1640 p.162⎬

·6 $m \varepsilon \text{N}_1 . 2^m+1 \varepsilon \text{Np} .\supset. m \varepsilon 2 \uparrow \text{N}_0$ ⎨FERMAT a.1640 t.2 p.205⎬

[$p \varepsilon 2\text{N}_1+1 . m \varepsilon \text{N}_1 . §\Sigma\ 4{\cdot}2 .\supset. 2^{mp}+1 \varepsilon (\text{N}_1+1)\times(2^m+1) .\supset. \text{P}$]

·7 $a,b,m \varepsilon \text{N}_1 . a^m+b^m \varepsilon \text{Np} .\supset. m \varepsilon 2 \uparrow \text{N}_0$

·8 $7\times 2^{50}+1 \varepsilon \text{Np}$ ⎨ SEELHOFF Zm. a.1886 t.31 p.380 ⎬

❋ 7·1 $2^2-1, 2^3-1, 2^5-1, 2^7-1 \varepsilon \text{Np}$ ⎨EUCLIDE IX P36 scolia⎬

$2^{13}-1, 2^{17}-1, 2^{19}-1 \varepsilon \text{Np}$

$2^{31}-1 \varepsilon \text{Np}$ ⎨MERSENNE a.1644; EULER BerlinM. a.1772 p.35⎬

$2^{61}-1 \varepsilon \text{Np}$ ⎨PERVOUCHINE, Acad. S. Petersbourg, a.1883 ⎬

·2 $m \varepsilon \text{N}_1 . 2^m-1 \varepsilon \text{Np} .\supset. m \varepsilon \text{Np}$ ⎨FERMAT a.1640 t.2 p.198⎬

·3 $m \varepsilon \text{Np} . a,b \varepsilon \text{N}_1 - \text{N}_1 \times m . a>b .\supset. a^{m-1}-b^{m-1} \varepsilon \text{N}_1 \times m$

❋ 8·1 $a,b,m \varepsilon \text{N}_1 . \text{D}(a,b)=1 .\supset. \exists\, \text{Np} \frown (a+\text{N}_0 \times b) \frown (m+\text{N}_1)$

⎨ LEGENDRE a.1808 p.398; Dm. DIRICHLET a.1837 t.1 p.313, MERTENS WienA. a.1899; WarszawaP. t.11 p.194 ⎬

·2 $a \varepsilon N_1+3 . \supset . \exists Np \frown (a+N_1) \frown (2a-N-2)$
{ BERTRAND JP. a.1845 Cahier 30 p.129.
Dm. TCHEBYCHEF a.1852 Œuvres t.1 p.52 {

·3 $2(N_1+1) \supset Np+Np$ { GOLDBACH a.1742 CorrM. t.1 p.135 {
Demonstratione defice.
G. Cantor (Congrès de Caen de l'A.F. a.1894) verifica que
$2\times(2 \cdots 500) \supset Np+Np$
V. Aubry (IdM. t.3 a.1896 p.75) que $2\times(2 \cdots 1000) \supset Np+Np$
R. Haussner (Jahresversammlung der Deutschen Math. Verein. a.1900 p.7) que $2\times(2 \cdots 5000) \supset Np+Np$.

·4 $m \varepsilon 2 \uparrow N_1 . \supset : 2^m+1 \varepsilon Np .=. 3 \uparrow (2 \uparrow (m-1))+1 \varepsilon N_1 \times [(2 \uparrow m)+1]$
{ PROTH CorrN. a.1878 t.4 p.210 {

·5 $q \varepsilon N_0 . 4q+3 , 8q+7 \varepsilon Np . \supset . 2^{4q+3}-1 \varepsilon (8q+7)N_1$
{ LUCAS TorinoA. a.1878 t.13 p.283 {

❋ 9·1 $p \varepsilon Np . \supset . N_1 \frown (2^p-1)/N_1 \supset N_0 \times p+1$

·2 $m,a,b \varepsilon N_1 . a^m-b^m \varepsilon Np . \supset . m \varepsilon Np . a=b+1$

·3 $a,b \varepsilon N_1 . p \varepsilon Np - \iota 2 . \supset . N_1 \frown (a^p-b^p)/N_1 \supset [N_1 \frown (a-b)/N_1] \cup (N_1 \times p+1)$

·4 » $. a^{p-1}+b^{p-1} \varepsilon N_1 \times p . \supset . a,b \varepsilon N_1 \times p$
{ ·1-·4 EULER PetrNC. a.1747-48 I p.20 {

·5 $m \varepsilon N_1 . \supset : m^4+4 \varepsilon Np .=. m=1$
{ GOLDBACH a.1742 CorrM. t.1 p.139 ; S. GERMAIN a.1772 p.296 {

·6 $a \varepsilon Np . b,c \varepsilon N_1 . \supset . (b+c)^a - b^a - c^a \varepsilon N_1 \times a$
{ EULER PetrNC. a.1747 t.1 p.20 {

❋ 10·1 $m \varepsilon N_1 . 4m+1 \varepsilon Np . \supset . m^m-1 \varepsilon N_0 \times (4m+1)$
{ BIKMORE a.1896 *Ed. Times*, t.65 p.78 {

$m,n,p \varepsilon Np . \supset .$

·2 $a^m+b^n \varepsilon Np - \iota 2 . \supset . \mathrm{Dvr}(m,n) \varepsilon 2 \uparrow N_0$ { LUCAS a.1891 p.342 {

·3 $a^m-b^m \varepsilon N_1 \times p . \supset . a \uparrow \mathrm{Dvr}(m, p-1) - b \uparrow \mathrm{Dvr}(m, p-1) \varepsilon N_1 \times p$
{ EULER PetrNC. a.1747-48 t.1 p.20 {

❋ 11·1 $a \varepsilon Np . \supset . (a-2)! -1 \varepsilon N_0 \times a$

·2 $a \varepsilon Np . \supset . (a-1)! +1 \varepsilon N_1 \times a$
{ LEIBNIZ Mss. Math. t.3 B11 fol.10:

« Productus continuorum usque ad numerum qui antepraecedit datum divisus per datum relinquit 1, ... si datus sit primitivus. Si datus sit deri-

vativus relinquet numerus qui cum dato habeat communem mensuram unitate majorem. » |

| WILSON, (WARING a.1770 p.218).
Dm. LAGRANGE a.1771 t.3 p.425 |

·3 $a \varepsilon \mathrm{Np} .=. a \varepsilon \mathrm{N}_1+1 . (a-1)!+1 \varepsilon \mathrm{N}_1 \times a$

·4 $a \varepsilon \mathrm{N}_1 . 4a+1 \varepsilon \mathrm{Np} .\supset. [(2a)!]^2+1 \varepsilon \mathrm{N}_1 \times (4a+1)$

·5 » $. 4a-1$ » » $-$ » $-$ »

| WARING a.1770; a.1782 p.380 : « Sit n numerus primus ... $\dfrac{2^2 3^2 4^2 5^2 \dots \overline{\frac{n-1}{4}}^2 \pm 1}{n}$ (ubi erit $+1$, quando $\frac{n-1}{2}$ fit par numerus, sin aliter -1) integri erunt numeri. » |

| Dm. LAGRANGE a.1771 t.3 p.431 |

* 12·1 $a \varepsilon \mathrm{Np} . b \varepsilon 1 \cdots (a-1) .\supset. \mathrm{C}(a,b) \varepsilon \mathrm{N}_1 \times a$

| LEIBNIZ *Math. Schr.* t.7 p.102:

« Si numerus rerum sit primitivus, combinatio ejus quaelibet per ipsum dividi potest, dempta prima et ultima. » |

·2 $a \varepsilon \mathrm{Np} . b \varepsilon 0 \cdots (a-1) .\supset. \mathrm{C}(a-1, b) \varepsilon \mathrm{N}_0 \times a + (-1)^b$

·3 $a \varepsilon \mathrm{Np} - \iota 2 . b \varepsilon 2 \cdots (a-1) .\supset. \mathrm{C}(a+1, b) \varepsilon \mathrm{N}_1 \times a$

| ·2·3 LUCAS AJ. a.1878 t.1 p.229 |

·4 $\mathrm{Hp} \cdot 1 .\supset. \mathrm{C}(a-2, b-1) \varepsilon \mathrm{N}_0 \times a - (-1)^b \times b$

Existe plure tabula de Np, et de divisores:

J. Ch. Burckhardt, *Table des Diviseurs pour tous les nombres du deuxième million.* Paris a.1814; *troisième million...* Paris a.1816; *premier million.* Paris a.1817.

J. Glaisher, *Factor table for the fourth million.* London 1879; *fifth million,* a.1880; *sixth million* a.1883.

Z. Dase, *Factorentafeln für alle Zahlen der siebente million,* Hamburg a.1862; *achte million* a.1863; *neunte million* a.1865. (Nono millione es completato per Rosenberg); *zehnte million* inedito, conservato in Archivios de Academia de Berlin.

J. Kulik linque manuscripto non terminato de tabula de divisores de numeros $1 \cdots 10^8$ conservato in Academia de Wien. (Vide Encyclopädie a.1901 t.1 p.952).

Davis, *Les nombres premiers de* 100 000 001 à 100 000 699, JdM. a.1866 s.2 t.11 p.188.

§17 mp

* 1. $a,b \varepsilon \mathrm{N}_1 \,.\, p \varepsilon \mathrm{Np} \,.\supset.$

·0 $\mathrm{mp}(p,a) = \max[\mathrm{N}_0 \cap x \ni (a \varepsilon \mathrm{N}_1 \times p^x)]$ Df

Si a es numero naturale, et p es numero primo, $\mathrm{mp}(p,a)$ indica: maximo exponente de potestate de p que divide a.

·1 $\mathrm{mp}(p,a) \,\varepsilon \mathrm{N}_0 \;.\; a \varepsilon \mathrm{N}_1 \times p \upharpoonright \mathrm{mp}(p,a) \;.\; a/p \upharpoonright \mathrm{mp}(p,a) \,-\varepsilon\, \mathrm{N}_1 \times p$

[$m \varepsilon \mathrm{N}_1 \,.\, p \upharpoonright m > a \,.\, x \varepsilon\, m + \mathrm{N}_1 \,.\supset.\, a < p^x \,.\supset.\, a \,-\varepsilon\, \mathrm{N}_1 + x \,.\, 0 \varepsilon\, \mathrm{N}_0 \cap x \ni (a \varepsilon \mathrm{N}_1 \times p^x) \,.\, -\exists\, \mathrm{N}_0 \cap x \ni (a \varepsilon \mathrm{N}_1 \times p^x) \cap (m + \mathrm{N}_1) \,.\, \S\max 1{\cdot}8 \,.\supset.\, \mathrm{P}$]

·2 $\mathrm{mp}(x,a) = \iota\, \mathrm{N}_0 \cap x \ni (a \varepsilon \mathrm{N}_1 \times p^x \,.\, a/p^x \,-\varepsilon\, \mathrm{N}_1 \times p)$ Dfp

·3 $\mathrm{mp}(p, ab) = \mathrm{mp}(p,a) + \mathrm{mp}(p,b)$

[$x = \mathrm{mp}(p,a) \,.\, y = \mathrm{mp}(p,b) \,.\supset.\, a \varepsilon \mathrm{N}_1 \times p^x \,.\, b \varepsilon \mathrm{N}_1 \times p^y \,.\, a/p^x \,-\varepsilon\, \mathrm{N}_1 \times p \,.\, b/p^y \,-\varepsilon\, \mathrm{N}_1 \times p \,.\supset.\, a \times b \,\varepsilon\, \mathrm{N}_1 \times p^{x+y} \,.\, a/p^{x+y} \,-\varepsilon\, \mathrm{N}_1 \times p \,.\supset.\, \mathrm{mp}(p, a \times b) = x + y$]

·4 $a \varepsilon \mathrm{N}_1 \times b \,.\supset.\, \mathrm{mp}(p,a) \geqq \mathrm{mp}(p,b)$

[$c \varepsilon \mathrm{N}_1 \,.\, a = bc \,.\supset.\, \mathrm{mp}(p,a) = \mathrm{mp}(p,b) + \mathrm{mp}(p,c) \geqq \mathrm{mp}(p,b)$]

·5 $\mathrm{mp}(p, a^b) = b \times \mathrm{mp}(p,a)$

·6 $\mathrm{mp}[p, \mathrm{D}(a,b)] = \min[\iota\, \mathrm{mp}(p,a) \cup \iota\, \mathrm{mp}(p,b)]$

·7 $\mathrm{mp}[p, \mathrm{m}(a,b)] = \max[\iota\, \mathrm{mp}(p,a) \cup \iota\, \mathrm{mp}(p,b)]$

* 2. $a,b \varepsilon \mathrm{N}_1 \,.\supset.$

·1 $x \varepsilon \mathrm{Np} \,.\supset_x.\, \mathrm{mp}(x,a) = 0 \,:\supset.\, a = 1$

[$a \varepsilon \mathrm{N}_1 + 1 \,.\supset.\, \exists\, \mathrm{Np} \cap x \ni (a \varepsilon \mathrm{N}_1 \times x)$

$a \varepsilon \mathrm{N}_1 + 1 \,.\, x \varepsilon \mathrm{Np} \,.\, a \varepsilon \mathrm{N}_1 \times x \,.\supset.\, \mathrm{mp}(x,a) > 0$ [

·2 $a \varepsilon \mathrm{N}_1 \times b \,.=:\, x \varepsilon \mathrm{Np} \,.\supset_x.\, \mathrm{mp}(x,a) \geqq \mathrm{mp}(x,b)$

·3 $a \varepsilon \mathrm{N}_1^{\,b} \,.=:\, x \varepsilon \mathrm{Np} \,.\supset_x.\, \mathrm{mp}(x,a) \,\varepsilon\, \mathrm{N}_0 \times b$

* 3·1 $a \varepsilon \mathrm{N}_0^2 + \mathrm{N}_1^2 \,.=:\, x \varepsilon \mathrm{Np} \cap (4\mathrm{N}_0 + 3) \,.\supset_x.\, \mathrm{mp}(x,a) \,\varepsilon\, 2\mathrm{N}_0$

} GIRARD a.1634 p.156 {

·2 $n \varepsilon \mathrm{N}_1 + 1 \,.\supset.\, \exists\, \mathrm{Np} \cap x \ni [\mathrm{mp}(x, n!) = 1]$

} LIOUVILLE JdM. s.2 t.2 a.1857 p.278 {

§18 Φ

✻ $a,b \varepsilon N_1 . \supset .$

·0 $\Phi a = \text{Num} \{ 1 \cdots a \frown x \ni [\text{Dvr}(x,a) = 1] \}$ Df

·01 $\Phi 1 = 1 . \Phi 2 = 1 . \Phi 3 = 2 \ldots$ ·02 $\Phi a \, \varepsilon \, N_1$

Euler (a.1760) voca Φa « numerus partium ad a primarum », et PetrA. t.4 II a.1780 p.18, indica per πa.

Gauss, a. 1801, Werke t.1 p.30 introduce symbolo Φ

Lucas a.1891, voca illo « indicatore », nomen introducto per Cauchy s.1 t.6 p.124 in sensu pauco differente.

·1 $\text{Dvr}(a,b) = 1 . \supset . \Phi(ab) = (\Phi a)(\Phi b)$

·2 $\text{Dvr}(a,b) = 1 . \supset . (a \uparrow \Phi b) - 1 \, \varepsilon \, b N_0$

·3 $a \varepsilon \text{Np} . \supset . \Phi a = a - 1$

·4 —— $. m \varepsilon N_1 . \supset . \Phi a^m = a^{m-1}(a-1)$

VOCABULARIO II.

101. ***Arithmetica*** G ἀριθμητική, A arithmetic, D arithmetik, F arithmétique, HI aritmetica, R arifmetica. = scientia de numeros.
⊂ arithmetico (102) — -o + -a (37).

102. ***arithmetico*** G, A arithmetical, D arithmetisch, F arithmétique, HI aritmetica, R arifmetic'. = numerico.
⊂ arithmo — -o + -etico (104).

103. arithmo G, = numero. ⊃ (101), log-arithmo.
(Ultimo elemento de arithmo es -mo || L -mo (125).
In primo elemento arith-, Vanic'ek vide radice ar- || L ar- = ar-te ⌒ ar-mo ⌒ .... Fick vide radice ra- || L ra- = ra-tione — -tione, || D reihe, rede. Ergo G arithmo || D reim ⊃ I rima).

104. -etico G -ητικό-ς = (102) ⌒ po-etico ⌒ phon-etico.
arithmo — -o + -e- ⊃ arithme- G = numera,
arithme- + -to (135) ⊃ arithmeto G = numerato,
arithmeto — -o + -ico (35) ⊃ arithmetico.

105. ***numero*** HI, F nombre, numéro, A number, nùmero, D nummer, numerus, R numer'. ⊃ numer-abile AFHI, numer-ico ADFHI, numer-a aDfHIR, numera-tione ADFHIR, ...
⊂ nume- (106) + -ro (107).

106. nume-, numera, eme. ⊃ nume-ro, num-mo.
|| G neme, nomo; D nehme.

107. -ro = nume-ro ⌒ ag-ro ⌒ mace-ro ⌒ integ-ro ⌒ libe-ro ⌒ rub-ro.
|| G -ro = ag-ro ⌒ eryth-ro ⌒ cylind-ro; S -ra.

108. ***zero*** AFHI (non L). ⊂ Arabo : sihron.

109. ***plus, plure ;*** ADF plus, R pljus' (in Mathematica, cum valore +); I più. ⊃ plur-ale (vide nota ad N. 56), F plus-ieurs.
|| G poly, G.antiquo plos; S puru, D viel.
⊂ ple — -e + -us.

110. -us, -ure, -ore, -iore ⊃ (109), maj-us, min-us,...;
⊃ maj-ore, min-ore, infer-iore, poster-iore,...
|| A -er ; D -er, gröss-er — gross, jüng-er — jung ...; R -jejs', S -jas.
S nav-jas = L nov-iore = D neu-er = R nov-jes'-ij
Transformatione de E *s* in *r*, ut in N. 56, es commune ad LAD.
Vocabulo D gross + suffixo E -ios = grösser, ubi litera *i* es expresso per transformatione de -o- in -ö-.

111. ***ple*** (L.antiquo). ⊃ ple-no, F plein, H lleno, I pieno, com-ple-to AFHIr., sup-ple-mento ADFhI, ...
|| G ple-, A full, D viel, voll, R pol-no, S par.

112. ***inductione*** ADFHIR. ⊂ in (113) + duc (20) + -tione (12)

113. ***in***, I in, FH en, || A in, D in, G en, R v'. ⊃ in-stituto ADEHIR.

114. NUMEROS.

1 ***uno***, un-, F un, HI uno.
⊃ un-iforme ADFHI, un-ione ADFHI, un-iversitate ADFHIR...
|| a, an, one, D ein, G oino, Slavo ino. ⊂ E oino.

2 ***duo,*** du-, F deux, H dos, I due. ⊃ du-alismo ADFHIR.
|| A two, D zwei, G dyo, di-, R dva, dv', S dva. ⊂ E dvo.

3 ***tres,*** tri-, F trois, H tres, I tre. ⊃ tri-angulatione ADFHiR.
|| G treis, tri-. ⊃ tri-gonometria ADFHIR.
|| A three, D drei, R tri, S trajas. ⊂ E tri.

4 ***quatuor,*** quar-, quadr-, quater, F quatre, H cuatro, I quattro.
⊃ quar-to ADFHIR, quadr-atura ADFHIR, quater-nione.
|| G tettares, tetra-. ⊃ tetra-hedro ADFHI.
|| A four, D vier, R c'etyre, S c'atvar-, c'atur-. ⊂ E quetur.

5 ***quinque,*** quin-, F cinq, H cinco, I cinque. ⊃ quin-ario ADFHI, quin-to ADFHIR. || G pente. ⊃ penta-metro ADFHIR.
|| A five, D fünf, R pjatị, S panc'a. ⊂ E penque.

6 ***sex,*** F six, H seis, I sei.
|| A six, D sechs, G hex. R s'es-tị, S s'as'. ⊂ E sex.

7 ***septem,*** F sept, H siete, I sette.
|| A seven, D sieben, G hepta, R semị, sedịm-, S sapta. ⊂ E septem

8 ***octo***, F huit, H ocho, I otto. ⊃ octo-bre ADFHIR.
|| A eight, D acht, G octo. R vos-emị, osịm-, S as'ta. ⊂ E octo.

9 ***novem,*** F neuf, H nueve, I nove.
|| A nine, D neun, G ennea, ena, S nava. ⊂ E neun

10 ***decem,*** dec-, F dix, H diez, I dieci. ⊃ dec-i-metro ADFHIR.
|| G deca. ⊃ deca-litro ADFHIR.
|| A then, D zehn, R desja-tị, S daça.

100 ***centum,*** cent-, F cent, H ciento, I cento, AD cent (moneta).
|| G he-caton. ⊃ hecto-gramma ADFHIR.
|| A hund-red, D hund-ert, R sto, S çata.
⊂ E cento ⊂ cen- (= decem, L -gin- de tri-gin-ta, G -con-) + -to (135, = ⌈2).

1000 ***mille,*** mil-, F mil, mille, H mil, I mille.
⊃ L mil-ia, mill-ia, A mile, D meile, R milja.
⊂ (secundo Brugmann) smi (L sim-, G mia, =1) + hili (|| G chili-oi, S sa-hasra-m, E gheslo).

1000000 ***milione*** (non L) I, ADF million, H millon, R millịon'.
⊂ mil- + -ione (accrescitivo I).

115. ***cifra*** (Stifel a. 1544), cyphra (Euler) ⊂ Arabo çifr.
A cipher, cypher, F chiffre, HI cifra, D ziffer, R tsifra, s'ifr'.

116. ***additione*** ADFHI. ⊂ addito (117) — -o + -ione (118).

117. ***addito*** ⊂ adde (118) — -e + -i- (9) + -to (135).

118. -ione = (116) ⌒ un-ione ⌒ leg-ione ⌒ opin-ione ⌒ ...
-tione (12) = -to (135) — -o + -ione.

119. ***adde***, A add, D addire. ⊂ ad (41) + -de.

120. -de = ad-de ⌒ con-de ⌒ e-de ⌒ per-de ... = pone, fac (137).
|| S dha-, G the-, A do, D thu-e.

121. ***summa*** (nomen), A sum, D summe, F somme, H suma, I somma, R summa. ⊂ summo — -o + -a (126).

122. ***summa*** (verbo), A sum, D summire, F somme, H suma, I somma.
= adde. ⊂ summo — -o + -a (4).

123. ***summo***, I sommo, H sumo. ⊃ summ-ario, summ-itate.
|| S upama. ⊂ sup- + -mo.

124. sup-, ⊃ sup-er, sum-mo, sus- sus-cita, sus-pende.
|| G hyp- = hyp-er ⌒ hyp-s ⌒ hyp-so.
A up, over; D ob-, über, ober, oben, auf,...
(S upa = L ad), S upari = L super (257).
Es ligato cum suo contrario: sub (95).

125. -mo = sum-mo ⌒ pri-mo ⌒ infi-mo ⌒ supre-mo ⌒ extre-mo ⌒ ...
septi-mo ⌒ deci-mo ... ⌒ ani-mo ⌒ fir-mo ⌒ gru-mo ...
|| G -mo = thy-mo (= L fumo) ⌒ ther-mo ⌒ hebdo-mo.
S sapta-ma ⌒ das'a-ma. D war-m ⌒ rau-m. R dym'.

126. -a, indica feminile naturale aut artificiale.
= (121) ⌒ fili-a ⌒ ordinat-a ⌒ recta. || G -a (37).

§2.

127. ***multiplicato*** ⊂ multiplica + -to (135).

128. ***multiplica*** H, A multiply, D multiplicire, F multiplie, I moltiplica.
⊂ multo — -o + -i- + plica.

129. ***multo***, I molto, H mucho. ⊃ mult-iforme AFHI, mult-itudo,...
|| secundo Vanic'ek, A many, D manche, menge; R mnogo.
Secundo Fick, L meliore, mille. Secundo Corsen, L mole.

130. ***plica***, F plie, H plega, I piega.
⊃ ex-plica ⌒ com-plica ⌒ du-plica ...
|| G plece πλέκε, S praç-na, D flech-te, L plecte (295).

131. ***multiplicatione***, ADFHI. ⊂ multiplica (128) + -tione (12).

132. ***producto*** H, A product, F produit, D produkt, I prodotto
R product'. ⊂ produce — -e + -to.

133. ***produce*** AHI, F produi-re. ⊂ pro + duce (20).

134. ***pro***, A pro, F pour, H por, pro, I per, prò.
⊃ pro-cessu ADFHIR, pro-fessore. || G pro, pro-blema ADFHIR.
|| A for, far, D für, ver-, vor; R pro; S pra; L per.

135. -to, ADF -t, H -do, I -to, R -t'.
= ama-to ⌢ dele-to ⌢ uni-to ⌢ no-to ⌢ produc-to ⌢ fac-to ...
= acu-to ⌢ denta-to ⌢ onus-to.
|| G -to, cathe-to, epithe-to.
|| A -ed, -d = unit-ed ⌢ add-ed ⌢ subtract-ed ...
D -t, -et, R -tyĭ, -toe, S -ta, -ita. ⊂ E -to.

136. ***factore***, AH factor, D faktor, F facteur, I fattore, R (non math.) factor'. ⊂ fac + -tore

137. ***fac***, F fai-re, fai-t; H hace; I fa.
⊃ fac-tura DFHiR, fac-ultate ADFHIR.
|| A do, D thue, G the- (23), ti-the-mi, S da-dha-mi, R dje-tı̣, dje-lo.
= L -de (120). ⊂ E dhê.

Linguistas pone :
E *dh* = S *dh* = G *th* = A *d* = D *th* = R *d* = L *f*, si initiale, L *d* si medio, et L *b* si cum *r*, aut ante *l*, aut post *u*.
L fac ⊂ L feci = G thê-ce.

138. -tore A -tor, D -tor, F -teur, H -dor, -tor, I -tore, R -tor'.
⊃ ora-tore ADFHIR, doc-tore ADFHIR.
|| G -tor, -τωρ, -τορ- = rhe-tore, his-tor-ico. S -tar.

§3.

139. ***potentia, potestate***, A potency, power, D potenz, F puissance, H potencia, I potenza. ⊃ potentia-le ADFHIR.
potentia ⊂ potente (140) — -e + -ia (143) = pote (141) + -ntia (144).
potestate ⊂ potes (141) + -tate (10).

140. ***potente*** HI, A potent. ⊂ pote (141) + -nte (142).

141. ***pote-***, potes, posse, F peut, puisse, H puede, poder, I può, pote-re.
|| G posi-s, des-pot-a =|| S pati =|| L potis, pote.

142. -nte, -ente, -iente, AD -ent, F -ant, -ent, HI -ente, R -ent'.
⊃ stud-ente ADfhIR, differ-ent-iale ADFHIR, expon-ente,...
|| A -ing, D -end, G -onti, S -anta.

Exemplo: S bhar-anta, G pher-onti, L fer-ente, A⊂L (dif)fer-ent, A || L bear-ing, D || L (ge)bär-end, HI (con)fer-ente, DR (dif)fer-ent(ial).

143. -ia HI, A -y, -e, D -ie, -ien, F -e, -ie, R -ija.
= (139), concord-ia, scient-ia, famil-ia ADfHiR, patr-ia, Ital-ia.
|| G -ia = analog-ia ⌢ geometr-ia ⌢ ...

144. -entia, -ntia, A -ence, -ency, D -enz, F -ance, H -encia, I -enza, R -entsija. ⊃ corresponde-ntia ADFHIR, differ-entia ADFHI.
⊂ -ente (142) — -e + -ia (143)

145. ***basi***, G βάσι-ς, ADFHIR. ⊂ ba- (146) + -si (24).

146. ba- G, || L va- ⊃ FI va; S ga, A go, D gehe. ⊂ E gva.
⊃ ba-si = pede, ba-sileus = dux, rex.

147. ***exponente*** H, ADR exponent, I esponente, F exposant.
⊂ expone (148) + -nte (142).

148. ***expone*** H, I espone. ⊂ ex (75) + pone (149).

149. ***pone*** HI ⊃ (148), com-pone, dis-pone; = F pose.

§5.

150. ***naturale*** I, AH natural, F naturel, D natürlich, R naturalṇyj.
⊂ natura (151) + -le (6).

151. ***natura*** HIR, AF nature, D natur. ⊂ nato (151) — -o + -ura (153)

152. ***nato*** I, F né. ⊃ (151), nat-ale, nat-ione ADFHIR, nat-ivo.
⊂ -gnato = co-gnato ⌒ a-gnato ⌒ ... Vide 249.
= genito ⊂ gene (154) — -e + -i (9) + -to.

153. -ura = (151) ⌒ flex-ura ⌒ fig-ura ⌒ ...

154. ***gene*** L.antiquo. = genera.
⊃ (152), gene-re, gene-rale ADFHIR, gen-ito, gen-itore, gen-te.
|| G gene- = gene-si ⌒ oxy-gen-o ⌒ homo-gene-o...
|| AD kin-d. S g'an-, g'anita, g'anitri.

155. ***majore***, A major, D major, meier, F majeur (⊂ majòre), maire (⊂ màjor), H mayor, I maggiore; || S mahijas.
⊂ mag- (156) + -iore (110).

156. mag- = pote, cresce. ⊃ (155) ⌒ mag-istro aDfhiR ⌒ mag-no.
|| A may, make; D möge, mache; R mogy. ⊂ E magh-, mag-.
(Es ligato ad G mega, mega-lo, mech-anica, man-gano. A more much; D mehr; S maha;..).

157. ***minore*** I, A minor, F mineur, H menor, D minor (Determinante) R (non math.) minor'. ⊂ min- (158) + -iore (110).

158. mino (L. raro). ⊃ (157), min-imo, min-istro ADFHIR.
|| G min-ythe, D min-dern, R men-ṣs'ĭj. ⊂ mi- (159) + -no (160).

159. mi- = minue. ⊃ (158). || S mi.

160. -no = mag-no ⌒ ple-no ⌒ Roma-no ⌒ pater-no ...
|| G -no = co-no ⌒ hyp-no ⌒ chro-no.
A -ne = do-ne ⌒ go-ne, D -n, geworde-n, gebisse-n, R -nyj, S -na.
R pol-nyj = S pra-na = L ple-no.

§6.

161. ***minus*** AD, F moins, I meno, R minus'. ⊂ min- (158) + -us (110).

Operatione — es dicto «subtractione». *a—b* es «differentia» inter *a* et *b*.

162. ***subtractione*** ADFHI. ⊂ sub (95) + tracto — -o + -ione (118).

163. ***subtrahe***, A subtract, D subtrahire, F soustraire, H substrahe I sottrae. ⊂ sub + trahe (165).

164. ***tracto*** H, A tract, trait, F trait, I tratto. ⊂ trahe — -e + -to.
⊃ (162), con-tracto, abs-tracto, tract-ato, tract-ione, tract-rice.
-h- + -to ⊃ -cto: trah- + -to ⊃ tracto, veh- + -to ⊃ vecto.

165. ***trahe***, F trai-re, HI trae.

166. ***differentia***, A difference, D differenz, F différence, H diferencia, I differenza. ⊃ differentia-le ADFHIR. ⊂ differ + -entia (144).

167. ***differ*** A, DI differ-ire, F differ-er, H difer-ir.
⊂ dis- (51) + fer (168).

168. ***fer*** ⊃ (166), con-fer-ente, fer-tile.
|| G phere ⊃ phere-tro, phos-phor-o, reo-phor-o, peri-pher-ia.
A bear, D (ge)bäre, R bra-tị, berị, S bhar. ⊂ E bher.
E bh = S bh = G ph = A b = D b = R b = L f-, -b- (L f- initiale, -b- medio).

§7.

169. ***diviso*** HI, F divisé. ⊂ di- (51) + viso (174).

170. ***divisione*** I, ADFH division. ⊂ diviso (169) — -o + -ione (118).

171. ***divide*** AHI, D dividire, F divise. ⊂ di- (51) + vide (175).

172. ***dividendo*** I, AD dividend, F dividende. = que debe es diviso.
= divide (171) + -ndo (176).

173. ***divisore*** I, ADH divisor, F diviseur. ⊂ diviso (169) + -re (177).

174. viso HI, = vide (175) — -e + -to (135).
-d- + -t- ⊃ -s- : claud- + -to ⊃ clauso, plaud- + -to ⊃ plauso, ...

175. ***vide,*** F voi-r, H ve-r, I vede. ⊃ (171), e-vide-nte, in-vid-ia, pro-vide-ntia. || G eid-, ide ⊃ id-ea, parabol-o-ide. A wot, (wisse = sci) R vid-jetị, vid', S vid, vidh.

176. -endo, -ndo ⊃ (172), minu-endo, multiplica-ndo, secu-ndo.

177. -re parte de suffixo -tore (138), -sore (173).

§9,10.

178. ***maximo*** H, ADFRL maximum, I massimo.
⊂ mag- (156) + -si- + -mo (125). Elemento -si- non es simplice.

179. ***minimo*** HI, ADFRL minimum.
⊂ mino (158) — -o + -i- (9) + -mo (125).

180. ***multiplo*** HI, AF multiple. ⊂ multo — -o + -i- (9) + -plo (181).

181. -plo = (180) ⌢ sim-plo ⌢ du-plo ⌢ ... || G -plo ⊃ di-plo-ma, D -fal, -fel, zwei-fel. ⊂ ple (111) — -e + -o.

182. -o = (181) ⌢ profug-o ⌢ luci-fer-o ⌢ prod-ig-o ⌢ bene-vol-o. = -ente. super-o ⌢ infer-o. || G -o, strat-eg-o, anthropo-phago. tele-graph-o.

§11.

183. ***quoto*** I, quot, quoties, quotiens, ADF quotient; H cociente, I quoziente. ⊃ quot-a, quot-idiano. || S cati, catitha.

184. ***resta*** HI, A rest, F reste, D restire. ⊂ re- (186) + sta (77).

185. ***resto*** (non L) HI, AD rest, F reste.
⊂ resta (184) — -a + -o (182).

186. re-, red- = red-de ⌢ re-trahe ⌢ re-clama ⌢ re-pugna ⌢ ...

Continua post Algebra.

III

ALGEBRA

III. ALGEBRA.

§1 f ɟ (functione)

(§1-4 contine omni complemento ad Logica mathematica, occurrente in continuatione de Formulario).

✻ 1.

« Functio, operatio, correspondentia » es vocabulo identico, ant simile inter se. In lingua commune, omni vocabulo relativo, ut « patre, filio,... » indica functio.

Functio es expresso in plure casu per signo que praecede variabile, que me voca signo de « prae-functio » Ita, in « log$x$, sin$x$ », « log, sin » es signo de prae-functio.

In alios casu, signo de functio seque variabile; me voca illo signo de « post-functio ». Ita in $a!$ (pag. 52), « ! » es signo de post-functio.

Nos considera duo classe a et b; et nos scribe $u \varepsilon a ɟ b$, et lege « u es signo de post-functione, que transforma omni a in b » vel « u es transformatore de a in b » vel « u es a ef b », si signo u scripto post omni individuo arbitrario x de classe a, produce novo elemento xu, que pertine ad classe b.

In symbolos:

·0 $a,b \varepsilon \text{Cls} . \supset \therefore u \varepsilon a ɟ b . =: x \varepsilon a . \supset_x . xu \varepsilon b$ Df ɟ

Et nos scribe $u \varepsilon b \mathrm{f} a$, et lege « u es signo de prae-functione, que ad omni a fac corresponde aliquo b » vel « u es b functione des a », si signo u scripto prae omni a produce elemento de classe b:

·01 $a,b \varepsilon \text{Cls} . \supset \therefore u \varepsilon b \mathrm{f} a . =: x \varepsilon a . \supset_x . ux \varepsilon b$ Df f

Pro brevitate, nos enuntia propositiones super uno solo de duo signo f et ɟ.

·1 $a,b\varepsilon$ Cls . $u\varepsilon a$ʃb . $x,y\varepsilon a$. $x=y$.Ɔ. $xu=yu$ Oper u

[§= P·1 .Ɔ. $x\varepsilon$ $z\ni(xu=zu)$. §ε 5·2 .Ɔ. $y\varepsilon$ $z\ni(xu=zu)$.Ɔ. Ths]

Si duo objecto x et y es aequale inter se, et si super illo nos fac identico operatione u, et duo resultatu fi aequale.

Nos « opera per u » quando nos transforma aequalitate $x=y$ in $ux=uy$.

·2 $a,b,c\varepsilon$ Cls . $u\varepsilon a$ʃb . cƆa .Ɔ. $u\varepsilon c$ʃb

[Hp . $x\varepsilon c$.Ɔ. $x\varepsilon a$.Ɔ. $xu\,\varepsilon b$:Ɔ. P]

·3 $a,b,c\varepsilon$ Cls . $u\varepsilon a$ʃb . bƆc .Ɔ. $u\varepsilon a$ʃc

[Hp .Ɔ: $x\varepsilon a$.Ɔ. $xu\,\varepsilon b$.Ɔ. $xu\,\varepsilon c$:Ɔ. Ths]

Si u transforma to-s a in b, et si c es subclasse de a, tunc u transforma to-s c in b. Si u es semper a ef b, et classe b continere in c, tunc u es a ef c.

* 2·1 $+\ \varepsilon\ N_0$ʃN_0 [= §+ 1·2]

$+$ « successivo » es operatione que transforma numero in numero. Es propositione primitivo ·2 de §+, scripto per signo ʃ.

·2 $s\varepsilon$ Cls . $0\varepsilon s$. $+\ \varepsilon\ s$ʃs .Ɔ. N_0Ɔs [= Induct]

·3 $a\varepsilon N_0$.Ɔ. $+a\ \varepsilon\ N_0$ʃN_0 . $\times a\ \varepsilon\ N_0$ʃ$N_0$. ↑$a\ \varepsilon\ N_0$ʃN_0
$-a\ \varepsilon\ (a+N_0)$ʃN_0 : $a\varepsilon N_1$.Ɔ. $/a\ \varepsilon\ (a\times N_1)$ʃ$N_1$

[= §+ 4·1 . §× 1·1 . §↑ 1·1 . §− 1·1 . §/1·1]

Si a es numero, tunc operationes :

$+a$, « plus a », « additione de a », « to adde a »,

$\times a$, « per a », « to multiplica per a »,

↑a, « ad a », « to eleva ad potestate a »,

transforma numero in numero.

Operatione :

$-a$, « minus a », « to subtrahe a »

es possibile supra numeros superiore ad a, et

$/a$, « in a », « to divide per a »

es possibile supra multiplos de a.

Plure Auctore moderno claude variabile inter ( ). Sed parenthesi jam habe in Arithmetica usu determinato, de collega plure elemento, et nos non pote ute illo in novo sensu. In vero, in **scriptura** (x), **litera** x **non** es ligato ad aliquo elemento. Omni Auctore scribe $\log x$, $\sin x$, et non $\log(x)$, $\sin(x)$; $f(x+h)$ et non $f((x+h))$. Lagrange, Abel,... non scribe parenthesi in hoc novo sensu, introducto verso a.1823.

Nota differentia inter f et f; f es symbolo constante, que nos lege « functione »; f es litera variabile (I §1), que pote repraesenta omni objecto, p. ex. aliquo functione.

✻ 3. $a,b,c,d \varepsilon \mathrm{Cls} \,.\supset:$

·0 $u \varepsilon a\,\text{ɟ}\,b \,.\, v \varepsilon b\,\text{ɟ}\,c \,.\, x \varepsilon a \,.\supset.\, x(uv) = (xu)v = xuv$ Df

·01 $u \varepsilon b\mathrm{f}a \,.\, v \varepsilon c\mathrm{f}b \,.\, x \varepsilon a \,.\supset.\, (vu)x = v(ux) = vux$ Df

·1 $u \varepsilon a\,\text{ɟ}\,b \,.\, v \varepsilon b\,\text{ɟ}\,c \,.\supset.\, uv \;\varepsilon\; a\,\text{ɟ}\,c$

·2 $u \varepsilon a\,\text{ɟ}\,b \,.\, v \varepsilon b\,\text{ɟ}\,c \,.\, w \varepsilon c\,\text{ɟ}\,d \,.\, x \varepsilon u \,.\supset.\, (xu)(vw) = (xuv)w$

Signo vu, de P ·01, indica operatione composito per operationes u et v.

sim rcp idem

✻ 4. $a,b,c \varepsilon \mathrm{Cls} \,.\supset\therefore$

·0 $u \varepsilon (b\mathrm{f}a)\mathrm{sim} \,.=:\, u \varepsilon b\mathrm{f}a : x,y \varepsilon a \,.\, ux = uy \,.\supset_{x,y}.\, x = y$ Df

·1 $u \varepsilon (b\mathrm{f}a)\mathrm{sim} \,.\, c \supset a \,.\supset.\, u \varepsilon (b\mathrm{f}c)\mathrm{sim}$

·2 » $.\, b \supset c \,.\supset.\, u \varepsilon (c\mathrm{f}a)\mathrm{sim}$

·3 » $.\, x,y \varepsilon a \,.\supset: x = y \,.=.\, ux = uy$

·4 » $.\, v \varepsilon (c\mathrm{f}b)\mathrm{sim} \,.\supset.\, vu \;\varepsilon\; (c\mathrm{f}a)\mathrm{sim}$

·5 $+ \;\varepsilon\; (N_0\,\text{ɟ}\,N_0)\mathrm{sim}$ [= §+ 1·4]

$a \varepsilon N_0 \,.\supset.\, +a \;\varepsilon\; (N_0\,\text{ɟ}\,N_0)\mathrm{sim}$ [= §+ 4·5]

$a \varepsilon N_1 \,.\supset.\, \times a \;\varepsilon\; (N_1\,\text{ɟ}\,N_1)\mathrm{sim} \,.\, \upharpoonleft a \;\varepsilon\; (N_1\,\text{ɟ}\,N_1)\mathrm{sim}$

·6 $u \varepsilon (b\mathrm{f}a)\mathrm{rcp} \,.=:\, u \varepsilon (b\mathrm{f}a)\mathrm{sim} : y \varepsilon b \,.\supset_y.\, \exists\, a \frown x \ni (ux = y)$ Df

·7 $u \varepsilon (b\mathrm{f}a)\mathrm{rcp} \,.\, v \varepsilon (c\mathrm{f}b)\mathrm{rcp} \,.\supset.\, vu \;\varepsilon\; (c\mathrm{f}a)\mathrm{rcp}$

·8 $\mathrm{idem}x = x$ Df

Nos dice que u es operatione simile (sim) inter classes a et b, si u es operatione tale que, si duo resultatu es aequale, et objecto primitivo es aequale inter se.

Operatione u es « reciproco » (rcp), si es simile, et si ad omni b corresponde aliquo a.

« idem » indica identitate.

✻ 5. $s \varepsilon \mathrm{Cls} \,.\, u \varepsilon s\,\text{ɟ}\,s \,.\, a \varepsilon s \,.\, b,c \varepsilon N_0 \,.\supset.$

·1 $au0 = a$ Df ·2 $au(b+) = (aub)u$ Df

·3 $aub \;\varepsilon s$

[Hp . $b=0$. P·1 $.\supset.$ Ths : Hp . $aub\, \varepsilon s$. P·2 $.\supset.$ $au(b+)\, \varepsilon s$: Induct $:\supset.$ P]

·4 $(ub) \;\varepsilon\; s\,\text{ɟ}\,s$ [= P·3]

·5 $u \varepsilon (s\,\text{ɟ}\,s)\mathrm{sim} \,.\supset.\, (ub) \;\varepsilon\; (s\,\text{ɟ}\,s)\mathrm{sim}$

[Hp . $b=0$ $.\supset.$ Ths (1)

Hp . $ub\, \varepsilon\, (s\,\text{ɟ}\,s)\mathrm{sim} \,.\, x,y \varepsilon s \,.\, x -= y \,.\supset.\, xub -= yub \,.$

$\supset.\, (xub)u -= (yub)u \,.\supset.\, xu(b+) -= yu(b+)$ (2)

Hp . $ub\, \varepsilon\, (s\,\text{ɟ}\,s)\mathrm{sim}$. (2) $.\supset.$ $u(b+)\, \varepsilon\, (s\,\text{ɟ}\,s)\mathrm{sim}$ (3)

(1).(3). Induct $.\supset.$ P]

·6 $v \varepsilon s\mathrm{J}s : x \varepsilon s . \supset_x . xuv = xvu : \supset . (aub)v = (av)ub$

[Hp . $b=0$. P·1 .$\supset$. Ths (1)

Hp . $(aub)v = (av)ub$. P·2 .$\supset$. $[au(b+)]v = [(aub)u]v =$
$[(av)ub]u = (av)u(b+)$ (2)

(1) . (2) . Induct .$\supset$. P]

·61 Hp P·6 .$\supset$. $(aub)vc = (avc)ub$

[Hp . $(ub,v,c)|(v,u,b)$ P·6 .$\supset$. Ths]

·62 Hp P·6 .$\supset$. $a[(uv)b] = a(ub)(vb)$

[Hp . $b=0$.$\supset$. Ths (1)

Hp . $a[(uv)b] = a(ub)(vb)$.$\supset$. $a[(uv)(b+)] = a[(uv)b]uv =$
$a(ub)(vb)uv = a(ub)u(vb)v = a[u(b+)][v(b+)]$ (2)

(1) . (2) . Induct .$\supset$. P]

·7 $a(ub)(uc) = a(uc)(ub)$ [P·61 . $v=u$.$\supset$. P]

·8 $(aub)uc = au(b+c)$

[Hp . $c=0$. P·1 .$\supset$. Ths (1)

Hp . $(aub)uc = au(b+c)$.$\supset$. $[(aub)uc]u = [au(b+c)]u$ (2)

» .$\supset$. $(aub)u(c+) = au(b+c+)$ (3)

Hp . (1) . (3) . Induct .$\supset$. Ths]

Definitione de operatione repetito. Si s es classe, et u es tranformatione des s in s, si a es individuo de classe s, et b es numero, tunc aub indica resultatu de operatione u, repetito b vice, super a.

Additione, multiplicatione, et potestate, in Arithmetica, es operatione repetito, et principale theoremas seque de theoria de operatione repetito.

* 6. $(N_0, +)|(s, u)$ P5·1 ·2 ·3 ·7 ·8 .$\supset$.
§+ P3·1 ·2 4·1 ·3 ·2

* 7·0 $a,b \varepsilon N_0$.$\supset$. $a \times b = 0[(+a)b]$ Dfp [= §×P1·0·01]

·1 $(N_0, +a, 0)|(s, u, a)$ P5·3 .$\supset$. §×P1·1

·2 $(N_0, +a, 0)|(s, u, a)$ P5·8 .$\supset$. §×P1·2

·3 $(N_0, +a, +b, 0, c)|(s, u, v, a, b)$ P5·62 .$\supset$. §×P1·3

·4 $s \varepsilon$ Cls . $u \varepsilon s\mathrm{J}s$. $x \varepsilon s$.$\supset$. $x[(ua)b] = x[u(a \times b)]$

[Hp . $b=0$.$\supset$. Ths (1)

Hp . $x[(ua)b] = x[u(a \times b)]$.$\supset$. $x[(ua)(b+1)] = x[(ua)b]ua =$
$[u(a \times b)]ua = xu(a \times b+a) = xu[a \times (b+1)]$ (2)

(1) . (2) . Induct .$\supset$. P]

·5 $(N_0, +a, 0)|(s,u,x)$ P·4 .$\supset$. §×P1·5

✻ 8·0 $a, m \varepsilon N_0 .\supset. a \uparrow m = a^m = 1[(\times a)m]$ Dfp

$(N_0, \times a, 1)|(s,u,a)$P5·1·2 $.\supset.$ §$\uparrow$P1·0·01
$(N_0, \times a, 1)|(s,u,a)$ P5·3 $.\supset.$ §$\uparrow$P1·1
» » » ·2
$(N_0, \times a, \times b, 1)|(s,u,v,a)$ P5·62 $.\supset.$ §$\uparrow$P1·3
$(N_0, \times a, m, n, 1)|(s,u,a,b,x)$ P8·4 $.\supset.$ §$\uparrow$P1·4

✻ 9. $s \varepsilon$ Cls . $v \varepsilon$ sfs . $a \varepsilon s$. $b \varepsilon N_0 .\supset.$
·5 $v^0 a = a$ ·2 $v^{b+1} a = v(v^b a)$ Df

Si signo de functione praecede variabile, plure Auctore per exponente indica functione repetito. Nos scribe exponente ad operatione D.

§ 2 | ‘ ’

Si A es expressione que contine litera x, per $A|x$, nos indica signo de praefunctione, que calculato super x, da pro resultatu A. Ce notatione es frequente in §Σ, Π, lim, D, S,...

$(|x) A$ indica signo de postfunctione, que ad x fac corresponde A. Nos lege signo $A|x$ « A inverso x », vel « A, ubi varia x »; et nos defini illo ut seque:

✻ 1·1 $a, b \varepsilon$ Cls . $u \varepsilon$ bfa $.\supset. (ux)|x = u$ Df |
·2 » $u \varepsilon$ afb $.\supset. (|x)(xu) = u$ Df

✻ 2. $a, b, c, d \varepsilon$ Cls . $u \varepsilon$ bfa $.\supset.$

·0 $u‘a = y \ni [\exists\, a \frown x \ni (ux = y)]$ Df‘

·01 $y \varepsilon u‘a .=. \exists\, x \ni (x \varepsilon a . ux = y)$ [P·0 . Oper $y \ni .\supset.$ P]

·1 $x \varepsilon a .\supset. ux \,\varepsilon\, u‘a$

·2 $c \supset a .\supset. u‘c \supset u‘a$ Oper $u‘$

[Hp $.\supset. c \frown x \ni (ux = y) \supset a \frown x \ni (ux = y)$. Oper $\exists$. Oper $y \ni .\supset.$ P]

·3 $u‘a \supset c .=: x \varepsilon a .\supset_x. ux \,\varepsilon c$

·4 $c \supset a . d \supset a .\supset. u‘(c \cup d) = u‘c \cup u‘d$ Distrib(‘ ∪)

[Df‘ $.\supset.$ $u‘(c \cup d) = y \ni \exists [(c \cup d) \frown x \ni (ux = y)]$
Distrib($\frown$,∪) $.\supset.$ » $y \ni \exists [c \frown x \ni (ux = y) \cup d \frown x \ni (ux = y)]$
Distrib($\exists$,∪) $.\supset.$ » $y \ni [\exists\, c \frown x \ni (ux = y)] \cup [\exists\, d \frown x \ni (ux = y)]$
Distrib($\ni$, ∪) $.\supset.$ » $y \ni [\exists\, c \frown x \ni (ux = y)] \cup y \ni [\exists\, d \frown x \ni (ux = y)]$
Df‘ $.\supset.$ » $u‘c \cup u‘d$]

·5 $\exists\,(u\text{‘}a)\frown c \;.=.\; \exists\, a\frown x\ni(ux\,\varepsilon c)$

[Df‘ $.\supset:$ $\exists\,(u\text{‘}a)\frown c \;.=.\; \exists\, c\frown y\ni[\exists\, a\frown x\ni(ux=y)$
§$\exists$ 3·4 $.\supset:$ » $\exists\, a\frown x\ni[\exists\, c\frown y\ni(y=ux)]$
Dfι $.\supset:$ » » $(\exists\, c\frown\iota ux)$
» » $(ux\,\varepsilon c)$]

·6 $v\varepsilon\, \mathrm{cf}b \;.\supset.\; v\text{‘}(u\text{‘}a)=(vu)\text{‘}a$

[Hp $.\supset.$ $v\text{‘}(u\text{‘}a) = z\ni\exists\, u\text{‘}a\frown y\ni(vy=z)]$
P·5 $.\supset.$ » $= z\ni[\exists\, a\frown x\ni[ux\,\varepsilon\, y\ni(vy=z)]\}$
» $= z\ni[\exists\, a\frown x\ni(vux=z)]$]

·7 $u\text{‘}\Lambda=\Lambda$

·8 $x\varepsilon a \;.\supset.\; u\text{‘}\iota x=\iota\, ux$

Si a, b es classe, et si u es signo de praefunctione de a ad b, tunc $u\text{‘}a$, lege « u de aliquo a », vel « u de omni a », vel « u des a », repraesenta classe de differente valore de expressione ux, ubi x sume omni valore in classe a; vel es classe de omni y reductibile ad forma $y=ux$, ubi x es aliquo individuo de classe a; id es, omni y tale que existe aliquo a et x tale que $y=ux$.

$u\text{‘}a$ es « imagine de classe a, in correspondentia u ».

Per definitiones opportuno, nos tace signo ‘ quando non existe periculo de ambiguitate. (Ex. P3·2).

❋ 3·0 $a,b\varepsilon\,\mathrm{Cls}\;.\;u\varepsilon\, a\text{ɟ}b \;.\supset.\; a\text{’}u=y\ni\exists\, a\frown x\ni(xu=y)$ Df

·1 $k\varepsilon\,\mathrm{Cls}\;.\supset.\;\mathrm{Cls}\text{’}k=y\ni\exists\,\mathrm{Cls}\frown x\ni(xk=y)=\mathrm{Cls}\frown y\ni(y\supset k)$ Dfp

Si u es signo de post-functione, nos inverte signo ‘ ; formula $a\text{’}u$, in lingua commune vale circa « des a to u ». Seque valore de $\mathrm{Cls}\text{’}k$, valores de xk, vel $x\frown k$, ubi x es classe arbitrario, conforme ad Df II §4 P 1·0.

·2 $u\varepsilon\,\mathrm{Cls}\text{’}\mathrm{N}_0\;.\;a\varepsilon\mathrm{N}_0 \;.\supset.\; u+a=u\text{’}+a=(x+a)|x\text{‘}u$ Dfp
» $a+u=(a+x)|x\text{‘}u$ Dfp
[= II§4P1·1·2]

Si u es classe de numero, et a es numero, $u+a$ indica valores de $x+a$ ubi varia x in classe u. Es definitione praecedente scripto sub forma plus breve, per symbolos de praesente §.

❋ 4·1 $a,b,c,d\varepsilon\,\mathrm{Cls}\;.\;u\varepsilon\,(b\mathrm{f}a)\mathrm{sim}\;.\;c\supset a\;.\;d\supset a\;.\supset.$
$u\text{‘}(c\frown d)=u\text{‘}c\frown u\text{‘}d$

·2 $u\varepsilon\,(b\mathrm{f}a)\mathrm{rcp} \;.=.\; u\varepsilon\,(b\mathrm{f}a)\mathrm{sim}\;.\;b\supset u\text{‘}a$ Dfp

§3 :

✻ $a,b,c,d \varepsilon \text{Cls} \,.\supset\!:$ ·0 $a{:}b = (x;y) \ni (x \varepsilon a \,.\, y \varepsilon b)$ Df

·01 $(x;y) \,\varepsilon\, (a{:}b) \,.=.\, x \varepsilon a \,.\, y \varepsilon b$ Dfp

·02 $a{:}b{:}c = (a{:}b){:}c = (x;y;z) \ni (x \varepsilon a \,.\, y \varepsilon b \,.\, z \varepsilon c)$ Df

Si a, b es classe, $a{:}b$ indica omni systema $x;y$ (considerato in I §2 P6), ubi x es elemento de classe a, et y es elemento de classe b. Illo es differente de $a;b$, systema formato per classe a et classe b. In defectu, in nostro linguas commune, de vocabulo cum valore proximo ad signo $:$, nos lege illo « virgula et puncto ».

Si $p_{x;y}$ es propositione cum duo variabile x et y, vel relatione inter duo variabile, et si x varia in classe a, et y in classe b, tunc $(x;y) \ni p_{x;y}$ es Cls'$(a{:}b)$.

·1 $(a{:}b) \supset (c{:}d) \,.=.\, a \supset c \,.\, b \supset d$

·2 $(a{:}b) = (c{:}d) \,.=.\, a = c \,.\, b = d$

·3 $(a{:}b) = (c{:}d) \,.=.\, (a;b) = (c;d)$

·4 $(a \cap c){:}(b \cap d) = (a{:}b) \cap (c{:}d)$

·5 $(a \cup c){:}b = (a{:}b) \cup (c{:}b) \;.\; a{:}(b \cup d) = (a{:}b) \cup (a{:}d)$ Distrib($:$, $\cup$)

·6 $(a \cup c){:}(b \cup d) = (a{:}b) \cup (c{:}b) \cup (a{:}d) \cup (c{:}d)$

·7 $\exists (a{:}b) \,.=.\, \exists a \,.\, \exists b$

·8 $(\iota x : \iota y) = \iota(x;y) \;.\; x;y = \imath(\iota x : \iota y)$

·9 $\iota x : \iota y = \iota z : \iota t \,.=.\, x;y = z;t$

{ ·1-·9 Padoa RdM. a.1900 t.6 p.120 }

✻ 2·1 $u,v \varepsilon \text{Cls'N}_0 \,.\supset.\, u+v = (y+z) | (y;z)\text{‘}(u{:}v)$ Dfp

[= II §4 P1·3]

§4 F (functione definito)

Vocabulo « functione » in Mathematica habe saepe valore de symbolo « f ». Resulta de suo definitione, que si u es transformatore des a in b, et si c es classe parte de a, u es etiam

transformatore des c in b (§1 P1·3). Per exemplo, si nos suppone operatione « mod » (valore absoluto) definito pro numero imaginario, illo resulta definito pro numero reale. Et post definitione de « mod » super numero imaginario, nos pote defini modulo de numeros complexo de ordine superiore ad 2, et modulo de substitutiones, et modulo de vectores, etc.

In Mathematica non existe uno definitione p. ex. de « multiplicatione », neque in Formulario existe aequalitate de forma:

$\times =$ (expressione composito per alios signo).

Sed existe definitione de multiplicatione inter duo N_0, post inter duo n (numero relativo), inter duo R (numero rationale), etc. In Formulario non es difficile de inveni plus que 30 definitiones de $x \times y$, cum hypothesi-s differente.

Ergo ad signo de functione non es ligato campo in quo functione es determinato, dicto campo de variabilitate, nam nos pote semper restringe et dilata illo.

In consequentia, nos non pote loque de aequalitate de duo functione; nam duo functione pote produce identicos resultatu in uno campo, et differentes in altero; duo functione arbitrario u et v habe semper campo de coincidentia, expresso per: $x \ni (ux = vx)$. Nos non pote loque de numero de functione que satisfac ad aliquo conditione. Nullo functione es invertibile; etc.

Quando mathematicos loque de aequalitate, numero, inversione de functione, vocabulo « functione » responde ad systema $(u;a)$, ubi u es functione, considerato in §1, et a es campo de variabilitate. Nos voca illo « functione definito », et indica per F, aut Funct, et nos pone per Df:

✻ 1·1 $a,b \varepsilon \mathrm{Cls} \,.\, u \varepsilon b\mathrm{f}a \,.\, x \varepsilon a \,.\supset.\, (u;a)x = ux$ Df

Si a et b es classe, et u es b functione des a, et x es a, tunc per $(u;a)x$ nos indica valore ux.

·2 $a,b \varepsilon \mathrm{Cls} \,.\, u \varepsilon b\mathrm{f}a \,.\supset.\, \mathrm{Variab}(u;a) = a$

Variabilitate de $(u;a)$ es classe a de valores de variabile.

·3 $a,b \varepsilon \mathrm{Cls} \,.\supset.\, b\mathrm{F}a = v \ni \{ \exists\, b\mathrm{f}a \frown u \ni [v = (u;a)] \}$ Df

Si a et b es classe, nos voca « functione definito des a » omni ente v reducibile ad forma $v = (u;a)$, ubi u es aliquo b functione des a, ut considerato in §1.

·4 Funct $=$ $v\ni\ \exists(a;b)\ni(a,b\varepsilon\,\mathrm{Cls}\,.\,v\,\varepsilon\,b\mathrm{F}a)$ Df

·5 $u\varepsilon$ Funct $.\supset.$ Variab$u$ $= \iota\,\mathrm{Cls}\frown a\ni[\exists\,\mathrm{Cls}\frown b\ni(u\varepsilon\,b\mathrm{F}a)]$ Dfp

·6 $u,v\varepsilon$ Funct $.\supset\therefore u=v\ .=:$
Variab$u$ $=$ Variab$v$ $: x\varepsilon$ Variab$u$ $.\supset_x.\ ux=vx$ Df

Duo functione definito u et v es dicto aequale inter se, quando campo de variabilitate de u coicide cum campo de variabilitate de v; et si pro omni x que pertine ad campo commune de variabilitate semper es $ux=vx$.

·7 $a,b\varepsilon\,\mathrm{Cls}\,.\supset.\ b\mathrm{F}a\supset b\mathrm{f}a$ [P1·1 . §f 1·01 .⊃P]

De definitione ·1 et de definitione de f resulta que omni functione definito es functione.

✻ 2. $a,b\varepsilon\,\mathrm{Cls}\,.\supset::$

·0 $b\mathrm{F}a=\mathrm{Cls}'(b{:}a)\frown u\ni\{x\varepsilon a\,.\supset_x.\ \exists\ y\ni[(y;x)\varepsilon u]:$
$x\varepsilon a\,.\,(y;x)\varepsilon u.\,(z;x)\varepsilon u\,.\supset_{x,y,z}.\ y=z\,\}$ Dfp

·01 $b\mathrm{F}a=\mathrm{Cls}'(b{:}a)\frown u\ni[x\varepsilon a\,.\supset_x.\ \mathrm{num}\ y\ni[(y;x)\varepsilon u]=1\}$ Dfp

·1 $u\varepsilon\,b\mathrm{F}a\,.\,x\varepsilon a\,.\supset.\ ux=\iota\ y\ni[(y;x)\varepsilon u]$ Df

Si a,b es classe, $b\mathrm{F}a$ indica omni relatione inter b et a, tale que ad omni a responde uno solo b. Definitione possibile de signo F per signo : sine usu de signo f, que occurre in P 1.

✻ 3·0 $u\varepsilon$ Funct sim $.\supset.\ u^{-1}=$
$\iota\,(\mathrm{Variab}u)\mathrm{F}(u\text{‘}\mathrm{Variab}u)\frown v\ni[x\varepsilon\,\mathrm{Variab}u\,.\supset_x.\ vu=x]$ Df

·1 $u\varepsilon$ Funct sim $.\supset.\ (u^{-1})^{-1}=u$

Si u es functione definito simile, tunc per u^{-1}, lege: u inverso, nos indica illo functione definito, que ad valore que assume ux, quando x varia in campo de variabilitate de u, fac corresponde valores de variabilitate de u, et tale que successione de operatione u et de suo inverso produce identitate.

·2 $a,b,c\varepsilon\,\mathrm{Cls}\,.\,u\varepsilon\,(b\mathrm{f}a)\mathrm{sim}\,.\,v\varepsilon\,(c\mathrm{F}b)\mathrm{sim}\,.\supset.\ (vu)^{-1}=v^{-1}u^{-1}$

·3 $a\varepsilon\,\mathrm{Cls}\,.\,u,v\varepsilon\,(a\mathrm{F}a)\mathrm{sim}\,.\,uv=vu\,.\supset.$
$u^{-1}v=v\,u^{-1}\ .\ u^{-1}v^{-1}=v^{-1}u^{-1}$

·4 $a\varepsilon\mathrm{N}_0\,.\supset.\ (+a,\,\mathrm{N}_0)^{-1}=(-a,\,a+\mathrm{N}_0)$

Inverso de operatione $+a$, in campo de numero, es operatione $-a$, in campo de numero superiore aut aequale ad a.

·5 $a\varepsilon\mathrm{N}_1\,.\supset.\ (\times a,\,\mathrm{N}_1)^{-1}=(/a,\,a\times\mathrm{N}_1)$

num ✻ 4·0 $m,n \varepsilon N_1 .\supset. \text{num}(1\cdots m \text{ F } 1\cdots n) = m^n$

·1 $m \varepsilon N_1 .\supset. \text{num}(1\cdots m \text{ F } 1\cdots m)\text{rcp} = m!$

{ Luca Paciuolo a.1494 fol.43 v. }

·2 $n \varepsilon N_1 . m \varepsilon n+N_0 .\supset. \text{num}(1\cdots m \text{ F } 1\cdots n)\text{sim} = m!/(m-n)!$

$1\cdots m$ F $1\cdots n$ = « variationes cum repetitione de objectos $1\cdots m$ ad n ».
$(1\cdots m$ F $1\cdots n)$sim = « variationes simplo ».
Si $m=n$, illos es vocato « permutationes ».

§5 $\bigcap$ $\bigcup$

✻ 1. $u,v \varepsilon$ Cls'Cls $.\supset.$

·0 $\bigcap u = y \ni \{x \varepsilon u .\supset_x. y \varepsilon x\} . \bigcup u = x \ni \exists \{u \frown y \ni (x \varepsilon y)\}$ Df

Si u es classe de classe, tunc $\bigcap u$, lege « parte commune des u » vel « producto logico des u » indica omni objecto pertinente ad omni classe de systema u.

$\bigcup u$, lege « universo des u » vel « summa des u » indica objectos pertinente ad aliquo classe u.

P·1 es analogo ad syllogismo.

·1 $x \varepsilon a . a \varepsilon u .\supset. x \varepsilon \bigcap u$

·2 $\bigcup(u \smile v) = \bigcap u \smile \bigcap v$ Distrib ($\bigcup$, $\smile$)

·3 $\bigcap(u \smile v) = \bigcap u \frown \bigcap v$

·4 $u \supset v .\supset. \bigcup u \supset \bigcup v . \bigcap v \supset \bigcap u$

·5 $\bigcup(u \frown v) \supset \bigcup u \frown \bigcup v$

·6 $\bigcap u \smile \bigcap v \supset \bigcap(u \frown v)$

{ ·2-·6 C. Burali-Forti, MA. t.48 }

·7 $u,v \varepsilon$ Cls $.\supset. \bigcup\{[(x;y)|x`u]|y`v = (u\text{:}v)$

Nos considera systema $x;y$. Nos varia x in classe u; $(x;y) \mid x`u$ repraesenta classe de systema. Nos varia $y$ in classe $v$; $[(x;y) \mid x`u] \mid y`v$ repraesenta classe de classe de systema. Suo universo vale $(u\text{:}v)$.

· §6 n

✻ 1·0 $n = +N_0 \cup -N_0$ Df

Expressiones, ut +5 et −3, jam ad nos occurre in III §1 P2·3·6. Illo es dicto:

$+N_0$ = numero positivo

$-N_0$ = numero negativo.

Vocabulo « positivo » et « negativo » habe valore de « additivo » et « subtractivo ».

Phrasi « numero negativo » habe forma grammaticale de « numero pari » sed valore differente. « Numero pari » es classe de numero, et inter vocabulo « numero » et « pari » es tacito signo logico $\cap$. « Numeros negativo » non es numero, sed systema composito per numero et signo —.

n, lege « numero relativo » indica « numeros positivo et negativo ». Vocabulo « et » vale $\cup$.

·1 $a,b\varepsilon N_0 . \supset . a+(+b) = a+b$ Df

·2 $b\varepsilon N_0 . a\varepsilon b+N_0 . \supset . a+(-b) = a-b$ Df

Prop. ·1·2·7 exprime conventiones commune.

·3 $x,y\varepsilon n . \supset \therefore$

$x=y . =: u\varepsilon N_0 . u+x, u+y\, \varepsilon N_0 . \supset_u . u+x = u+y$ Df

« Duo numero relativo x et y es aequale inter se, si pro omni numero u, super que nos pote fac operationes $+x$ et $+y$, semper fi $u+x = u+y$ ».

·4 $a,b\varepsilon N_0 . \supset : +a = +b . = . a=b$

[$a,b,u\varepsilon N_0 . \S+ 4{\cdot}6 . \supset : u+a = u+b . = . a=b$ (1)

$a,b\varepsilon N_0 . (1) . \text{Export} . \supset \therefore u\varepsilon N_0 . \supset_u . u+a = u+b := . a=b$ (2)

(2) . Df·3 . $\supset$. P]

·5 $a,b\varepsilon N_0 . \supset : -a = -b . = . a=b$

[$a,b,u\varepsilon N_0 . u-a, u-b\varepsilon N_0 . \supset : u-a = u-b . = . u+a+a+b =$
$u-b+a+b . = . u+b=u+a . = . a=b$]

·6 $a,b\varepsilon N_0 . \supset : +a = -b . = . a=0 . b=0$

[$a,b,u\varepsilon N_0 . u-b\, \varepsilon N_0 . \supset : u+a = u-b . = . u+a+b = u .$
$= . a+b = 0 . = . a=0 . b=0$]

·7 $a\varepsilon N_0 . \supset . a = +a$ Df

$*$ 2·0 $x,y\varepsilon n .\supset. x+y =$
$\iota\, n \frown z\ni[u\varepsilon N_0 . u+x, u+x+y\ \varepsilon N_0 .\supset_u. u+x+y=u+z]$ Df

« Si x et y es numero relativo, $x+y$ indica illo numero relativo z tale que, si nos sume numero (absoluto) arbitrario u, super que nos pote fac operatione $+x$, et super resultatu operatione $+y$, semper fi:

$$u+x+y=u+z \text{ ».}$$

·1 $a,b\varepsilon N_0 .\supset. +a+b=+(a+b)$ [§+4·2 $\supset$ P]

·2 » $-a-b=-(a+b)$ [§−1·5 $\supset$ P]

·3 $b\varepsilon N_0 . a\varepsilon\, b+N_0 .\supset. +a-b=-b+a=+(a-b)$
[§−1·6·7 $\supset$ P]

$a,b,c\varepsilon n .\supset.$ ·4 $a+b\ \varepsilon n$ ·5 $a+0=a$

·6 $a+b=b+a$ ·7 $a+(b+c)=(a+b)+c$

·8 $a+c=b+c .=. a=b$

$*$ 3·0 $x\varepsilon n .\supset. -x=\iota\, n\frown y\ni(x+y=0)$ Df

·1 $a\varepsilon N_0 .\supset. -(+a)=-a \ :\ -(-a)=+a$

$a,b\varepsilon n .\supset.$ ·2 $-a\ \varepsilon n$ ·3 $-(-a)=a$

·4 $a-b=a+(-b)$ Df ·5 $-(a+b)=-a-b$

·6 $x\varepsilon n . a+x=b .=. x=b-a$

$*$ 4. $a,b\varepsilon N_0 . x,y\varepsilon n .\supset.$

·0 $xa=\iota\, n\frown z\ni[u\varepsilon N_0 . u+x\ \varepsilon N_0 .\supset_u. (u+x)a=ua+z)]$ Df

·1 $ax=$ » » $a(u+x)=au+z)]$ Df

·2 $xy=$ » » $(u+x)y=ux+z)]$ Df

·3 $ax=xa$ Dfp

·4 $(+b)\times a=+ba$

[Hp . Df·0 $.\supset. (+b)\times a=\iota n\frown z\ni[u\varepsilon N_0 .\supset_u .(u+b)\times a=u\times a+z]$
Distrib$(\times,+)$ $.\supset.$ » $=$ » $u\times a+b\times a=u\times a+z]$
Oper $-u\times a .\supset.$ » $=$ » $+b\times a=z]$
Distrib$(\ni,\frown)$ $.\supset.$ » $=\iota z\ni[z\varepsilon n . z=+b\times a]$
Df n $.\supset.$ » $=\iota z\ni(z=+b\times a)$
Df ι $.\supset.$ » $=\iota z\ni[z\varepsilon\iota +b\times a]$
Df $\ni$ $.\supset.$ » $=\iota\iota(+b\times a)$
Df ι $.\supset.$ » $=+b\times a$]

·5 $(-b)a = -ba$

·6 $(+a)\times(+b) = +(a\times b)$. $(-a)\times(+b) = -(a\times b)$.
$(+a)\times(-b) = -(a\times b)$. $(-a)\times(-b) = +(a\times b)$

{ DIOPHANTO I 9: « *Λεῖψις ἐπὶ λεῖψιν πολλαπλασιασθεῖσα ποιεῖ ὕπαρξιν. Λεῖψις δὲ ἐπὶ ὕπαρξιν, ποιεῖ λεῖψιν.* » }

* 5. $a,b,c\,\varepsilon\,\mathrm{n} \,.\supset.$ ·1 $a\times b\,\varepsilon\,\mathrm{n}$ ·2 $0\times a = 0$. $1\times a = a$

·3 $a(b+c) = ab+ac$ ·4 $ab = ba$

·5 $a(bc) = (ab)c = abc$

·6 $ab = 0 \,.=.\, a=0 \,.\cup.\, b=0$

·7 $ac = bc \,.\, c \mathrel{-\!\!=} 0 \,.\supset.\, a=b$

* 6. $a,b,c,d\,\varepsilon\,\mathrm{n} \,.\supset.$ §× P2

·1 $a(b-c)+b(c-a)+c(a-b) = 0$

·2 $(a-b)(c-d)+(b-c)(a-d)+(c-a)(b-d) = 0$
[$(a-d,\ b-d,\ c-d) \mid (a,b,c)$ P·1 $.\supset.$ P]

·3 $ab(a-b)+bc(b-c)+ca(c-a)+(a-b)(b-c)(c-a) = 0$

·4 $(a+b)(b+c)(c-a)+(b+c)(c+a)(a-b)+(c+a)(a+b)(b-c) =$
$-(a-b)(b-c)(c-a)$

·5 $(a-b)(b+c-a)(a-b+c)+(b-c)(a-b+c)(a+b-c)+$
$(c-a)(a+b-c)(b+c-a) = -4(a-b)(b-c)(c-a)$
[$[3a-b-c,\ 3b-c-a,\ 3c-a-b] \mid (a,b,c)$ P·3 $.\supset.$ P]

·6 $a(b+c)(b+c-a)+b(c+a)(c+a-b)+c(a+b)(a+b-c) = 6abc$

·7 $a(b-c)(b+c-a)+b(c-a)(c+a-b)+c(a-b)(a+b-c) =$
$2(a-b)(b-c)(c-a)$

* 7·0 $ab(-a+b+c)+bc(a-b+c)+ca(a+b-c)-3abc =$
$(a-b)(b-c)(c-a)$

·1 $ab(a-b+c)+bc(a+b-c)+ca(-a+b+c)-3abc =$
$-(a-b)(b-c)(c-a)$

·2 $ab(a+b-c)+bc(-a+b+c)+ca(a-b+c)+5abc =$
$(a+b)(b+c)(c+a)$

·3 $a(a-b+c)(a+b-c)+b(a+b-c)(-a+b+c)+c(-a+b+c)(a-$
$b+c) = 4abc-(-a+b+c)(a-b+c)(a+b-c)$

·4 $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) =$
$abc-(-a+b+c)(a-b+c)(a+b-c)$

·5 $(a+b)(b-c)(c-a)+(b+c)(c-a)(a-b)+(c+a)(a-b)(b-c) =$
$8abc-(a+b)(b+c)(c+a)$

·6 $(a+b+c)\{(-a+b+c)(a-b+c)+(a-b+c)(a+b-c)+(a+b-c)(-a+b+c)\} = 8abc+(-a+b+c)(a-b+c)(a+b-c)$

·7 $(aa'+bb')(cc'+dd')+(ab'-a'b)(cd'-c'd) = (ac+bd)(a'c'+b'd')+(ad-bc)(a'd'-b'c')$

n ↾

* 10. $a,b\varepsilon$n . $m,n\varepsilon N_0$.⊃. ·0-·02 = §↾P1·0-·02 ·1 a^m εn

·2-·4 = §↾P1·2-·4 ·5 $(-a)^2 = a^2$

·6 $(-a)$↾$(2m) = a$↾$(2m)$ ·7 $(-a)$↾$(2m+1) = -a$↾$(2m+1)$

§↾ P2 . P3.

* 11. $a,b\varepsilon$n .⊃.

·0 $(a-b)^2 = a^2-2ab+b^2$

·1 $(a+b)(a-b) = a^2-b^2$

·2 $(a+b)^2+(a-b)^2 = 2(a^2+b^2)$ { Euclide ii P9 }

·3 $(a+b)^2-(a-b)^2 = 4ab$ { » » P5 }

* 12. $a,b,c\varepsilon$n .⊃.

·1 $(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2-ab-bc-ca)$
$= 2[(a-b)(a-c)+(b-a)(b-c)+(c-a)(c-b)]$
[$(b-c,c-a,a-b)$ | (a,b,c) §↾ 3·1 .⊃. P]

·2 $(a-b)^2+(b-c)^2+(c-a)^2+(a+b+c)^2 = 3(a^2+b^2+c^2)$

·3 $(a+b+c)^2+(a+b-c)^2+(a-b+c)^2+(-a+b+c)^2 = 4(a^2+b^2+c^2)$

·4 $(a+2b)(b+c-a)+(b+2c)(a-b+c)+(c+2a)(a+b-c)=(a+b+c)^2$

·5 $3(a+b+c)^2 = (a+b-c)^2+(b+c-a)^2+(c+a-b)^2+8(ab+ac+bc)$

·6 $(-a+2b+2c)^2+(2a-b+2c)^2+(b-c)^2 = 9(a^2+b^2+c^2)$ $+2a2$
[$(2b+2c-a,\ 2c+2a-b,\ 2a+2b-c)$ | (a,b,c) P·2 .⊃. P]
{ Euler a.1750 CorrM. t.1 p.515 }

❋ 13. $a,b,c,d\varepsilon n\ .\supset.$

·1 $(a+b+c+d)^2+(a+b-c-d)^2+(a+c-b-d)^2+(a+d-b-c)^2 = (-a+b+c+d)^2+(a-b+c+d)^2+(a+b-c+d)^2+(a+b+c-d)^2 = 4(a^2+b^2+c^2+d^2)$ } LEGENDRE a.1816 p.8 }

❋ 14. $a,b\varepsilon n\ .\supset.$

·1 $(a^2+ab+b^2)(a-b) = a^3-b^3$

·2 $2(a^3+b^3)-(a+b)(a^2+b^2) = (a-b)^2(a+b)$

·3 $(a+b)^3 = a(a-3b)^2 + b(b-3a)^2$

❋ 15. $a,b,c\varepsilon n\ .\supset.$

·1 $a^2(b-c)+b^2(c-a)+c^2(a-b) = (a-b)(a-c)(b-c)$

·2 $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2) = (b-a)(c-a)(c-b)$

·3 $(a+b)^2(a-b)+(b+c)^2(b-c)+(c+a)^2(c-a) = -(a-b)(b-c)(c-a)$
[$(b+c, c+a, a+b) | (a,b,c)$ P·1 $.\supset.$ P]

·4 $a^2(b+c)+b^2(c+a)+c^2(a+b)-6abc = a(b-c)^2+b(c-a)^2+c(a-b)^2$

·5 $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3 = 24abc$
[$(b+c-a, c+a-b, a+b-c) | (a,b,c)$ §↑ 3·3 $.\supset.$ P]

·6 $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$

·7 $2(a^3+b^3+c^3-3abc) = (a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

·8 $3(a^3+b^3+c^3)-(a+b+c)(a^2+b^2+c^2) = (a-b)^2(a+b)+(b-c)^2(b+c)+(c-a)^2(c+a)$

·9 $2[(a+b+c)^3-27abc] = (a-b)^2(a+b+7c)+(b-c)^2(7a+b+c)+(c-a)^2(a+7b+c)$

❋ 16. $a,b,c,d\varepsilon n\ .\supset.$

·1 $(a+b-c)(a-b+c)(-a+b+c) = a^2(b+c-a)+b^2(a+c-b)+c^2(a+b-c)-2abc$

·2 $(a-b)^3+(b-c)^3+(c-a)^3 = 3(a-b)(b-c)(c-a)$
[$(b-c, c-a, a-b) | (a,b,c)$ §↑ 3·3 $.\supset.$ P]

·3 $a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc = (a+b)(b+c)(c+a)$

·4 $(a+b+c)^3+(-a+b+c)^3+(a-b+c)^3+(a+b-c)^3 = 8(a^3+b^3+c^3)+6(-a+b+c)(a-b+c)(a+b-c)$

·5 $a^3+b^3+c^3+d^3-3(abc+abd+acd+bcd) = (a+b+c+d)(a^2+b^2+c^2+d^2-ab-bc-ca-ad-bd-cd)$

·6 $(-a+b+c)(a-b+c)^2+(a-b+c)(a+b-c)^2+(a+b-c)$
$(-a+b+c)^2+(-a+b+c)(a-b+c)(a+b-c) = 4\{abc-$
$(a-b)(b-c)(c-a)\}$

·7 $(-a+b+c)^2(a-b+c)+(a-b+c)^2(a+b-c)+(a+b-c)^2$
$(-a+b+c)+(-a+b+c)(a-b+c)(a+b-c) = 4\{abc+$
$(a-b)(b-c)(c-a)\}$

* 17. $a,b\varepsilon n$.⊃.

·1 $(a^3+a^2b+ab^2+b^3)(a-b) = a^4-b^4$

·2 $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2 = 4ab(a^2+b^2)$

·3 $a(a-2b)^3-b(b-2a)^3 = (a-b)(a+b)^3$

·4 $(a^2+b^2)^2 = (a^2-b^2)^2+(2ab)^2$ { EUCLIDE X lemma P29 }

·5 $a^4+4b^4 = (a^2+2ab+2b^2)(a^2-2ab+2b^2)$
{ EULER a.1742 CorrM. t.1 p.145 }

·6 $a^4+a^2b^2+b^4 = (a^2+ab+b^2)(a^2-ab+b^2)$

·7 $3(a^4+a^2b^2+b^4)-(a^2+ab+b^2)^2 = 2(a-b)^2(a^2+ab+b^2)$

·8 $2(a^4+a^2b^2+b^4)-3ab(a^2+b^2) = (a-b)^2(2a^2+ab+2b^2)$

* 18. $a,b\varepsilon n$.⊃.

·1 $a^4+b^4-ab(a^2+b^2) = (a-b)^2(a^2+ab+b^2)$

·2 $(a+b)^4-(a-b)^4 = 8ab(a^2+b^2)$ [P11·2·3 .⊃. P]
{ CAUCHY Exerc. a.1841 t.2 p.144 }

·3 $(a+b)^4 = (a^2-6ab+b^2)^2 + 16ab(a-b)^2$

* 19. $a,b,c\varepsilon n$.⊃.

·1 $a(b-c)^3+b(c-a)^3+c(a-b)^3 = (a-b)(b-c)(c-a)(a+b+c)$
$= a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$

·2 $a^3(b-c)+b^3(c-a)+c^3(a-b) = (a-b)(b-c)(a-c)(a+b+c)$

·3 $(a+b-2c)(a-b)^3+(b+c-2a)(b-c)^3+(c+a-2b)(c-a)^3 = 0$
[$(b+c-2a, c+a-2b, a+b-2c)|(a,b,c)$ P·1 .⊃. P]

·4 $(a+b+c)(a+b-c)(a-b+c)(-a+b+c) =$
$2(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4) = (a^2+b^2+c^2)^2-2(a^4+b^4+c^4)$

·5 $[(a-b)^2+(b-c)^2+(c-a)^2]^2 = 2[(a-b)^4+(b-c)^4+(c-a)^4]$

·6 $a^4+b^4+c^4 = (a+b+c)(a-b-c)(b-c-a)(c-a-b)+$
$2(a^2b^2+b^2c^2+c^2a^2)$ [P·4 ⊃ P]

·7 $(a+b+c)(a^3+b^3+c^3)-(a^2+b^2+c^2)^2 =$
$ab(a-b)^2+ac(a-c)^2+bc(b-c)^2$

·8 $(a+b)(a-b)^3+(b+c)(b-c)^3+(c+a)(c-a)^3 =$
$2(a+b+c)(a-b)(b-c)(c-a)$ [$(b+c, c+a, a+b) | (a,b,c)$ P·1 .⊃. P]

✻ 20. $a,b,c\varepsilon$n .⊃.

·1 $ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2) = -(a+b+c)(a-b)(b-c)(c-a)$

·2 $ab(a+b)^2+bc(b+c)^2+ca(c+a)^2 =$
$(a+b+c)[(a+b)(b+c)(c+a)-4abc]$

·3 $a(b-c)(b+c-a)^2+b(c-a)(c+a-b)^2+c(a-b)(a+b-c)^2 = 0$

·4 $(a+b+c)^4 + (-a+b+c)^4 + (a-b-c)^4 + (a+b-c)^4 =$
$4(a^4+b^4+c^4) + 24(b^2c^2+c^2a^2+a^2b^2)$

·5 $a^2(a-b+c)(a+b-c)+b^2(a+b-c)(-a+b+c)+c^2(-a+b+c)(a-b+c = (a+b+c)\{2abc-(-a+b+c)(a-b+c)(a+b-c)\}$

·6 $(a+b+c)^2\{(-a+b+c)^2+(a-b+c)^2+(a+b-c)^2\}+(-a+b+c)^2(a-b+c)^2+(a-b+c)^2(a+b-c)^2+(a+b-c)^2(-a+b+c)^2 = 6(a^4+b^4+c^4)+4(a^2b^2+b^2c^2+c^2a^2)$

✻ 21. $a,b,c,d\varepsilon$n .⊃.

·1 $(a^2+b^2)(c^2+d^2) = (ac+bd)^2+(ad-bc)^2 = (ac-bd)^2+(ad+bc)^2$
{ Diophanto III P22 }

·2 $(a^2-b^2)(c^2-d^2) = (ac-bd)^2-(ad-bc)^2$

·3 $(a^2+b^2+c^2+d^2)^2 = (a^2+b^2-c^2-d^2)^2+(2ac\pm 2bd)^2+(2bc\mp 2ad)^2$
{ P. Tannery IdM. a.1898 p.282 }

✻ 22. $a,b,c,d,e,f,g,h,a',b',c',d',e',f',g',h',p,q\varepsilon$n .⊃.

·1 $(a^2+cb^2)(a'^2+cb'^2) = (aa'+cbb')^2+c(ab'-a'b)^2$

·2 » » $= (aa'-cbb')^2+c(ab'+a'b)^2$

·3 $(a^2+b^2+c^2)(a'^2+b'^2+c'^2)-(aa'+bb'+cc')^2 =$
$(ab'-a'b)^2+(ac'-a'c)^2+(bc'-b'c)^2$

·4 $(a^2+b^2+c^2+d^2)(a'^2+b'^2+c'^2+d'^2)=(aa'+bb'+cc'+dd')^2+(ab'-a'b+cd'-c'd)^2+(ac'-a'c-bd'+b'd)^2+(ad'-a'd+bc'-b'c)^2$

{ EULER PetrNC. t.5 a.1754 p.54 }

·5 $(a^2-pb^2-qc^2+pqd^2)(a'^2-pb'^2-qc'^2+pqd'^2)=$
$(aa'+pbb'\pm q(cc'+pdd'))^2-p(ab'+a'b\pm q(cd'+c'd))^2$
$-q(ac'-pbd'\pm(a'c-pb'd))^2+pq(bc'-ad'\pm(a'd-b'c))^2$

{ LAGRANGE a.1770 t.3 p.201 }

·6 $(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)(a'^2+b'^2+c'^2+d'^2+e'^2+f'^2+g'^2+h'^2)$
$=(aa'+bb'+cc'+dd'+ee'+ff'+gg'+hh')^2$
$+(ab'-ba'+cd'-dc'+ef'-fe'+gh'-hg')^2$
$+(ac'-bd'-ca'+db'+eg'-fh'-ge'+hf')^2$
$+(ad'+bc'-cb'-da'+eh'+fg'-gf'-he')^2$
$+(ae'+bf'+cg'+dh'+ea'+fb'+gc'+hd')^2$
$+(af'-be'+ch'-dg'+eb'-fa'+gd'-hc')^2$
$+(ag'-bh'-ce'+df'+ec'-fd'-ga'+hb')^2$
$+(ah'+bg'-cf'-de'+ed'+fc'-gb'-ha')^2$

{ DEGEN, Mém. de l'Acad. de St. Pétersbourg, a.1822 t.8 p.4 }

·7 $(ab)^4+[(a+b)b]^4+[a(a+b)(a^2+ab+2b^2)]^2=(a^2+ab+b^2)^4$

❋ 23. $a,b,c\varepsilon\mathrm{n}$.⊃.

·1 $a^5+b^5-ab(a^3+b^3)=(a+b)(a-b)^2(a^2+b^2)$

·2 $(a+b)^5=a(a^2-10ab+5b^2)^2+b(5a^2-10ab+b^2)^2$

·3 $ab(a^3-b^3)+bc(b^3-c^3)+ca(c^3-a^3)+$
$(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)=0$

·4 $a^3(b^2-c^2)+b^3(c^2a^2-)+c^3(a^2-b^2)=(a-b)(a-c)(b-c)(ab+ac+bc)$
[$(bc, ca, ab)\,|\,(a,b,c)$ P20·1 .⊃. P]

·5 $(a+b+c)^5=(a+b-c)^5+(b+c-a)^5+(c+a-b)^5+80abc(a^2+b^2+c^2)$
{ CAUCHY Exercices a.1841 t.2 p.144 }
[$(b+c-a, c+a-b, a+b-c)\,|\,(a,b,c)$ §↑3·8 .⊃. P]

·6 $2[(a-b)^5+(b-c)^5+(c-a)^5]=$
$5(a-b)(b-c)(c-a)[(a-b)^2+(b-c)^2+(c-a)^2]$
[$(b-c, c-a, a-b)\,|\,(a,b,c)$ P·5 .⊃. P]

❋ 24. $a,b,c\varepsilon\mathrm{n}$.⊃.

·0 $(a+b)^6=(a-b)^2(a^2-14ab+b^2)^2+4ab(3a-b)^2(3b-a)^2$

·1 $(a^3+a^2b+ab^2+b^3)^2+(a^3-a^2b+ab^2-b^3)^2=2(a^2+b^2)^3$

·2 $(a^3+a^2b-ab^2-b^3)^2-(a^3-a^2b-ab^2+b^3)^2=4ab(a^2-b^2)^2$

·3 $(a^3+a^2b-ab^2+b^3)^2-(a^3-a^2b-ab^2-b^3)^2=4ab(a^2+b^2)(a^2-b^2)$

·4 $4(a^2+ab+b^2)^3-27(a^2b+ab^2)^2=(a-b)^2[2(a^2+ab+b^2)+3ab]^2$

} LAGRANGE a.1777, Œuvres t.4 p.346 {

·5 $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3=3abc(a-b)(b-c)(c-a)$

[$(ab-ac, bc-ba, ca-cb)|(a,b,c)$ P14·6 .$\supset$. P]

·6 $(a+b)^2(b+c)^2(c-a)^2+(b+c)^2(c+a)^2(a-b)^2+(c+a)^2(a+b)^2(b-c)^2$
$+(a-b)^2(b-c)^2(c-a)^2=4(a^2+b^2)(b^2+c^2)(c^2+a^2)-32a^2b^2c^2$

·7 $(a+b)^2(b-c)^2(c-a)^2+(b+c)^2(c-a)^2(a-b)^2+(c+a)^2(a-b)^2(b-c)^2$
$+(a+b)^2(b+c)^2(c+a)^2=4(a^2+b^2)(b^2+c^2)(c^2+a^2)+32a^2b^2c^2$

·8 $a(b-c)(a-b+c)^2(a+b-c)^2+b(c-a)(a+b-c)^2(-a+b+c)^2+$
$c(a-b)(-a+b+c)^2(a-b+c)^2=-16abc(a-b)(b-c)(c-a)$

·9 $(a+b+c)^2\{(-a+b+c)^2(a-b+c)^2+(a-b+c)^2(a+b-c)^2+(a+$
$b-c)^2(-a+b+c)^2\}+(-a+b+c)^2(a-b+c)^2(a+b-c)^2=$
$64a^2b^2c^2-4(a^2+b^2+c^2)(a+b+c)-a+b+c)(a-$
$b+c)(a+b-c)$

❋ 25. $a,b,c \varepsilon n$.$\supset$.

·1 $a^7+b^7-ab(a^5+b^5)=(a+b)(a-b)^2(a^2+ab+b^2)(a^2-ab+b^2)$

·2 $(a^6+b^6)(a+b)-2ab(a^5+b^5)=(a-b)^2(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$
[P·1 .$\supset$. P]

·3 $(a^6+b^6)(a+b)-(a^2+b^2)(a^5+b^5)=(a-b)^2(a+b)(a^2+b^2)ab$

·4 $(a+b)^7=a(a^3-21a^2b+35ab^2-7b^3)^2+b(7a^3-35a^2b+21ab^2-b^3)^2$

·5 $(a+b+c)^7-(a+b-c)^7-(a-b+c)^7-(-a+b+c)^7=$
$56abc[3(a^4+b^4+c^4)+10(a^2b^2+b^2c^2+c^2a^2)]$

} LAMÉ a.1840 JdM. t.5 p.197 {

❋ 26. $a,b \varepsilon n$.$\supset$.

·1 $(a+b)^8=(a^4-28a^3b+70a^2b^2-28ab^3+b^4)^2+$
$64ab(a-b)^2(a^2-6ab+b^2)^2$

·2 $(a^4+a^3b-a^2b^2+ab^3+b^4)^2-(a^4-a^3b-a^2b^2-ab^3+b^4)^2=$
$4ab(a^2+b^2)(a^4-a^2b^2+b^4)$

·3 $(a^4+a^3b-a^2b^2-ab^3+b^4)^2-(a^4-a^3b-a^2b^2+ab^3+b^4)^2 =$
$4ab(a^2-b^2)(a^4-a^2b^2+b^4)$

·4 $a^9+b^9-ab(a^7+b^7)=(a+b)(a-b)^2(a^2+b^2)(a^4+b^4)$

·5 $(a+b)^9 = a(a^4-36a^3b+126a^2b^2-84ab^3+9b^4)^2 +$
$b(9a^4-84a^3b+126a^2b^2-36ab^3+b^4)^2$

n $>$

✻ 30. $x,y\varepsilon$n $.\supset$: ·0 $x>y .=. y<x .=. x\varepsilon\, y+N_1$ Df

·01 $x>y .=: u\varepsilon N_0 . u+x, u+y\, \varepsilon N_0 .\supset_u . u+x>u+y$ Dfp

·02 $a,b\varepsilon N_1 .\supset: +a>+b .=. a>b : +a>-b : -a>-b .=. a<b$

$a,b,c,d\varepsilon$n $.\supset.$

·1 $a>b . b>c .\supset. a>c$

·2 $a>b .=. a+c > b+c$

·3 $a>b . c>d .\supset. a+c > b+d$

·4 $a>b .\cup. a=b .\cup. a<b$

·5 $a>b .=. -a<-b$

✻ 31. $a,b\varepsilon$n $. c\varepsilon N_1 .\supset: a>b .=. ac>bc$

✻ 32. $a,b,c\varepsilon N_1 . -(a=b=c) .\supset.$

·1 $(a+b-c)^2+(a+c-b)^2+(b+c-a)^2 > ab+bc+ca$

·2 $abc>(a+b-c)(a+c-b)(b+c-a)$ ¦ Bertrand a.1855 p.142 ¦
[$(b+c-a, a+c-b, a+b-c)|(a, b, c)$ §N_1 12·25 $.\supset.$ P]

·3 $(a-b)^2(a+b-c)+(b-c)^2(b+c-a)+(c-a)^2(c+a-b)>0$

✻ 33·0 $a\varepsilon\,(2N_0+1)-(5N_0) .\supset. a^4-1\, \varepsilon\, 80N_0$

$a\varepsilon$n $.\supset.$ ·1 $(2a-1)^2-1\ \varepsilon$ 8n ·2 $a(a^2-1)\,\varepsilon$ 6n

·3 $a^2(a^2-1)\ \varepsilon$ 12n ¦ Leibniz MathS. t.7 p.101 ¦

·4 $a(a^2-1)(a^2-4)\,\varepsilon$ 120n

·5 $a^2(a^4-1)\ \varepsilon$ 60n

·6 $a^2(a^2-2)(a^4-1)(a^4-16)\ \varepsilon$ 25200n ·7 $a(a^{12}-1)\,\varepsilon$ 2730n

·8 $a^2(a^4-1)(a^4-9)(a^4-16)\ \varepsilon$ 46800n

❋ 34. $a,b\varepsilon n . \supset .$

·1 $ab(a^2-b^2) \varepsilon 6n$ ·2 $ab(a^2+b^2)(a^2-b^2) \varepsilon 30n$

$a\varepsilon 2n+1 . \supset .$ ·3 $a(a^4-1) \varepsilon 240n$

·4 $a^2(a^2-1)(a^4-1) \varepsilon 5760n$

·5 $a^2(a^2-1)(a^6-1) \varepsilon 4032n$ ·6 $a^2(a^4-1)(a^8-1) \varepsilon 115200n$

·7 $a,b\varepsilon n . m\varepsilon N_1 . \supset . a^m-b^m \varepsilon n\times(a-b) . a^{2m}-b^{2m} \varepsilon n\times(a+b)$

·8 $n\varepsilon 6N_1-1 . a,b\varepsilon N_1 . \supset .$
$(a+b)^n-a^n-b^n \varepsilon nab(a+b)(a^2+ab+b^2)\times N_1$

·9 $n\varepsilon 6N_1+1 . a,b\varepsilon N_1 . \supset .$
$(a+b)^n-a^n-b^n \varepsilon nab(a+b)(a^2+ab+b^2)^2\times N_1$

{ CAUCHY a.1839 Œuvres s.1 t.4 p.501; Exerc. a.1841 t.2 p.137 }

❋ 35·1 $2n+1 \supset n^2-n^2 . 4n \supset n^2-n^2$
[$a\varepsilon n . \supset . 2a+1=(a+1)^2-a^2 . 4a=(a+1)^2-(a-1)^2$]

·2 $n \supset n^3+n^3+n^3+n^3+n^3$ { OLTRAMARE IdM. a.1895 p.25,166 }

·3 $a\varepsilon n . \supset . a(a^4-1) \varepsilon 30n$

❋ 40. $(n \mid N_0)$§max

·1 $u\varepsilon$ Cls'n . max$u$ $\varepsilon u . \supset .$ min$(-u) = -$max$u$

❋ 41. $u\varepsilon$ Cls'n . $\exists\, u-\iota 0 . \supset .$

·0 $Du = \max[N_1 \frown x \ni (u \supset n\times x)] . D(\iota 0) = 0$ Df

·1 $D(u \cup \iota 0) = Du$

❋ 42.

·1 $a,b\varepsilon N_1 . D(a,b) = 1 . c\varepsilon n . \supset . \exists\, 0\cdots(b-1) \frown x\ni (ax-c \varepsilon nb)$

·2 $a,b\varepsilon N_1 . D(a,b) = 1 . c\varepsilon n . u,v\varepsilon n . au+bv=c . \supset :$
$x,y\varepsilon n . ax+by=c . = . \exists\, n \frown z\ni[x=u+bz . y=v-az]$

·3 $a,b,c\varepsilon n . D(a,b,c) = 1 . \supset . (x;y;z)\ni(ax+by+cz=0) =$
$(bw-cv , cu-aw , av-bu)|(u,v,w)$ ‘ (n⁏n⁏n)

{ CAUCHY a.1826, Œuvres s.2 t.6 p.287 }

❋ 43·1 $a\varepsilon N_1 . 2a+1 \varepsilon Np . b\varepsilon n - n(2a+1) . \supset .$
$b\varepsilon n^2 + (2a+1)n . = . (-b)^a-1 \varepsilon n(2a+1)$

{ LEGENDRE a.1797 N.134 }

Numeros n^2+an vocare « residuo quadratico ».

§7 mod sgn

* 1·0 $x \varepsilon n \,.\supset.\, \mathrm{mod}x = \iota\, N_0 \frown a\ni(x = +a \,.\cup.\, x = -a)$ Df

mod$a$, lege « modulo de a », vel « valore absoluto de a » habe forma mol$a$ « moles a » in Leibniz; et forma que nos adopta in Argand a. 1814, in Cauchy,...

Voce « modulo » habe alio sensu in theoria de congruentias, et de functiones elliptico. Ergo Weierstrass a. 1851 substitue illo per $|a|$, ubi signo de functione es scripto parte ante et parte post variabile.

·1 $a \varepsilon N_0 \,.\supset.\, \mathrm{mod}(+a) = \mathrm{mod}(-a) = a$

·2 $\mathrm{mod}(a+b) \leqq \mathrm{mod}a + \mathrm{mod}b$

·3 $\mathrm{mod}(-a) = \mathrm{mod}a$

·4 $\mathrm{mod}(a \times b) = (\mathrm{mod}a) \times (\mathrm{mod}b)$ Distrib(mod, ×)

·5 $(\mathrm{mod}a)^m = \mathrm{mod}(a^m)$

* 2·0 $a \varepsilon N_0 \,.\supset.\, \mathrm{sgn}a = 1 \,.\, \mathrm{sgn}(-a) = -1 \,.\, \mathrm{sgn}0 = 0$ Df

Notatione « signum a » es introducto per Kronecker *Werke*, t.2 p.39.

$a,b \varepsilon n \,.\supset.$ ·1 $\mathrm{sgn}a \;\varepsilon\; \iota 1 \cup \iota 0 \cup \iota(-1)$ ·2 $\mathrm{sgn}(-a) = -\mathrm{sgn}a$

.3 $\mathrm{sgn}a = \mathrm{sgn}b \,.\supset.\, \mathrm{sgn}(a+b) = \mathrm{sgn}a$

·4 $\mathrm{sgn}(a \times b) = \mathrm{sgn}a \times \mathrm{sgn}b$ Distrib(sgn, ×)

·5 $a \varepsilon n \,.\supset.\, a = \mathrm{sgn}a \times \mathrm{mod}a$

·6 $a \varepsilon n \,.\supset.\, \mathrm{mod}a = a \times \mathrm{sgn}a$ Dfp

§8 R

$*$ 1·1 $a,b \varepsilon N_1 . \supset . b/a = (\times b)(/a)$ Df

·2 $R = N_1/N_1$ Df R

·3 $x,y \varepsilon R . \supset \therefore x = y . =: u \varepsilon N_1 . ux, uy \, \varepsilon N_1 . \supset_u . ux = uy$ Df

R = « Numero rationale ».

Illo es omni expressione de forma b/a, ubi a et b es numero naturale; b/a indica operatione « multiplica per b et divide per a », operatione factibile supra numeros de determinato classe.

Numero rationale es operatione, ut numero relativo.

Per conveniente substitutione nos transforma aliquo Prop. de §n in Prop. de §R:

$(\times, /, N_1, R) \mid (+, -, N_0, n)$ §n 1·3 2·0 ·4·6·7·8 3·0 ·2 ·3·4·5·6
$\supset$ §R 1·3 5·0 6·1·2·3·5 7·0·8·1·2·3·4·5

Duo operatione $n = +N_0 \cup -N_0$ genera semper novo operatione n: $+3-5 = -2$. Duo operatione $\times N_0 \cup /N_0$ non semper genera operatione de identico forma. Systema vel « gruppo de omni operatione $\times N_0 \cup /N_0$ es reductibile ad forma $\times N_0/N_0 = R$.

In usu commune 2/5 indica operatione « divide per 5, et multiplica per 3 ». 2/5 de metro, es quod resulta si nos divide metro in 5 parte, et multiplica resultatu per 3. Nos praefer inverte ordine de operatione, ut nos rende operatione possibile in maximo numero de casu.

Consideratione de fractione ut operatione es naturale et antiquo.

Ahamesu, calculatore aegyptio de anno —2000 circa, scribe (papyro Rhind, columna 12):

« $1-(2/3+/15) = /5+/15$

In vero, opera supra 15, et te habe:

$15-10-1 = 3+1$ ».

Nota, in ce papyro, suppressione de numeratore, quando vale unitate.

Super historia de differentes Df de numero rationale, vide Formul. t.4.

$*$ 2. $a,b,c,d,e,f \, \varepsilon N_1 . \supset$:

·0 $a/b = c/d . = . ad = bc$ { Euclide vii P19 }

[Hp . $a/b = c/d . \supset . bd$, $(bd)(a/b)$, $(bd)(c/d) \, \varepsilon N_1$. P1·2 $. \supset .$
$(bd)(a/b) = (bd)(c/d) . \supset . ad = bc$ (1)

Hp . $ad = bc . u \varepsilon N_1 . u(a/b)$, $u(c/d) \, \varepsilon \, N_1 . \supset . uad = ubc . \supset .$
$(ua/b)bd = (uc/d)bd$. §$\times$ 1·7 $. \supset . ua/b = uc/d$ (2)

Hp . $ad = bc$. (2) . Export $. \supset . a/b = c/d$ (3)

(1) . (3) $. \supset .$ P]

2·1 $a/b = (ac)/(bc)$

·2 $a/b = c/b \;.{=}.\; a = c$ ·3 $a/b = a/d \;.{=}.\; b = d$

·4 $a/b = c/d \;.{=}.\; a/c = b/d \;.{=}.\; b/a = d/c$

·5 $a/b = d/e \,.\, b/c = e/f \,.\supset.\, a/c = d/f$

·6 $a/b = e/f \,.\, b/c = d/e \,.\supset.\, a/c = d/f$

·7 $a/b = c/d \;.{=}.\; (a+b)/b = (c+d)/d$

·8 $a/b = c/d \;.{=}.\; a/b = (a+c)/(b+d)$

·9 $a/b = c/d \,.\, e/b = f/d \,.\supset.\, (a+e)/b = (c+f)/d$

[P·0 $\supset$ P·1-·9]

{ Euclide v P12, 18, 19, 22, 24, vii P 13, 17. }

✻ 3·0 $x, y \varepsilon \mathrm{R} \,.\supset.$

$x+y = \iota\, \mathrm{R} \cap z \ni (u \varepsilon \mathrm{N}_1 \,.\, ux, uy\, \varepsilon \mathrm{N}_1 \,.\supset_u.\, ux+uy = uz)$ Df

3·1 $a, b, c \varepsilon \mathrm{N}_1 \,.\supset.\, a/c + b/c = (a+b)/c$

[$a/c+b/c = \iota\, \mathrm{R} \cap z \ni [u \varepsilon \mathrm{N}_1 \,.\, ua/c, ub/c\, \varepsilon \mathrm{N}_1 \,.\supset_u.\, ua/c + ub/c = uz]$

$=$ » » » » $(ua+ub)/c = uz]$

$=$ » » » » $u(a+b)/c = uz]$

$=$ » » » » $(a+b)/c = z]$

$=$ » $[z = (a+b)/c] = (a+b)/c$]

·2 $a, b, c, d \varepsilon \mathrm{N}_1 \,.\supset.\, a/b + c/d = (ad+bc)/(bd)$

[$a/b + c/d = (ad)/(bd) + (bc)/(bd) = (ad+bc)/(bd)$]

✻ 4. $a, b, c \varepsilon \mathrm{R} \,.\supset:$ ·1 $a+b\, \varepsilon \mathrm{R}$

·2 $a+b = b+a$

[P3·0 $.\supset.\; a+b = \iota \mathrm{R} \cap z \ni (u \varepsilon \mathrm{N}_1 \,.\, ua, ub \varepsilon \mathrm{N}_1 \,.\supset_u.\, ua+ub = uz)$

(Comm+) = (§+4·4) $.\supset.$ » » » » $.\; ub+ua = uz)$

P3·0 $.\supset.$ » $= b+a$]

·3 $a+(b+c) = (a+b)+c = a+b+c$ ·4 $a+c = b+c \;.{=}.\; a = b$

✻ 5·0 $x, y \varepsilon \mathrm{R} \,.\supset.$

$x \times y = xy = \iota\, \mathrm{R} \cap z \ni (u \varepsilon \mathrm{N}_1 \,.\, ux, uxy\, \varepsilon \mathrm{N}_1 \,.\supset_u.\, uxy = uz)$ Df

$a, b, c, d \varepsilon \mathrm{N}_1 \,.\supset.$

·1 $(a/b)(b/c) = a/c$

[Df·0 $.\supset.\; (a/b)(b/c) = \iota\, \mathrm{R} \cap z \ni [u \varepsilon \mathrm{N}_1 \,.\, ua/b, ua/b \times b/c\, \varepsilon \mathrm{N}_1 \,.\supset_u.\, u \times a/b \times b/c = uz]$

§/ ·1 $.\supset.$ » » » » » $ua/c = uz]$

Df 1·2 $.\supset.$ » $= a/c$]

·2 $(a/b)(c/d) = (ac)/(bd)$ { Euclide viii P5 }

❋ 6. $a,b,c\varepsilon R .\supset:$

·1 $a\times b\ \varepsilon R$ ·2 $ab=ba$ ·3 $a(bc)=(ab)c=abc$

·4 $a(b+c)=ab+ac$

·5 $ac=bc .=. a=b$

$(R \mid N_0)$§× P2

❋ 7·0 $x\varepsilon R .\supset. /x = \iota R \frown y \ni (x\times y=1)$ Df

$a,b\varepsilon N_1 .\supset.$ ·1 $/(a/b)=b/a$

$[/(a/b)=\iota R\frown y\ni[(a/b)\times y=1]$

$= \text{»} (a\times y=b)$

$= \text{»} (y=b/a)$

$=b/a\ ]$

❋ 8. $a,b\varepsilon R .\supset:$

·1 $/a\ \varepsilon R$ ·2 $/(/a)=a$

·3 $b/a=b\times(/a)$ Df ·4 $/(ab)=(/a)(/b)$

·5 $x\varepsilon R . ax=b .=. x=b/a$ [Oper/a .⊃. P]

·6 $a,b,c\,\varepsilon R .\supset:$

$x,y\varepsilon R . x/a=y/b . x+y=c .=. x=ac/(a+b) . y=bc/(a+b)$

$[\ x,y\varepsilon R . x/a=y/b . x+y=c .=. x\varepsilon R . y=bx/a . x+bx/a=c$

$.=. x\varepsilon R . x(1+b/a)=c . y=bx/a$

$.=. x=ac/(a+b) . y=bc/(a+b)\ ]$

❋ 9. $f\varepsilon\ RfR : y,z\varepsilon R .\supset_{y,z}. f(y+z)=fy+fz : x\varepsilon R :\supset.$

$fx=(f1)\times x$

[Hp .⊃. $f(1\times x)=1\times fx$ (1)

Hp . $n\varepsilon N_1$. $f(nx)=n\times fx$.⊃. $f[(n+1)x]=f(nx+x)=$

$f(nx)+fx=nfx+fx=(n+1)fx$ (2)

Hp . $n\varepsilon N_1$. (1) . (2) . Induct .⊃. $f(nx)=nfx$ (3)

» . $(x/n)|x$ P(3) . Oper/n .⊃. $f(x/n)=(fx)/n$ (4)

Hp . $m,n\varepsilon N_1$. (3) . (4) .⊃. $f(mx/n)=mf(x/n)=(m/n)fx$ (5)

» . $(1|x)$ P(5) .⊃. $f(m/n)=(m/n)f1$ (6)

(6) .⊃. P]

R ↾

❋ 11. $a,b\varepsilon R . m,n\varepsilon N_0 .\supset.$ ·0 $a\upharpoonright 0=1$ Df

·01 $a\upharpoonright(m+1)=(a\upharpoonright m)\times a$ Df

·02 $a^m=a\upharpoonright m$ Df

·1 $a \upharpoonright m \, \varepsilon \mathrm{R}$ ·2 $a^m a^n = a^{m+n}$

·3 $(ab)^m = a^m b^m$ ·4 $(a^m)^n = a^{mn}$

·5 $/(a^m) = (/a)^m$ Comm$(/, \upharpoonright)$

·6 $p,q \varepsilon \mathrm{N}_1 . \supset . (p/q)^m = p^m/q^m$

$(\mathrm{R} \mid \mathrm{N}_0)$ §$\upharpoonright$P2. 3.

R >

✻ 21. $a,b \varepsilon \mathrm{R} . \supset :$ ·0 $b>a .=. a<b .=. b \varepsilon a+\mathrm{R}$ Df

·01 $b>a .=. u \varepsilon \mathrm{N}_1 . ua, ub \, \varepsilon \, \mathrm{N}_1 . \supset_u . ub > ua$ Dfp

§N_1 P2, 3, 5, 6, 7, 11, 12, 13, 14

✻ 22. $a,b,c,d \varepsilon \mathrm{N}_1 . \supset :$

·1 $a/b > c/b .=. a>c : a/b > a/d .=. b<d$ | Eucl. v P10 |

·2 $a/b > c/d .=. ad > bc$

·3 $a/b = c/d . a>b . a>c . \supset . a+d > b+c$ | Eucl. v P25 |

·4 $a/(a+b) < (a+c)/(a+b+c)$

·5 $a/b < c/d . \supset . a/b < (a+c)/(b+d) < c/d$
| Pappus vii P8 p.691 |

✻ 23. $a,b,c,d \varepsilon \mathrm{R} . \supset .$

·1 $b>a .=. /b < /a$

·2 $a<b . m,n \varepsilon \mathrm{R} . \supset . a < (ma+nb)/(m+n) < b$

·3 $a<b .=. \exists \, \mathrm{R} \frown x \ni (a<x<b)$ [P·2 ⊃ P]

·4 $a \varepsilon \mathrm{R} - \iota 1 . \supset . a + /a > 2$

✻ 24·1 $m \varepsilon \mathrm{N}_1+1 . a \varepsilon \mathrm{R} . \supset . (1+a)^m > 1+ma$

[$(1+a) = 1+2a+a > 1+2a$ (1)

$m \varepsilon \mathrm{N}_1+1 . (1+a)\upharpoonright m > 1+ma$. Oper $\times(1+a) . \supset . (1+a)\upharpoonright(m+1) >$
$(1+ma)(1+a) = 1+(m+1)a+ma > 1+(m+1)a$ (2)
(1) . (2) . Induct $. \supset .$ P]

·2 $m \varepsilon \mathrm{N}_1 . a \varepsilon \mathrm{R} . \supset . (1+a)^m < 1+m(1+a)^m a$

[$1+a < 1+a(1+a)$ (1)

$m \varepsilon \mathrm{N}_1 . (1+a)\upharpoonright m < 1+ma(1+a)\upharpoonright m$. Oper $\times(1+a) . \supset .$
$(1+a)\upharpoonright(m+1) < 1+a+ma(1+a)\upharpoonright(m+1) < 1+a(1+a)\upharpoonright(m+1)+$
$ma(1+a)\upharpoonright(m+1) = 1+(m+1)a(1+a)\upharpoonright(m+1)$ (2)
(1) . (2) . Induct $. \supset .$ P]

·3 $m \varepsilon \mathrm{N}_1 . a \varepsilon \mathrm{R} . \supset . (1+a)^{m+1} < 1+(m+1)(1+a)^m a$

[P·2 . Oper $+a(1+a)\upharpoonright m . \supset .$ P]

R —

✻ 31. $a,b \varepsilon N_1 \,.\, c \varepsilon\, a+N_1 \,.\, d \varepsilon\, b+N_1 \,.\supset.$

·1 $c/a = d/b \,.=.\, (c-a)/a = (d-b)/b$ ∤ Euclide vii P11 ∤

·2 $c/a = d/b \,.=.\, (c+a)/(c-a) = (d+b)/(d-b)$

✻ 32·0 $x\varepsilon R \,.\, y\varepsilon\, x+R \,.\supset.\, y-x = \iota\, R \frown z \ni (x+z=y)$ Df

·1 $b,c\varepsilon N_1 \,.\, a\varepsilon\, b+N_1 \,.\supset.\, a/c-b/c = (a-b)/c$

·2 $a,b,c,d,ad-bc\, \varepsilon N_1 \,.\supset.\, a/b-c/d = (ad-bc)/(bd)$

$(R \mid N_0)$ §—

✻ 33·1 $m\varepsilon\, N_1+1 \,.\, a\varepsilon R \,.\, a<1 \,.\supset.\, (1-a)^m > 1-ma$

·2 $m\varepsilon N_1 \,.\, a\varepsilon R \,.\supset.\, (1-a)^m < 1-m(1-a)^m a$

[$[a/(1-a) \mid a]$P24·1·2 $\supset$ P]

·3 » $.\supset.\, (1-a)^m < /(1+ma)$

·4 » $.\, ma<1 \,.\supset.\, (1+a)^m < /(1-ma)$

[P24·2.33·2 $.\supset.\, (1+a)^m(1-ma)<1 \,.\, (1-a)^m(1+ma)<1 \,.\supset.$ P·3·4]

·5 $m,n\varepsilon N_1 \,.\, x\varepsilon R \,.\, x<n \,.\supset.\, (1+x/m)^m(1-x/n)^n <1$

[$1> [1+x/(mn)]\times[1-x/(mn)]$ (1)

(1) . Oper $\upharpoonright(mn) \,.\supset.\, 1> [1+x/(mn)]\upharpoonright(mn)\times[1-x/(mn)]\upharpoonright(mn)$ (2)

P24·1 $.\supset.\, [1+x/(mn)]\upharpoonright n \geqq 1+x/m \,.\supset.\, [1+x/(mn)]\upharpoonright(mn) \geqq (1+x/m)\upharpoonright m$ (3)

P33·1 $.\supset.\, [1-x/(mn)]\upharpoonright m > 1-x/n \,.\supset.\, [1-x/(mn)]\upharpoonright(mn) > (1-x/n)\upharpoonright n$ (4)

(2) . (3) . (4) $.\supset.$ P]

·6 $x,y\varepsilon R \,.\, x -= y \,.\, m,n\varepsilon N_1 \,.\supset.\, x^m y^n < [(mx+ny)/(m+n)]^{m+n}$

[$mn(y-x)/(m+n) \mid x$ P·5 $.\supset.$ P]

✻ 34·0 $a\varepsilon R \,.\, m\varepsilon N_1 \,.\supset.\, a^{-m} = /a^m$ Df

$a,b\varepsilon R \,.\, m,n\varepsilon n \,.\supset.$ ·1 $a^m\, \varepsilon R$ P11·2-·4

·5 $a^{-m} = /a^m$ ·6 $a^n/a^m = a^{n-m}$

✻ 40. $(R \mid N_0)$§max

·1 $u\varepsilon$ Cls'R . max$u\, \varepsilon R \,.\supset.$ min $/u = /$max$u$

✻ 41·0 $u \varepsilon \mathrm{Cls'R} .\supset. \mathrm{m}u = \min[\mathrm{R} \frown x \ni (u \supset x/\mathrm{N}_1)]$ Df

·01 » » $= /\mathrm{D}/u$ Dfp

·1 $a,b,c \varepsilon \mathrm{N}_1 .\supset. \mathrm{m}(a/c\,, b/c) = [\mathrm{m}(a,b)]/c$

{ Bertrand a.1849 p.107 }

✻ 42·0 $u \varepsilon \mathrm{Cls'R} .\supset. \mathrm{D}u = \max[\mathrm{R} \frown x \ni (u \supset \mathrm{N}_1 \times x)]$ Df

·1 $a,b,c \varepsilon \mathrm{N}_1 .\supset. \mathrm{D}(a/c, b/c) = [\mathrm{D}(a,b)]/c$

{ Bertrand a.1849 p.105 }

$u,v \varepsilon \mathrm{Cls'R} . \mathrm{D}u, \mathrm{D}v \,\varepsilon \mathrm{R} .\supset.$ ·2 $\mathrm{D}(u \times v) = (\mathrm{D}u) \times (\mathrm{D}v)$

·3 $a \varepsilon \mathrm{R} .\supset. \mathrm{D}(au) = a\mathrm{D}u$ ·4 $n \varepsilon \mathrm{N}_1 .\supset. (\mathrm{D}u)^n = \mathrm{D}(u^n)$

·5 $a \varepsilon \mathrm{R} . n \varepsilon \mathrm{N}_1 .\supset. \mathrm{D}[a \upharpoonright (0 \cdots n)] = [\mathrm{D}(1,a)]^n$

{ Barrieu AnnN. a.1895 t.14 p.214 }

§9 r

✻ 1·0 $\mathrm{r} = +\mathrm{R} \cup -\mathrm{R} \cup \iota 0$ Df ·01 $\mathrm{r} = \mathrm{n}/\mathrm{N}_1$ Dfp

r = « numero rationale relativo ».

$x,y \varepsilon \mathrm{r} .\supset\therefore$

·1 $x = y .=: u \varepsilon \mathrm{R} . u+x, u+y \,\varepsilon \mathrm{R} .\supset_u. u+x = u+y$ Dt

·2 $x+y =$

$\bar{\iota}\, \mathrm{r} \frown z \ni (u \varepsilon \mathrm{R} . u+x, u+x+y \,\varepsilon \mathrm{R} .\supset_u. u+x+y = u+z)$ Df

·3 $-x = \bar{\iota}\, \mathrm{r} \frown y \ni (x+y = 0)$ Df

·4 $x-y = x+(-y)$ Df

·5 $x = y .=: u \varepsilon \mathrm{n} . ux, uy \,\varepsilon \mathrm{n} .\supset_u. ux = uy$ Dfp

·6 $x \times y = xy = \bar{\iota}\, \mathrm{r} \frown z \ni (u \varepsilon \mathrm{n} . ux, uxy \,\varepsilon \mathrm{n} .\supset_u. uxy = uz)$ Df

·7 $/x = \bar{\iota}\, \mathrm{r} \frown y \ni (xy = 1)$ Df

·8 $x/y = x \times (/y)$ Df

·9 $x+y = \bar{\iota}\, \mathrm{r} \frown z \ni (u \varepsilon \mathrm{n} . ux, uy \,\varepsilon \mathrm{n} .\supset_u. ux+uy = uz)$ Dfp

✻ 2·0 $a,b \varepsilon \mathrm{r} .\supset. a+b \,\varepsilon \mathrm{r} \;.\; -a \,\varepsilon \mathrm{r} \;.\; a \times b \,\varepsilon \mathrm{r}$. §n

·1 $a \varepsilon \mathrm{r} - \iota 0 .\supset. /a \,\varepsilon \mathrm{r} \;.\; /-a = -/a$

✱ 3. $a,b,c,d\varepsilon\mathrm{r} \,.\supset\therefore$

·1 $a-c \mathrel{-\!=} 0 \,.\supset:\, x\varepsilon\mathrm{r} \,.\, ax+b = cx+d \,.=.\, x=(d-b)/(a-c)$

[Oper$-b \,.\supset:$ $x\varepsilon\mathrm{r} \,.\, ax+b = cx+d \,.=.\, x\varepsilon\mathrm{r} \,.\, ax = cx+d-b$
Oper$-cx \,.\supset:$ » $.=.\, x\varepsilon\mathrm{r} \,.\, ax-cx = d-b$
Distrib$(\times,+) \,.\supset:$ » $.=.\, x\varepsilon\mathrm{r} \,.\, (a-c)x = d-b$
Oper$/(a-c) \,.\supset:$ » $.=.\, x\varepsilon\mathrm{r} \,.\, x=(d-b)/(a-c)$
Hp . P2·1 $.\supset:$ » $.=.\, x=(d-b)/(a-c)$]

·2 $x,y\varepsilon\mathrm{r} \,.\, x+y=a \,.\, x-y=b \,.=.\, x=(a+b)/2 \,.\, y=(a-b)/2$
{ Diophanto I 1 }

[$x,y\varepsilon\mathrm{r} \,.\, x+y=a \,.\, x-y=b \,.=.\, y\varepsilon\mathrm{r} \,.\, x=y+b \,.\, y+b+y=a$
$.=.\, y=(a-b)/2 \,.\, x=(a-b)/2+b = (a+b)/2$]

[$x,y\varepsilon\mathrm{r} \,.\, x+y=a \,.\, x-y=b \,.$ Oper$+$. Oper$-$ $.\supset.$
$x,y\varepsilon\mathrm{r} \,.\, 2x=a+b \,.\, 2y=a-b \,.\supset.\, x=(a+b)/2 \,.\, y=(a-6)/2$ (1)
$x=(a+b)/2 \,.\, y=(a-b)/2 \,.\supset.\, x,y\varepsilon\mathrm{r} \,.\, x+y=a \,.\, x-y=b$ (2)
(1) (2) $\supset$ P]

·3 $x,y,z\varepsilon\mathrm{r} \,.\, y+z=a \,.\, z+x=b \,.\, x+y=c \,.=.\, x=(b+c-a)/2 \,.$
$y=(a+c-b)/2 \,.\, z=(a+b-c)/2$ { Diophanto I 16 }

·4 $x,y,z\varepsilon\mathrm{r} \,.\, y+z-x=a \,.\, z+x-y=b \,.\, x+y-z=c \,.=.$
$x=(b+c)/2 \,.\, y= \ldots$ { Diophanto I 18 }

·5 $x,y,z,t\varepsilon\mathrm{r} \,.\, y+z+t=a \,.\, x+z+t=b \,.\, x+y+t=c \,.\, x+y+$
$z=d \,.=.\, x=(b+c+d-2a)/3 \,.\, y= \ldots$ { Diophanto I 17 }

·6 $x,y,z,t\varepsilon\mathrm{r} \,.\, y+z+t-x=a \,.\, x+z+t-y=b \,.\, x+y+t-z=c \,.$
$x+y+z-t=d \,.=.\, x=(a+b+c+d)/4-a/2 \,.\, y= \ldots$

{ Diophanto I 19 }

✱ 4·1 $a,b,c,a',b',c'\varepsilon\mathrm{r} \,.\, ab'-a'b \mathrel{-\!=} 0 \,.\supset:$
$x,y\varepsilon\mathrm{r} \,.\, ax+by=c \,.\, a'x+b'y=c' \,.=.$
$x=(cb'-c'b)/(ab'-a'b) \,.\, y=(ac'-a'c)/(ab'-a'b)$

[Hp $.\supset.\, a \mathrel{-\!=} 0 \,.\cup.\, a' \mathrel{-\!=} 0$
Hp . $a \mathrel{-\!=} 0 \,.\supset:\, x,y\varepsilon\mathrm{r} \,.\, ax+by=c \,.\, a'x+b'y=c' \,.=.$
$y\varepsilon\mathrm{r} \,.\, x=(c-by)/a \,.\, a'(c-by)/a+b'y=c'$
$.=.$ » » $.\, a'(c-by)+ab'y=ac'$
$.=.$ » » $.\, a'c-a'by+ab'y=ac'$
$.=.$ » » $.\, (ab'-a'b)y=ac'-a'c$
$.=.\, y=(ac'-a'c)/(ab'-a'b) \,.\, x=[c-b(ac'-a'c)/(ab'-a'b)]/a \,.=. \ldots$]

·2 $a,b,c,a',b',c'\varepsilon\mathrm{r} \,.\, ab'-a'b=0 \,.\, ac'-a'c \mathrel{-\!=} 0 \,.\supset.$
$-\exists\, (x;y)\ni[\, x,y\varepsilon\mathrm{r} \,.\, ax+by=c \,.\, a'x+b'y=c' \,]$

·3 $a,b,c,a',b',c'\varepsilon\mathrm{r} \,.\, ab'-a'b=0 \,.\, ac'-a'c=0 \,.\, a \mathrel{-\!=} 0 \,.\supset:$
$x,y\varepsilon\mathrm{r} \,.\, ax+by=c \,.\, a'x+b'y=c' \,.=.\, y\varepsilon\mathrm{r} \,.\, x=(c-by)/a$

§10 E β dt nt

❋ 1. $x,y \varepsilon \mathrm{r} \ .\supset.$

·0 $\mathrm{E}x = \max \mathrm{n} \frown y \ni (y \leqq x)$ Df

Legendre a. 1808 introduce notatione Ex, lege « integro (Entier) de x ».

·1 $\mathrm{E}x \, \varepsilon \, \mathrm{n} \ . \ \mathrm{E}x \leqq x < \mathrm{E}x + 1$

·2 $\mathrm{E}x = \iota \, \mathrm{n} \frown a \ni (a \leqq x < a+1)$ Dfp

·3 $x \varepsilon \mathrm{n} \ .\supset. \ \mathrm{E}x = x$

·4 $x > y \ .\supset. \ \mathrm{E}x \geqq \mathrm{E}y$

·5 $a \varepsilon \mathrm{n} \ .\supset. \ \mathrm{E}(a+x) = a + \mathrm{E}x$

[$\mathrm{E}x \leqq x < \mathrm{E}x+1 \ .\supset. \ a+\mathrm{E}x \leqq a+x < a+\mathrm{E}x+1 \ . \ a+\mathrm{E}x \, \varepsilon \mathrm{N}_0 \ .\supset. \ \mathrm{P}$]

·6 $\mathrm{E}x + \mathrm{E}y \leqq \mathrm{E}(x+y) \leqq \mathrm{E}x + \mathrm{E}y + 1$

[$\mathrm{E}x \leqq x < \mathrm{E}x+1 \ . \ \mathrm{E}y \leqq y < \mathrm{E}y+1 \ .\supset. \ \mathrm{E}x+\mathrm{E}y \leqq x+y <$
$\mathrm{E}x+\mathrm{E}y+2 \ .\supset. \ \mathrm{P}$]

❋ 2. $x,y \varepsilon \mathrm{R} \ .\supset:$

·1 $a \varepsilon \mathrm{N}_1 \ .\supset. \ a \times \mathrm{E}x \leqq \mathrm{E}(a \times x) < a \times (\mathrm{E}x+1)$

·2 $\mathrm{E}x \times \mathrm{E}y \leqq \mathrm{E}(x \times y) < (\mathrm{E}x+1)(\mathrm{E}y+1)$

·3 $a \varepsilon \mathrm{N}_1 \ .\supset. \ \mathrm{E}(x/a) = \mathrm{E}\{(\mathrm{E}x)/a\}$

·4 $x > 1 \ .\supset. \ \mathrm{E}(\mathrm{E}y \,/\, \mathrm{E}x) \geqq \mathrm{E}(y/x) \geqq \mathrm{E}[\mathrm{E}y \,/\, (\mathrm{E}x+1)]$

[$(y/x) \mid y$ P·61 $.\supset. \ \mathrm{E}x \times \mathrm{E}(y/x) \leqq \mathrm{E}y < (\mathrm{E}x+1) \times \mathrm{E}(y/x+1)$.
Oper $/\mathrm{E}x \ .\supset. \ \mathrm{E}(y/x) \leqq \mathrm{E}y/\mathrm{E}x \ . \ \mathrm{E}(y/x) \geqq \mathrm{E}y/(\mathrm{E}x+1)$. OperE .
P·4 $.\supset.$ P]

·5 $a,b \varepsilon \mathrm{N}_1 \ .\supset. \ \mathrm{quot}(a,b) = \mathrm{E}(a/b)$ Dfp

·6 $x \varepsilon \mathrm{r}-\mathrm{n} \ .\supset. \ \mathrm{E}x + \mathrm{E}(-x) = -1$
$x \varepsilon \mathrm{n} \ .\supset. \ \mathrm{E}x + \mathrm{E}(-x) = 0$

·7 $x \varepsilon \mathrm{R} \ .\supset. \ \mathrm{E}[(\mathrm{E}x)/x] - 1 = \mathrm{E}x + \mathrm{E}(-x)$

❋ 3. $a \varepsilon \mathrm{R} \ . \ n \varepsilon \mathrm{n} \ .\supset.$

·0 $\mathrm{Cfr}_n a = \mathrm{Cfr}_0 \mathrm{E}(\mathrm{X}^{-n} a)$ Df

·1 $m,n \varepsilon \mathrm{N}_1 \ . \ m \varepsilon \mathrm{N}_1 \times n \ .\supset. \ \mathrm{Cfr}_{-m} / (\mathrm{X}^n - 1) = 1$

·2 » $. \ m -\varepsilon \mathrm{N}_1 \times n \ .\supset.$ » $= 0$

✻ 4. $a,b\varepsilon \mathrm{R} \,.\supset.$ ·0 $\mathrm{ord}a = \max \mathrm{n} \frown n\ni(\mathrm{X}^n \leqq a)$ Df

·1 $\mathrm{X}\upharpoonright\mathrm{ord}a \leqq a < \mathrm{X}\upharpoonright(\mathrm{ord}a+1)$

·2 $\mathrm{ord}(a+b) \geqq \max(\iota\,\mathrm{ord}a \cup \iota\,\mathrm{ord}b)$.

» $\leqq$ » $+1$

[$\mathrm{ord}a \geqq \mathrm{ord}b \,.\supset.\, \mathrm{X}\upharpoonright\mathrm{ord}a \leqq a < \mathrm{X}\upharpoonright(\mathrm{ord}a+1) \,.\, b < \mathrm{X}\upharpoonright(\mathrm{ord}a+1) \,.$
$\supset.\, \mathrm{X}\upharpoonright\mathrm{ord}a < a+b < 2\times\mathrm{X}\upharpoonright(\mathrm{ord}a+1) < \mathrm{X}\upharpoonright(\mathrm{ord}a+2) \,.\supset.\, \mathrm{P}$]

·3 $a>b \,.\supset.\, \mathrm{ord}a \geqq \mathrm{ord}b$

·4 $\mathrm{ord}(a\times b) \geqq \mathrm{ord}a + \mathrm{ord}b$.

» $\leqq \mathrm{ord}a + \mathrm{ord}b + 1$

[$\mathrm{X}\upharpoonright\mathrm{ord}a \leqq a < \mathrm{X}\upharpoonright(\mathrm{ord}a+1) \,.\, \mathrm{X}\upharpoonright\mathrm{ord}b \leqq b < \mathrm{X}\upharpoonright(\mathrm{ord}b+1) \,.\supset.\, \mathrm{X}\upharpoonright(\mathrm{ord}a+$
$\mathrm{ord}b) \leqq a\times b < \mathrm{X}\upharpoonright(\mathrm{ord}a+\mathrm{ord}b+2) \,.\supset.\, \mathrm{P}$]

·5 $m\varepsilon\mathrm{N}_1 \,.\supset.\, m\times\mathrm{ord}a \leqq \mathrm{ord}(a^m) < m(\mathrm{ord}a+1)$

·6 $a>b \,.\supset.\, \mathrm{ord}(a-b) \leqq \mathrm{ord}a$

·7 $a\varepsilon\, \mathrm{R}-(\mathrm{X}\upharpoonright\mathrm{n}) \,.\supset.\, \mathrm{ord}/a = -\mathrm{ord}a - 1$
[Hp $.\supset.\, \mathrm{X}\upharpoonright\mathrm{ord}a < a < \mathrm{X}\upharpoonright(\mathrm{ord}a+1) \,.\supset.\, \mathrm{X}\upharpoonright(-\mathrm{ord}a-1) < /a <$
$\mathrm{X}\upharpoonright(-\mathrm{ord}a) \,.\supset.\, \mathrm{P}$]

✻ 5. $x,y\varepsilon\mathrm{r} \,.\supset.$ ·0 $\beta x = x - \mathrm{E}x$ Df

·01 $\beta\beta x = \beta x \,.\, \mathrm{E}\beta x = 0 \,.\, \beta\mathrm{E}x = 0$

Zehfuss (Grunert Archiv, a.1850 t.27 p.12) introduce functione βx; litera β es initiale de voce « Bruchtheil ».

Et vocare « mantissa », id es « excedente ». Wallis, Opera a.1693 p.41: « Ejusque partes decimales abscissas, *appendicem* voco, sive *mantissam* ».

·1 $y\varepsilon\mathrm{n} \,.\supset.\, \beta(x+y) = \beta x$

·11 $\beta(x+y) = \beta(\beta x + \beta y)$

·2 $x\varepsilon\mathrm{n} \,.\supset.\, \beta x + \beta(-x) = 0$

·21 $x -\varepsilon\, \mathrm{n} \,.\supset.\, \beta x + \beta(-x) = 1$

·3 $0 \leqq \beta x < 1$

·31 $\beta(x+y) \leqq \beta x + \beta y$

·4 $y\varepsilon\mathrm{n} \,.\supset.\, \beta(xy) = \beta(y\,\beta x)$

·5 $a\varepsilon\mathrm{N}_1 \,.\supset.\, \mathrm{E}(ax) = a\times\mathrm{E}x + \mathrm{E}(a\beta x)$

·7 $a\varepsilon\mathrm{N}_0 \,.\, b\varepsilon\mathrm{N}_1 \,.\supset.\, \mathrm{rest}(a,b) = b\times\beta(a/b)$ Dfp

✻ 6. $a,b\varepsilon\mathrm{R} \,.\, m,n\varepsilon\mathrm{N}_1 \,.\supset:$

·0 $\mathrm{dt}a = \min[\mathrm{N}_1 \frown \mathrm{N}_1/a]$ Df

·01 $\mathrm{nt}a = \mathrm{dt}/a = \min[\mathrm{N}_1 \frown \mathrm{N}_1\times a]$ Df

dt$a$ = « denominatore reducto de a »

nt$a$ = « numeratore » de a »

Theoria et notationes per Prof. A. Padoa RdM. a.1898 p.90-94.

·1 $\mathrm{dt}a\, \varepsilon N_1$. $\mathrm{dt}m = 1$. $\mathrm{nt}m = m$

·2 $b \varepsilon N_1 \times a$.=. $\mathrm{dt}a\, \varepsilon N_1 \times \mathrm{dt}b$. $\mathrm{nt}b\, \varepsilon N_1 \times \mathrm{nt}a$

·3 $a+b\, \varepsilon N_1$.⊃. $\mathrm{dt}a = \mathrm{dt}b$ ·31 $a-b\, \varepsilon N_1$.⊃. $\mathrm{dt}a = \mathrm{dt}b$

·4 $\mathrm{dt}(m+a) = \mathrm{dt}a$ ·41 $m > a$.⊃. $\mathrm{dt}(m-a) = \mathrm{dt}a$

·5 $\mathrm{dt}(a^m) = (\mathrm{dt}a)^m$

·6 $a \varepsilon R^m$.=. $\mathrm{dt}a, \mathrm{nt}a\, \varepsilon N_1^m$

Dvr mlt ·7 $\mathrm{Dvr}(\mathrm{dt}a, \mathrm{nt}a) = 1$

·8 $\mathrm{dt}a = /\mathrm{Dvr}(1, a) = /\mathrm{mlt}(1, /a)$ Dfp

·81 $\mathrm{dt}(m/n) = n/\mathrm{Dvr}(m,n) = \mathrm{mlt}(m,n)/m$

·82 $\mathrm{Dvr}(m,n) = 1$.=. $\mathrm{dt}(m/n) = n$

·9 $u \varepsilon \mathrm{Cls'R}$. $\mathrm{num}u\, \varepsilon N_1$.⊃. $\mathrm{Dvr}u = \mathrm{Dvr}(\mathrm{nt}`u)/\mathrm{mlt}(\mathrm{dt}`u)$

·91 » » » .⊃. $\mathrm{mlt}u = \mathrm{mlt}(\mathrm{nt}`u)/\mathrm{Dvr}(\mathrm{dt}`u)$

{ BARRIEU Mathesis a.1883 t.3 p.217 }

§11 η

✻ 1·0 $\eta = R \frown x \ni (x < 1)$ Df

·1 $\eta \times \eta = \eta$ ·2 $\eta N_1 = \eta R = R$

✻ 2. $a,b \varepsilon R$. $u,v,w\, \varepsilon \mathrm{Cls'R}$.⊃.

·1 $b < a$.=. $b \varepsilon \eta a$ ·1 $\eta a = R \smile x \ni (x < a)$

·2 $\eta a \supset \eta b$.⊃. $a \leqq b$

[Hp .⊃. $b -\varepsilon\, \eta b$.⊃. $b -\varepsilon\, \eta a$.⊃. $-(b < a)$.⊃. Ths]

·3 $\eta a = \eta b$.⊃. $a = b$ [Hp . P·4 .⊃. $a \leqq b$. $b \leqq a$.⊃. Ths]

·4 $\eta(u+v) = \eta u + \eta v$ Distrib$(\eta,+)$

·5 $\eta w(u+v) = \eta wu + \eta wv$

·6 $m,n \varepsilon N_1$.⊃. $\eta u^{m+n} = \eta u^m u^n$

✻ 3·1 $u \varepsilon \mathrm{Cls'R}$. $\max u\, \varepsilon R$.⊃. $\eta u = \eta \max u$

·2 $u \varepsilon \mathrm{Cls'R}$. $\min(R - \eta u)\, \varepsilon R$.⊃. $\eta u = \eta \min u$

·3 $u \varepsilon \mathrm{Cls'R}$. $a \varepsilon R$.⊃: $\eta a = \eta u$.=. $a = \min(R - \eta u)$

Signo η, lege « fractio proprio » vel « eta », indica « numero rationale minore de 1 ». Si $a \varepsilon R$, ηa repraesenta « classe de rationale minore de a ». Si $u \varepsilon$ Cls'R, ηu, vel $\eta \times u$ repraesenta « classe de rationale minore de aliquo u ».

Nota analogia de II §5 P4·4·5·6 cum aliquo P de ce §.

§12 Q l′ l, ∞ θ

✻ 1. $u \varepsilon \text{Cls'R} . a \varepsilon \text{R} . \supset :$ ·0 $a < \text{l}'u .=. a \varepsilon \eta u$ Df

·1 $\text{R} \frown x \ni (x < \text{l}'u) = \eta u$

·2 $a = \text{l}'u .=. \text{R} \frown x \ni (x < a) = \text{R} \frown x \ni (x < \text{l}'u)$ Df

·3 » $.=. \eta a = \eta u$

·4 $a > \text{l}'u .=. -(a < \text{l}'u) . -(a = \text{l}'u)$ Df

l′ = « limite superiore » | Guilmin a.1847; vide RdM. t.6 p.137|.
= « limite supero », « obere Grenze » | Weierstrass |.
= « maximo ideale » | Pringsheim, *Encyclopädie*, p. 72 |.

Nos ne pone Df de forma:
$u \varepsilon \text{Cls'R} . \supset : \text{l}'u =$ (expressione composito per signos praecedente) Df
sed nos defini solo relatione $a < \text{l}'u$.

✻ 2. $a,b \varepsilon \text{l}'\text{“Cls'R} . \supset :$

·1 $a = b .=. \text{R} \frown x \ni (x < a) = \text{R} \frown x \ni (x < b)$ Df

Si a et b es limite supero de classe de rationale, nos dice que $a = b$, quando classe de rationale minore de a (definito per P1·0) coincide cum classe de rationale minore de b.

·2 $a \leqq b .=. \text{R} \frown x \ni (x < a) \supset \text{R} \frown x \ni (x < b)$ Df

·3 $b \geqq a .=. a \leqq b$ Df

·4 $a < b .=. b > a .=. -(b \leqq a)$ Df

✻ 3. $u,v \varepsilon \text{Cls'R} . \supset :$

·0 $\text{l}'u = \text{l}'v .=. \eta u = \eta v$ [$(\text{l}'u, \text{l}'v) \mid (a,b)$ P2·1 $. \supset .$ P]

·1 $\text{l}'u \leqq \text{l}'v .=. \eta u \supset \eta v$ [» P2·2 $. \supset .$ P]

·2 $\text{l}'u < \text{l}'v .=. \exists \eta v - \eta u$

·3 $\text{l}'u = \text{l}'\eta u$ [$\eta(\eta u) = (\eta\eta)u = \eta u$. P·0 $. \supset$ P]

·4 $a \varepsilon \text{R} . \supset . a = \text{l}'\eta a = \text{l}'\iota a$

·5 $1 = \text{l}'\eta$

✻ 4·0 $u \varepsilon \text{Cls'R} . \exists u . \exists \text{R} - (u / \eta) . \supset . \text{l}_{,}u = \text{l}' \text{R} - (u / \eta)$ Df

·1 » » $. \supset : 0 = \text{l}_{,}u .=. u / \eta = \text{R}$ Df

$(\text{l}_{,}, /\eta, <) \mid (\text{l}', \eta, >)$ P1 . P2 . P3

$\text{l}_{,}u$ = « limite infero des u ».

❋ 5·0 $Q = l'`[Cls'R \frown u \ni (\exists u \,.\, \exists R - \eta u)]$ Df Q

Q, lege «quantitate reale positivo» es omni limite supero de aliquo classe u de rationale, existente, et tale que existe aliquo rationale maiore de omni u.

·1 $R \supset Q$ [P3·4 $\supset$ P]

·2 $u \varepsilon Cls'R \,.\, \exists u \,.\, \exists R - \eta u \,.\supset.\, l'u \,\varepsilon\, Q$

❋ 6·0 $\infty = l'R$ Df

∞, lege «infinito», es limite supero de rationales.

·1 $l'N_1 = \infty$ [$l'N_1 = l'\eta N_1 = l'R = \infty$]

·2 $u \varepsilon Cls'R \,.\supset.\, l'u \leqq \infty$

·3 $u \varepsilon Cls'R \,.\, \exists u \,.\, -\exists R - \eta u \,.\supset.\, l'u = \infty$

·4 $u \varepsilon Cls'R \,.\, \exists u \,.\supset.\, l'u \,\varepsilon\, Q \cup \iota\infty$ [P5·2. P6·3 $.\supset$ P]

❋ 7. $a,b,c \varepsilon Q \cup \iota\infty \,.\supset.$

·0 $a+b = l'[R \frown x \ni (x<a) + R \frown x \ni (x<b)]$ Df

·1 $u,v \varepsilon Cls'R \,.\, \exists u \,.\, \exists v \,.\supset.\, l'u + l'v = l'(u+v)$ Distrib$(l',+)$

[Hp $.\supset.$ $l'u, l'v \,\varepsilon\, Q \cup \iota\infty$. P·0 $.\supset.$ $l'u + l'v = l'[R \frown x \ni (x < l'u) + R \frown x \ni (x < l'v)] = l'(\eta u + \eta v) = l'\eta(u+v) = l'(u+v)$]

·2 $a+b = b+a$

[$u,v \varepsilon Cls'R \,.\, l'u=a \,.\, l'v=b \,.\supset.\, a+b = l'u+l'v = l'(u+v) = l'(v+u) = l'v+l'u = b+a$

·3 $a+(b+c) = (a+b)+c$

[$u,v,w \varepsilon Cls'R \,.\, a = l'u \,.\, b = l'v \,.\, c = l'w \,.\supset.$
$a+(b+c) = l'u+(l'v+l'w) = l'u+l'(v+w) = l'[u+(v+w)] =$
$l'[(u+v)+w] = l'(u+v)+l'w = (l'u+l'v)+l'w = (a+b)+c$]

·4 $a,b \varepsilon Q \,.\supset.\, a+b \,\varepsilon Q$ ·5 $a+\infty = \infty$

❋ 8. $a,b,c,d \varepsilon Q \,.\supset:$ ·1 $b>a \,.=.\, b \varepsilon a+Q$ Dfp

·2 $b>a \,.=.\, b+c > a+c$ ·3 $b>a \,.\, d>c \,.\supset.\, b+d > a+c$

❋ 9. $a,b,c \,\varepsilon Q \cup \iota\infty \,.\supset:$

·0 $a \times b = l'[R \frown x \ni (x<a) \times R \frown x \ni (x<b)]$ Df

·1 $u,v \varepsilon Cls'R \,.\, \exists u \,.\, \exists v \,.\supset.\, l'u \times l'v = l'(u \times v)$ Distrib$(l',\times)$

[Hp $.\supset.$ $l'u, l'v \varepsilon Q \cup \iota\infty \,.\supset.\, l'u \times l'v = l'[R \frown x \ni (x<l'u) \times R \frown x \ni (x<l'v)] =$
$l'[\eta u \times \eta v] = l'\eta(u \times v) = l'(u \times v)$]

·2 $ab = ba$ ·3 $a(bc) = (ab)c = abc$

[$(\times \mid +)$P7·2·3 $\supset$ P9·2·3]

·4 $a(b+c) = ab+ac$

[$u,v,w \varepsilon \text{Cls'R} \,.\, a = \text{l}'u \,.\, b = \text{l}'v \,.\, c = \text{l}'w \,.\supset.\, a(b+c) = \text{l}'u \times (\text{l}'v + \text{l}'w) = \text{l}'u \times \text{l}'(v+w) = \text{l}'[u \times (v+w)] = \text{l}'[\eta u(v+w)] = \text{l}'(\eta uv + \eta uw) = \text{l}'(\eta uv) + \text{l}'(\eta uw) = \text{l}'(uv) + \text{l}'(uw) = (\text{l}'u)(\text{l}'v) + (\text{l}'u)(\text{l}'w) = ac+bc$]

·5 $a,b \varepsilon \text{Q} \,.\supset.\, a \times b \,\varepsilon \text{Q}$ ·6 $a \times \infty = \infty$

✻ 10. $a,b,c,d \varepsilon \text{Q} \,.\supset:$

·1 $a > b \,.=.\, ac > bc$ ·2 $a > b \,.\, c > d \,.\supset.\, ac > bd$

·3 $a > b \,.\, c > d \,.\supset.\, ac+bd > ad+bc$ [$\S\text{N}_1\text{P}6 \supset \text{P}{\cdot}1{\cdot}{\cdot}3$]

✻ 11. $a,b \varepsilon \text{Q} \,.\supset.$ ·0 $/a = \iota\, \text{Q} \frown x \ni (x \times a = 1)$ Df

·1 $/a \,\varepsilon \text{Q}$ ·2 $/(/a) = a$ ·3 $/(ab) = (/a)(/b)$

·4 $b/a = b \times (/a)$ Df

·5 $a = b \,.=.\, /a = /b \,.=.\, a/b = 1$

·6 $x \varepsilon \text{Q} \,.\, ax = b \,.=.\, x = b/a$

✻ 12·0 $\theta = \text{Q} \frown x \ni (x < 1)$ Df

$(\theta, \text{Q}) \mid (\eta, \text{R})$ $\S\eta$

lege « theta », indica quantitate minore de 1.

✻ 13. $u,v \varepsilon \text{Cls'Q} \,.\, \exists u \,.\, \exists v \,.\, a \varepsilon \text{Q} \,.\supset:$

·0 $\text{l}'u = \text{l}'(\text{R} \frown \theta u)$: $\text{l}_{,}u = \text{l}_{,}(\text{R} \frown u/\theta)$ Df

·1 $a = \text{l}'u \,.=.\, \theta a = \theta u$: $a = \text{l}_{,}u \,.=.\, a/\theta = u/\theta$

·2 $\infty = \text{l}'u \,.=.\, \theta u = \text{Q}$: $0 = \text{l}_{,}u \,.=.\, u/\theta = \text{Q}$

·3 $\text{l}'(u+v) = \text{l}'u + \text{l}'v$: $\text{l}_{,}(u+v) = \text{l}_{,}u + \text{l}_{,}v$

·4 $\text{l}'(u \times v) = \text{l}'u \times \text{l}'v$: $\text{l}_{,}(u \times v) = \text{l}_{,}u \times \text{l}_{,}v$

·5 $\text{l}'u \,\varepsilon \text{Q} \,.\supset.\, \text{l}_{,}(/u) = /\text{l}'u$

·6 $\text{l}'u = \infty \,.=.\, \text{l}_{,}/u = 0$

·7 $\text{l}'(u \cup v) = \infty \,.=.\, \text{l}'u = \infty \,.\cup.\, \text{l}'v = \infty$

·8 $\text{l}_{,}(u \cup v) = 0 \,.=.\, \text{l}_{,}u = 0 \,.\cup.\, \text{l}_{,}v = 0$

✻ 14·0 $a \varepsilon \text{Q} \,.\, b \varepsilon\, a+\text{Q} \,.\supset.\, b-a = \iota\, \text{Q} \frown x \ni (a+x = b)$ Df

Hp·0 $.\supset.$ ·1 $b-a \,\varepsilon \text{Q}$ ·2 $b-a+a = b$ ·3 $(\text{Q} \mid \text{N}_0)$ $\S-$

Q ↾

❋ 20. $a,b \varepsilon Q \cup \iota\infty \,.\, m,n \varepsilon N_0 \,.\supset.$

·0 $a^0 = 1 \quad . \quad a^{m+1} = a^m \times a$ Df

·1 $u \varepsilon \text{Cls'R} \,.\, \exists u \,.\supset.\, (l'u)^m = l'(u^m)$

[$u^0 = \iota 1 \,.\, l'(u^0) = 1 \,.\, (l'u)^0 = 1 \,.\supset.\, l'(u^0) = (l'u)^0$ (1)

$m \varepsilon N_0 \,.\, l'(u^m) = (l'u)^m \,.\supset.\, (l'u)^{m+1} = (l'u)^m \times (l'u) =$

$l'(u^m) \times l'u = l'(u^m \times u) = l'(\eta u^m u) = l'\eta u^{m+1} = l'u^{m+1}$ (2)

(1) . (2) . Induct $.\supset.$ P]

·2 $a^m a^n = a^{m+n}$

·3 $(ab)^m = a^m b^m$ ·4 $(a^m)^n = a^{mn}$

·5 $a \varepsilon Q \,.\supset.\, a^m \varepsilon Q$ ·6 $\infty^m = \infty$

·7 $u \varepsilon \text{Cls'Q} \,.\, \exists u \,.\supset.\, l'(u^m) = (l'u)^m \quad . \quad l_{\iota}(u^m) = (l_{\iota}u)^m$

§↾ P2. 3. §N_1 P11-14. §R P24.

❋ 21. $a \varepsilon Q \,.\, m,n,p,q \varepsilon N_1 \,.\supset.$

·0 $a ↾ /m = {}^m\sqrt{a} = \imath\, Q \frown x \ni (x^m = a)$ Df

·01 $\sqrt{a} = {}^2\sqrt{a}$ Df

·1 ${}^m\sqrt{a}\, \varepsilon Q$ ·2 $({}^m\sqrt{a})^m = a$ ·3 ${}^1\sqrt{a} = a$

·4 ${}^m\sqrt{(ab)} = ({}^m\sqrt{a})({}^m\sqrt{b})$ Distrib(√, ×)

[$({}^m\sqrt{a}\; {}^m\sqrt{b})^m = ({}^m\sqrt{a})^m ({}^m\sqrt{b})^m = ab$. Oper ${}^m\sqrt{}$ $.\supset.$ P]

·5 ${}^m\sqrt{/a} = /\, {}^m\sqrt{a}$ Comm(√, /)

[${}^m\sqrt{/a} \times {}^m\sqrt{a} = {}^m\sqrt{(a/a)} = {}^m\sqrt{1} = 1 \,.\supset.\, {}^m\sqrt{/a} = /\,{}^m\sqrt{a}$

·6 ${}^m\sqrt{(a^n)} = ({}^m\sqrt{a})^n$ Comm(√, ↾)

[$[({}^m\sqrt{a}) ↾ n] ↾ m = ({}^m\sqrt{a}) ↾ (mn) = [({}^m\sqrt{a}) ↾ m] ↾ n = a ↾ n$. Oper ${}^m\sqrt{}$ $.\supset.$ P]

·7 ${}^n\sqrt{({}^m\sqrt{a})} = {}^{mn}\sqrt{a}$

[$[{}^n\sqrt{({}^m\sqrt{a})}] ↾ (mn) = \{[{}^n\sqrt{({}^m\sqrt{a})}] ↾ n\} ↾ m = ({}^m\sqrt{a}) ↾ m = a$. Oper ${}^{mn}\sqrt{}$ $.\supset.$ P]

·8 $n/m = q/p \,.\supset.\, {}^m\sqrt{a^n} = {}^p\sqrt{a^q}$

[Hp $.\supset.$ $({}^m\sqrt{a} ↾ n) ↾ (mp) = a ↾ (np)$. $({}^p\sqrt{a} ↾ q) ↾ (mp) = a ↾ (mq)$. $(np) =$
mq $.\supset.$ $({}^m\sqrt{a} ↾ n) ↾ (mp) = ({}^p\sqrt{a} ↾ q) ↾ (mp)$. Oper ${}^{mp}\sqrt{}$ $.\supset.$ P]

❋ 22·0 $a \varepsilon Q \,.\, m \varepsilon R \,.\supset.$

$a ↾ m = \imath\, y \ni [p,q \varepsilon N_1 \,.\, m = p/q \,.\supset_{p,q}\, .\, y = (a ↾ p) ↾ /q]$ Df

·1 $a \varepsilon Q \,.\, p,q \varepsilon N_1 \,.\supset.\, a ↾ (p/q) = {}^q\sqrt{(a^p)}$

·2 $a \varepsilon 1 + Q \,.\, m \varepsilon Q \,.\supset.\, a ↾ m = l'a ↾ (R \frown \theta m)$ Df

·3 $a\varepsilon\, 1+Q \,.\, m\varepsilon Q \,.\supset.\, a\upharpoonleft m = 1_{\prime}\, a\upharpoonleft(R\frown m/\theta)$ Dfp

·4 $a\,\varepsilon\,\theta \,.$ » » $= 1_{\prime}\, a\upharpoonleft(R\frown \theta m)$ Df

·5 » » » » $= 1'\, a\upharpoonleft(R\frown m/\theta)$ Dfp

✻ 23. $a,b,m,n\varepsilon Q \,.\supset.$ ·1 $a\upharpoonleft m\;\varepsilon\; Q$

·2 $a^m a^n = a^{m+n}$

[$a\varepsilon\, 1+Q \,.\supset.\, a\upharpoonleft m \times a\upharpoonleft n = 1'a\upharpoonleft(R\frown\theta m)]\times[1'a\upharpoonleft(R\frown\theta n)]$
$= 1'\{[a\upharpoonleft(R\frown\theta m)]\times[a\upharpoonleft(R\frown\theta n)]\}$
$= 1'a\upharpoonleft[(R\frown\theta m)+(R\frown\theta n)]$
$= 1'a\upharpoonleft[R\frown\theta(m+n)]$
$= a\upharpoonleft(m+n)$]

·3 $(ab)^m = a^m b^m$ ·4 $(a^m)^n = a^{mn}$

·5 $a>b \,.=.\, a^m>b^m$ ·6 $a>1 \,.\supset:\, m>n \,.=.\, a^m>a^n$

·7 $a<1 \,.\supset:\, m>n \,.=.\, a^m<a^n$

✻ 24. $a,b,c,d\varepsilon R \,.\supset:$

·1 $a+\surd b = \surd c \,.\supset.\, b\varepsilon R^2 \,.\, c\varepsilon R^2$

[Hp . Oper $\upharpoonleft 2 \,.\supset.\, a^2+b+2a\surd b = c \,.\supset.\, \surd b = (c-a^2-b)/(2a) \,.\supset.$ P]

·2 $b\, -\varepsilon R^2 \,.\, a+\surd b = c+\surd d \,.\supset.\, a=c \,.\, b=d$

[$a>c \,.\, a+\surd b = c+\surd d \,.\supset.\, (a-c)+\surd b = \surd d$. P·1 $.\supset.\, b\varepsilon R^2$ (1)
$a<c \,.$ » » » $(c-a)+\surd d = \surd b$ » » » (2)
$a -= c \,.$ » » . (1) . (2) $.\supset.\, b\varepsilon\, R^2$. Transp $.\supset.$ P]

·3 $\surd a+\surd b = \surd c \,.\supset.\, a/b\;\varepsilon\; R^2$

[Hp $.\supset.\, a+b+2\surd(ab) = c$. P·1 $.\supset.\, a\times b\;\varepsilon\; R^2 \,.\supset.$ Ths]

·4 $a/b\, -\varepsilon R^2 \,.\, a>b \,.\, c>d \,.\, \surd a+\surd b = \surd c+\surd d \,.\supset.\, a=c \,.\, b=d$

[Hp $.\supset.\, a+b+2\surd(ab) = c+d+2\surd(cd) \,.\, ab\, -\varepsilon\, R^2 \,.\supset.\, a+b=c+d \,.\, ab=cd$
$.\supset.\, (a-b)^2=(c-d)^2 \,.\supset.\, a-b=c-d \,.\supset.$ Ths]

·5 $a/b\, -\varepsilon\, R^2 \,.\supset.\, \surd a+\surd b+\surd c -= \surd d$

·6 $b\, -\varepsilon\, R^2 \,.\, \surd(a+\surd b) = \surd c+\surd d \,.\, c>d \,.\supset.$
$a^2-b\;\varepsilon\; R^2 \,.\, c = [a+\surd(a^2-b)]/2 \,.\, d = [a-\surd(a^2-b)]/2$

{ ·1, ·2-·3, ·4, ·5, ·6 Euclide x P26, 42, 79, 111, 59-96 }

✻ 25. $a,b\varepsilon Q \,.\supset:$

·1 $a^2-b\;\varepsilon Q \,.\supset.\, \surd(a+\surd b) = \surd\{[a+\surd(a^2-b)]/2\} + \surd\{[a-\surd(a^2-b)]/2\}$

·2 » $-$ » $-$ »

·3 $a-b\;\varepsilon Q \,.\supset.\, \surd(\surd a+\surd b) =$
$\sqrt[4]{}[\{2a-b+2\surd[a(a-b)]\}/4] + \sqrt[4]{}[\{2a-b-2\surd[a(a-b)]\}/4]$

❋ 26. $a,b \varepsilon \mathrm{Q} \,.\supset:$

·1 $\sqrt{(ab)} \leqq (a+b)/2$ $\quad [\, (a+b)^2 > 4ab \,.\supset.\, \mathrm{P}]$

·2 $\sqrt{(a+b)} > \sqrt{a} + \sqrt{b} - \sqrt[4]{(ab/4)}$

·3 $b > a \,.\supset.\, (b-a)^2/(8b) < (a+b)/2 - \sqrt{(ab)} < (b-a)^2/(8a)$
$[\, (a+b)/2 - \sqrt{(ab)} = (a-b)^2/[2(a+b+2\sqrt{(ab)})] \,]$

❋ 30.

·0 $m,n,x,y \varepsilon \mathrm{Q} \,.\, x -= y \,.\supset.\, x^m y^n < [(mx+ny)/(m+n)]^{m+n}$
$[m,n \varepsilon \mathrm{R} \,.\, p \varepsilon \mathrm{N}_1 \,.\, mp,np \varepsilon \mathrm{N}_1 \,.\, \S\mathrm{R}33{\cdot}6 \,.\supset.\, x \upharpoonleft (mp) \times y \upharpoonleft (np) <$
$[(mpx+npy)/(mp+np)] \upharpoonleft (mp+np) \,.\, \mathrm{Oper} \upharpoonleft /p \,.\supset.\, \mathrm{P} \,]$

·1 $m,n,x \varepsilon \mathrm{Q} \,.\, x < n \,.\supset.\, (1+x/m)^m (1-x/n)^n < 1$
$[(1+x/m,\, 1-x/n) \mid (x,y)\ \mathrm{P}{\cdot}0 \,.\supset.\, \mathrm{P} \,]$

·2 $m,x \varepsilon \mathrm{Q} \,.\, mx\, \varepsilon \theta \,.\supset.\, (1+x)^m (1-mx) < 1$
$[\, (mx, 1) \mid (x, n)\ \mathrm{P}{\cdot}1 \,.\supset.\, \mathrm{P} \,]$

·3 $m \varepsilon \mathrm{Q} \,.\, x \varepsilon \theta \,.\supset.\, (1-x)^m (1+mx) < 1$
$[\, (x/m) \mid x\ \mathrm{P}{\cdot}1 \,.\supset.\, (1+x/m)^m (1-x) < 1$ (1)
$(/m) \mid m\ (1) \,.\supset.\, (1+mx) \upharpoonleft /m\, (1-x) < 1 \,.\, \mathrm{Oper} \upharpoonleft m \,.\supset.\, \mathrm{P} \,]$

·4 $m,x \varepsilon \mathrm{Q} \,.\supset.\, (1+x)^{m+1} > 1+(m+1)x$
$[\, x/(1+x) \mid x\ \mathrm{P}{\cdot}2 \,.\supset.\, [1-x/(1+x)]^m [1+mx/(1+x)] < 1$
$.\supset.\, /(1+x)^m\ [1+(m+1)x]/(1+x) < 1$
$.\supset.\, 1+(m+1)x < (1+x)^{m+1} \,]$

·5 $m \varepsilon\, 1+\mathrm{Q} \,.\, x \varepsilon \mathrm{Q} \,.\supset.\, (1+x)^m > 1+mx$
$[(m-1) \mid m\ \mathrm{P}{\cdot}3 \,.\supset.\, \mathrm{P} \,]$
$[(m-1,1,1,1+mx) \mid (m,n,x,y)\ \mathrm{P}{\cdot}0 \,.\supset.\, \mathrm{P} \,]$

·6 $m \varepsilon \theta \,.\, x \varepsilon \mathrm{Q} \,.\supset.\, (1+x)^m < 1+mx$
$[\, (1/m,\, mx) \mid (m,x)\ \mathrm{P}{\cdot}5 \,.\supset.\, \mathrm{P}{\cdot}6 \,]$

·7 $m \varepsilon\, 1+\mathrm{Q} \,.\, x, mx \varepsilon \theta \,.\supset.\, (1-x)^m > 1-mx$

·8 $m,x \varepsilon \theta \,.\supset.\, (1-x)^m < 1-mx$

❋ 31.

·1 $m,n \varepsilon \mathrm{Q} \,.\, m < n \,.\, x \varepsilon \mathrm{Q} \,.\supset.\, (1+x/m)^m < (1+x/n)^n$
$[\, \mathrm{Hp} \,.\supset.\, n/m > 1 \,.\supset.\, (1+x/n) \upharpoonleft (n/m) > 1+x/m \,.\, \mathrm{Oper} \upharpoonleft m \,.\supset.\, \mathrm{P} \,]$

·2 $m < n \,.\supset.\, (1+/m)^m < (1+/n)^n$ $\quad [\, \mathrm{P}{\cdot}1 \,.\, x=1 \,.\supset.\, \mathrm{P} \,]$
$[(n-m, 1+/m, 1) \mid (n,x,y)\ \mathrm{P}30{\cdot}0 \,.\supset.\, \mathrm{P} \,]$

·3 $m,n \varepsilon \mathrm{Q} \,.\, n > m \,.\, x \varepsilon \mathrm{Q} \,.\, x < m \,.\supset.\, (1-x/m)^m < (1-x/n)^n$
$[\, \mathrm{Hp} \,.\supset.\, n/m > 1 \,.\supset.\, (1-x/n) \upharpoonleft (n/m) > 1-x/m \,.\, \mathrm{Oper} \upharpoonleft m \,.\supset.\, \mathrm{P} \,]$

·4 $m,n \varepsilon \mathrm{Q} \,.\, n > m > 1 \,.\supset.\, (1-/m)^m < (1-/n)^n$

·5 $m,n,x\varepsilon Q \,.\, m<n \,.\supset.\, (1+x/m)^{m+x} > (1+x/n)^{n+x}$

[$(m+x,\, n+x) \mid (m,n)$ P·3 $.\supset.$ $[m/(m+x)]^{m+x} < [n/(n+x)]\restriction(n+x)$.
Oper/ $.\supset.$ P]

·6 $m,n\varepsilon Q \,.\, m<n \,.\supset.\, (1+/m)^{m+1} > (1+/n)^{n+1}$

·7 $m,n\varepsilon Q \,.\, n>1 \,.\supset.\, (1+/m)^m(1-/n)^n < 1$

[P30·1 . $x=1$ $.\supset.$ P]

·8 $m,n\varepsilon Q \,.\supset.\, (1+/m)^m < (1+/n)^{n+1}$

[$(n-1) \mid n$ P·7 $\supset$ P]

[$[n+1,\, (m+1)/m,\, n/(n+1)] \mid (n,x,y)$ P30·0 $.\supset.$ P]

·9 $a,b\varepsilon Q \,.\, a -= b \,.\supset.\, (a+b)\restriction(a+b) < a\restriction a\; b\restriction b\; 2\restriction(a+b)$

[$(b,a,a,b) \mid (m,n,x,y)$ P30·0 $.\supset.$ P]

✻ 32.

·1 $a,b\varepsilon Q \,.\, m\varepsilon\, 1+Q \,.\supset.\, (a+b)^m > a^m+ma^{m-1}b$

[$(b/a) \mid x$ P30·5 . Oper $\times(a\restriction m)$ $.\supset.$ P]

·2 $a\varepsilon Q \,.\, b\varepsilon\, \theta a \,.\, m\varepsilon\, 1+Q \,.\supset.\, (a-b)^m > a^m-ma^{m-1}b$

[$(b/a) \mid x$ P30·7 . Oper $\times(a\restriction m)$ $.\supset.$ P]

·3 $a,b\varepsilon Q \,.\, m\varepsilon 1+Q \,.\supset.\, (a+b)^m < a^m+m(a+b)^{m-1}b$

[$(a+b) \mid a$ P·2 $.\supset.$ $a^m > (a+b)^m - m(a+b)^{m-1}b$ $.\supset.$ P]

·4 $a\varepsilon Q \,.\, b\varepsilon\, \theta a \,.\, m\varepsilon 1+Q \,.\supset.\, (a-b)^m < a^m-m(a-b)^{m-1}b$

[$(a-b) \mid a$ P·1 $.\supset.$ P]

·5 $a,b\varepsilon Q \,.\, a<b \,.\, m\varepsilon\, 1+Q \,.\supset.$

$$ma^{m-1} < (b^m-a^m)/(b-a) < mb^{m-1}$$

·6 $a,b\varepsilon Q \,.\, m\varepsilon\, 1+N_1 \,.\supset.\, \sqrt[m]{}(a^m+b) < a+b/(ma^{m-1})$

·7 $a,b\varepsilon Q \,.\, m\varepsilon\, 1+N_1 \,.\, b<ma^m \,.\supset.\, \sqrt[m]{}(a^m-b) < a-b/(ma^{m-1})$

[$b/(ma^{m-1}) \mid b$ P·1·2 . Oper $\sqrt[m]{}$ $.\supset.$ P·6·7]

✻ 33. $a\varepsilon\, Q-\iota 1 \,.\, m\varepsilon\, Q+1 \,.\supset.$ ·1 $a^m+/a^m > a+/a$

·2 $(a^m-/a^m)/(a-/a) > m$

·3 $a^m+/a^m > m^2(a+/a)-2(m^2-1)$

✻ 34. $a,b,c\varepsilon Q \,.\supset:$

·0 $D(a,b) = l_{\prime}\mathrm{mod}[(\mathrm{n}a+\mathrm{n}b)-\iota 0]$ Df

·1 $D(a,b)\, \varepsilon Q_0$ ·2 $D(a,b) = D(b,a)$

·3 $a\varepsilon\, Q-R \,.=.\, D(1,a)=0$

·4 » $.\, b\varepsilon R \,.\supset.\, D(a,b)=0$

·5 $D(a,b)\, \varepsilon Q \,.\, D(a,c)=0 \,.\supset.\, D(b,c)=0$

·6 $D(ac,bc) = cD(a,b)$ ·7 $a>b \,.\supset.\, D(a,b) = D(a-b,b)$

·8 $m\varepsilon N_1 \,.\supset.\, D(a+mb,b) = D(a,b)$

§13 q

❋ 1·0 $q = Q \cup -Q \cup \iota 0$ Df q

$(Q_0, q) \mid (N_0, n)$ §n P1-32

q = « quantitate » vel « numero reale ». Es numero positivo, aut negativo aut nullo.

❋ 2·0 $a,b \varepsilon q \,.\supset:\, x = a \pm b \,.=.\, x = a+b \,.\cup.\, x = a-b$ Df±

·1 $a \varepsilon q \,.\supset:\, x \varepsilon q \,.\, x^2 = a \,.=.\, a \geqq 0 \,.\, x = \pm\sqrt{a}$

·2 $a,b \varepsilon q \,.\supset:\, x \varepsilon q \,.\, x^2+2ax+b=0 \,.=.\, a^2 \geqq b \,.\, x = -a \pm \sqrt{(a^2-b)}$

[$x^2+2ax+b=0 \,.=.\, (x+a)^2 = a^2-b$
$.=.\, a^2-b \geqq 0 \,.\, x+a = \pm\sqrt{(a^2-b)} \,.=.\, \dots$]

¦ Euclide vi P28, 29 ¦

¦ Leonardo Pisano a.1202 p.407 :

« (Si) volueris invenire quantitatem census [x^2], qui cum datis radicibus [$+2ax$] equetur numero dato [$= -b$], sic facias: accipe quadratum medietatis radicum [a^2], et adde eum super numerum datum [a^2-b]; et eius, quod provenerit, radicem accipe [$\sqrt{(a^2-b)}$]; de qua numerum medietatis radicum tolle [$\sqrt{(a^2-b)}-a$]; et quod remanserit erit radix quesiti census. » ¦

·3 $a,b,c \varepsilon q \,.\, a -= 0 \,.\supset:$

$x \varepsilon q \,.\, ax^2+bx+c=0 \,.=.\, b^2-4ac \geqq 0 \,.\, x = [-b \pm \sqrt{(b^2-4ac)}]/(2a)$

[$ax^2+bx+c=0 \,.=.\, 4a^2x^2+4abx+4ac=0 \,.=.\, (2ax+b)^2-b^2+4ac=0$
$.=.\, (2ax+b)^2 = b^2-4ac \,.=.\, \dots$]

¦ Brahmagoupta a.598, Versione de Rodet p. 75 :

« Mets le nombre connu dans le côté opposé à celui où sont... l'inconnue et son carré. Au nombre connu, multiplié par quatre fois le nombre des x^2 ajoute le carré du coefficient du terme moyen; la racine de ceci, moins le coefficient du terme moyen, étant divisée par deux fois le nombre des carrés est la valeur de x. » ¦

·4 $a,b \varepsilon Q \,.\supset:\, a(a-b) = b^2 \,.=.\, b = a(\sqrt{5}-1)/2 \,.=.\, a = b(\sqrt{5}+1)/2$
$.=.\, a^2+(a-b)^2 = 3b^2 \,.=.\, b(a+b) = a^2 \,.=.\, a-b = a(3-\sqrt{5})/2$

¦ Euclide xiii P1-6 ¦

✻ 3. $a,b,x,y \varepsilon \mathrm{q} \;.\supset:$

·1 $x,y \varepsilon \mathrm{q} \;.\; x+y=2a \;.\; xy=b \;.\; x>y \;.=.$
$$a^2>b \;.\; x=a+\sqrt{(a^2-b)} \;.\; y=a-\sqrt{(a^2-b)}$$

[$x+y=2a \;.\; xy=b \;.\supset.\; (x+y)^2-4xy=4(a^2-b)$ (1)
$x+y=2a \;.\; (x+y)^2-4xy=4(a^2-b) \;.\supset.\; xy=b$ (2)
(1) . (2) $.\supset:$
$x+y=2a \;.\; xy=b \;.\; x>y \;.=.\; x+y=2a \;.\; (x+y)^2-4xy=4(a^2-b) \;.\; x>y$
$.=.$ » $.\; (x-y)^2=4(a^2-b) \;.\; x>y$
$.=.$ » $.\; a^2-b>0 \;.\; x-y=2\sqrt{(a^2-b)}$
$.=.$ ...]

{ Diophanto I P27, 30 }

·2 $x,y \varepsilon \mathrm{q} \;.\; x+y=2a \;.\; x^2+y^2=2b^2 \;.\; x>y \;.=.$
$$b^2>a^2 \;.\; x=a+\sqrt{(b^2-a^2)} \;.\; y=a-\sqrt{(b^2-a^2)}$$

[$x+y=2a \;.\; x^2+y^2=2b^2 \;.\; x>y$
$.=.\; x+y=2a \;.\; (x+y)^2-(x^2+y^2)=4a^2-2b^2 \;.\; x>y$
$.=.$ » $.\; xy=2a^2-b^2 \;.\; x>y$
$.=.$ ...]

{ Diophanto I P28 }

·3 $a,b \varepsilon \mathrm{Q} \;.\supset:\; x,y \varepsilon \mathrm{Q} \;.\; x^2+y^2=a \;.\; xy=b \;.\; x>y \;.=.$
$$a>2b \;.\; x=\sqrt{(a+2b)}+\sqrt{(a-2b)} \;.\; y=\sqrt{(a+2b)}-\sqrt{(a-2b)}$$

[$x^2+y^2=a \;.\; xy=b \;.\; x>y \;.=.\; x^2+y^2+2xy=a+2b \;.\; x^2+y^2-2xy=a-2b$ $.\; x>y \;.=.\; (x+y)^2=a+2b \;.\; (x-y)^2=a-2b \;.\; x>y \;.=.\; x+y=\sqrt{(a+2b)} \;.\; a-2b>0 \;.\; x-y=\sqrt{(a-2b)} \;.=.$ ...]

{ Vieta Opera l.2 P2 }

·4 $x,y \varepsilon \mathrm{q} \;.\; x^3+y^3=a \;.\; x+y=b \;.=.\; x+y=b \;.\; 3bxy=b^3-a$

[$x^3+y^3=a \;.\; x+y=b \;.=.\; x+y=b \;.\; (x+y)^3-(x^3+y^3)=b^3-a$
$.=.$ » $.\; 3xy(x+y)=b^3-a$
$.=.$ ...] { Diophanto IV P1 }

·5 $x^4+y^4=a \;.\; x+y=b \;.=.\; x+y=b \;.\; 2(xy)^2-4b^2xy+b^4-a=0$

[$x^4+y^4=a \;.\; x+y=b \;.=.\; x+y=b \;.\; (x+y)^4-(x^4+y^4)=b^4-a$
$.=.$ » $.\; 4xy(x^2+y^2)+4(xy)^2=b^4-a$
$.=.$ » $.\; 4xy[(x+y)^2-2xy]+6(xy)^2=b^4-a$
$.=.$ » $.\; 4xy(b^2-2xy)+6(xy)^2=b^4-a$
$.=.$ ...]

·6 $x^5+y^5=a \;.\; x+y=b \;.=.\; x+y=b \;.\; 5bxy(b^2-xy)=b^5-a$

[$x^5+y^5=a \;.\; x+y=b \;.=.\; x+y=b \;.\; (x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)=a$
$.=.$ » $.\; b(x^4-x^3y+x^2y^2-xy^3+y^4)=a$
$.=.$ » $.\; b[(x+y)^4-5x^3y-5x^2y^2-5xy^3]=a$
$.=.$ » $.\; b^5-5bxy(x^2+xy+y^2)=a$
$.=.$ » $.\; 5bxy[(x+y)^2-xy]=b^5-a$
$.=.$ ...]

✻ 4·0 $a \varepsilon \mathrm{q} \,.\, m \varepsilon 2\mathrm{N}_1+1 \,.\supset.\, \sqrt[m]{a} = \iota\, \mathrm{q} \frown x \ni (x^m = a)$ Df

·1 $a,b,u,v\, \varepsilon \mathrm{Q} \,.\, u+v = b \,.\, uv = (a/3)^3 \,.\supset:$
$\iota\, \mathrm{Q} \frown x \ni (x^3 = ax + b) = \sqrt[3]{u} + \sqrt[3]{v}$
$[\, (\sqrt[3]{u}+\sqrt[3]{v})^3 = 3(\sqrt[3]{u}+\sqrt[3]{v})\sqrt[3]{(uv)}+u+v \,.\supset.\, (\sqrt[3]{u}+\sqrt[3]{v})^3 = a(\sqrt[3]{u}+\sqrt[3]{v})+b \,]$

{ N. Tartaglia a.1546 p.123:

... Quando che'l cubo restasse lui solo [$x^3 = ax + b$]
Tu osseruarai quest'altri contratti.
Del numer farai due tal part'a volo [$b = u+v$]
Che l'una in l'altra si produca schietto
El terzo cubo delle cose in stolo. [$uv = (a/3)^3$]
Delle qual poi, per commun precetto
Torrai li lati cubi insieme gionti
Et cotal summa sarà il tuo concetto. [$x = \sqrt[3]{u} + \sqrt[3]{v}$]
... Questi trovai, et non con passi tardi
Nel mille cinquecent'e quatro e trenta
Con fondamenti ben sald'e gagliardi
Nella città dal mar'intorno centa. }

$a,b \varepsilon \mathrm{q} \,.\supset:$ ·2 $b^2+a^3>0 \,.\supset.$
$\iota\, \mathrm{q} \frown x \ni (x^3+3ax+2b = 0) = \sqrt[3]{[-b+\sqrt{(b^2+a^3)}]} + \sqrt[3]{[-b-\sqrt{(b^2+a^3)}]}$

·3 $b^2+a^3 = 0 \,.\supset.\, \mathrm{q} \frown x \ni (x^3+3ax+2b = 0) = \iota(\sqrt[3]{b}) \cup \iota(-2\sqrt[3]{b})$

·4 $b^2+a^3 < 0 \,.\supset.\, \mathrm{num}[\mathrm{q} \frown x \ni (x^3+3ax+2b = 0)] = 3$

·5 $a,b \varepsilon \mathrm{Q} \,.\, 8a>b \,.\supset.$
$\sqrt[3]{\{a+(b+a)\sqrt{[(8a-b)/(27b)]}\}} + \sqrt[3]{[a-(b+a)\sqrt{[(8a-b)/(27b)]}]} = \sqrt[3]{b}$

·6 $a \varepsilon \mathrm{q} \,.\, b \varepsilon \mathrm{Q} \,.\supset.$
$\sqrt[3]{[a^3+3ab+(3a^2+b)\sqrt{b}]} + \sqrt[3]{[a^3+3ab-(3a^2+b)\sqrt{b}} = 2a$

✻ 5·0 $a,m \varepsilon \mathrm{Q} \,.\supset.\, a^{-m} = /(a^m)$ Df

$a,b \varepsilon \mathrm{Q} \,.\, m,n \varepsilon \mathrm{q} \,.\supset.$ ·1 $a^m \varepsilon \mathrm{Q}$ ·2 $a^m a^n = a^{m+n}$

·3 $(ab)^m = a^m b^m$ ·4 $(a^m)^n = a^{mn}$

·5 $a>b \,.=.\, a^m>b^m$ ·6 $a>1 \,.\supset:\, m>n \,.=.\, a^m>a^n$

·7 $a<1 \,.\supset:\, m>n \,.=.\, a^m<a^n$

✻ 6. $a,b \varepsilon \mathrm{q} \,.\supset.$

·0 $\mathrm{mod}a = \iota\, \mathrm{Q}_0 \frown x \ni (a = +x \,.\cup.\, a = -x)$

·1 $\mathrm{mod}a\, \varepsilon\, \mathrm{Q} \cup \iota 0$ ·2 $\mathrm{mod}(a+b) \leqq \mathrm{mod}a + \mathrm{mod}b$

·3 $\mathrm{mod}(-a) = \mathrm{mod}a$ ·4 $\mathrm{mod}(a \times b) = \mathrm{mod}a \times \mathrm{mod}b$

·5 $a -= 0 \,.\supset.\, \mathrm{mod}/a = /\mathrm{mod}a$

·6 $m\varepsilon \mathrm{N}_1 \,.\supset.\, \mathrm{mod}(a^m) = (\mathrm{mod}a)^m$ ·7 $\mathrm{mod}a = \sqrt{a^2}$

* 7·0 $x\varepsilon \mathrm{q} \,.\supset.\, \mathrm{E}x = \max \mathrm{n} \frown a\ni(a \leqq x)$ Df

(q | R) §E

* 8. $n\varepsilon\, \mathrm{N}_1+1 \,.\, x\varepsilon \mathrm{Q} \,.\supset.$

·0 $\mathrm{E}(\sqrt[n]{x}) = \iota\, \mathrm{N}_0 \frown z\ni[z^n \leqq x < (z+1)^n]$ Dfp

[$\mathrm{Df\,E} \,.\supset.\, \mathrm{E}\, \sqrt[n]{x} = \iota\, \mathrm{N}_0 \frown z\ni[z \leqq \sqrt[n]{x} < z+1] = \iota\, \mathrm{N}_0 \frown z\ni[z^n \leqq x < (z+1)^n]$]

·1 $\mathrm{E}(\sqrt[n]{x}) = \mathrm{E}(\sqrt[n]{\mathrm{E}x})$

[$\mathrm{P}{\cdot}0 \,.\supset.\, (\mathrm{E}\, \sqrt[n]{x})^n \leqq x < (\mathrm{E}\, \sqrt[n]{x}+1)^n \,.\, \mathrm{Oper E} \,.\supset.$
$(\mathrm{E}\, \sqrt[n]{x})^n \leqq \mathrm{E}x < \mathrm{E}\, \sqrt[n]{x}+1)^n \,.\, \mathrm{P}{\cdot}0 \,.\supset.\, \mathrm{P}$]

·2 $a\varepsilon\, 1+\mathrm{Q} \,.\supset.\, \sqrt{a} \leqq \mathrm{E}\sqrt{a} + [a-(\mathrm{E}\sqrt{a})^2]/(2\mathrm{E}\sqrt{a})$

·3 » » $> \mathrm{E}\sqrt{a} + [a-(\mathrm{E}\sqrt{a})^2]/(2\mathrm{E}\sqrt{a}+1)$

[$x = \mathrm{E}\sqrt{a} \,.\, y = \sqrt{a}-x \,.\supset.\, x\varepsilon \mathrm{N}_1 \,.\, 0 \leqq y \dot{<} 1 \,.\, \sqrt{a} = x+y \,.\supset.$
$a = x^2+2xy+y^2 \,.\supset.\, x^2+2xy \leqq a < x^2+(2x+1)y \,.\supset.$
$(a-x^2)/(2x) \geqq y > (a-x^2)/(2x+1)$]

·4 $a\varepsilon\, 1+\mathrm{Q} \,.\supset.\, \sqrt[3]{a} - \mathrm{E}\sqrt[3]{a} \lesseqgtr [a-(\sqrt[3]{a})^3]/[3(\mathrm{E}\sqrt[3]{a})^2]$

·5 » » $> [a-(\sqrt[3]{a})^3]/[3(\mathrm{E}\sqrt[3]{a})^2 + 3\mathrm{E}\sqrt[3]{a}+1]$

[$x = \mathrm{E}\sqrt[3]{a} \,.\, y = \sqrt[3]{a}-x \,.\supset.\, a = (x+y)^3 = x^3+3x^2y+3y^2+y^3 \,.\supset.$
$(a-x^3)/(3x^2) \geqq y > (a-x^3)/(3x^2+3x+1)$]

·6 $\mathrm{ord}\, \sqrt[n]{x} = \mathrm{E}\,(\mathrm{ord}\, x)/n$

[$n\, \mathrm{ord}\, \sqrt[n]{x} \leqq \mathrm{ord}\, x < n\, (\mathrm{ord}x+1)$]

* 9. $a\varepsilon \mathrm{q} \,.\supset.$ ·1 $a+\infty = \infty+a = \infty$ Df

·2 $a-\infty = (-\infty)+a = -\infty -\infty = -\infty$ Df

·3 $-\infty < a < +\infty \,.\, -\infty < +\infty$ Df

Ad expressione $\infty - \infty$ nos tribue nullo sensu.

·4 $\iota\underline{+}\infty = \iota+\infty \cup \iota-\infty$ Df

$u\varepsilon\, \mathrm{Cls}'(\mathrm{q} \cup \iota\underline{+}\infty) \,.\supset.$

·5 $\max u = \iota\, u \frown a\ni(x\varepsilon u \,.\supset_x.\, x \leqq a)$ Df

·6 $\min u =$ ————————————— $\geqq a)$ Df

·7 $\mathrm{num}u\, \varepsilon \mathrm{N}_1 \,.\supset.\, \max u, \min u\, \varepsilon u$

❋ 10. $u \varepsilon \text{Cls}'(\text{q} \cup \iota \pm \infty) . \supset .$

·0 $\text{l}'u = \min(\text{q} \cup \iota \pm \infty) \frown a \ni (x \varepsilon u . \supset_x . x \leqq a)$ Df

·01 $\text{l}_{,}u = \max($ » $\geqq a)$ Df

Si u es classe de quantitate finito aut infinito, tunc limite supero (infero) des u es minimo (maximo) de quantitates majore (minore) aut aequale ad omni u.

·1 $\text{l}'u, \text{l}_{,}u \; \varepsilon \; \text{q} \cup \iota \pm \infty$ [§Q P 6·4 . ⊃ . P]

Omni classe u de quantitates habe semper limite supero, et limite infero, finito aut infinito. Es ipso definitione de numero reale, rationale aut irrationale, considerato ut limite supero de classe de rationales.

·2 $u = \Lambda . \supset . \text{l}'u = -\infty . \text{l}_{,}u = +\infty$

·3 $\text{num}\, u = 1 . \supset . \text{l}'u = \text{l}_{,}u = \iota u$

·4 $\text{num}\, u > 1 . \supset . \text{l}'u > \text{l}_{,}u$

·5 $\text{l}'u \, \varepsilon u . \supset . \text{l}'u = \max u$: $\text{l}_{,}u \, \varepsilon u . \supset . \text{l}_{,}u = \min u$

❋ 11. $u, v \varepsilon \text{Cls}'\text{q} . a \varepsilon \text{q} . \supset :$

·1 $a = \text{l}'u .=. a - \text{Q} = u - \text{Q}$: $a = \text{l}_{,}u .=. a + \text{Q} = u + \text{Q}$

·2 $+\infty = \text{l}'u .=. u - \text{Q} = \text{q}$: $-\infty = \text{l}_{,}u .=. u + \text{Q} = \text{q}$

·3 $u \supset v . \supset . \text{l}'u \leqq \text{l}'v . \text{l}_{,}u \geqq \text{l}_{,}v$

·4 $u \supset v : y \varepsilon v . \supset_y . \exists\, u \frown x \ni (x \leqq y) : \supset . \text{l}_{,}u = \text{l}_{,}v$

·5 » » $\geqq$ » $\text{l}'u = \text{l}'v$

·6 $\text{l}'(u \cup v) = \max(\iota \text{l}'u \cup \iota \text{l}'v)$. $\text{l}_{,}(u \cup v) = \min(\iota \text{l}_{,}u \cup \iota \text{l}_{,}v)$

❋ 12. $u, v \varepsilon \text{Cls}'(\text{q} \cup \iota \pm \infty) - \iota \Lambda . \supset .$

·1 $\text{l}'(u+v) = \text{l}'u + \text{l}'v$. $\text{l}_{,}(u+v) = \text{l}_{,}u + \text{l}_{,}v$

[$\text{l}'u + \text{l}'v - \text{Q} = (\text{l}'u - \text{Q}) + (\text{l}'v - \text{Q}) = u - \text{Q} + v - \text{Q} = u + v - \text{Q}$]

·2 $\text{l}'(-u) = -\text{l}_{,}u$

[$-\text{l}_{,}u - \text{Q} = -(\text{l}_{,}u + \text{Q}) = -(u + \text{Q}) = (-u) - \text{Q}$]

·3 $m \varepsilon \text{Q} . \supset . \text{l}'(mu) = m\text{l}'u . \text{l}_{,}(mu) = m\text{l}_{,}u$

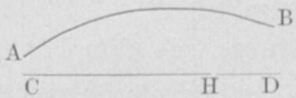
[$m\text{l}'u - \text{Q} = m(\text{l}'u - \text{Q}) = m(u - \text{Q}) = mu - \text{Q}$]

Inter differentes applicatione de limite supero et infero, nos cita sequente ad Geometria.

Si nos nosce area de polygono plano, nos pote defini area de omni figura in plano, ut « limite supero de area de polygonos interno ad figura dato », aut ut « limite infero de area de polygonos que contine in suo interno figura dato ».

Si nos nosce volumen de polyhedro, nos pote defini volumen de solido arbitrario ut limite supero de volumen de polyhedro interno ad illo, aut ut limite infero de volumen de polyhedro continente illo in proprio interno.

Si nos habe arcu de curva, longitudo de arcu es limite supero de longitudo de polygonales inscripto in illo.



Ita affirmatione « arcu AB in longitudine aequa segmento CD » vale « CD aequa limite supero de linea polygonale inscripto in AB », et, si nos elimina expressione « limite supero » nunc definito, affirmatione praecedente vale:

1º Nullo polygonale inscripto in AB supera CD.

2º Si nos sume segmento arbitrario CH minore de CD, nos pote inscribe in AB polygonale majore de CH.

Si arcu AB es plano, et convexo, suo longitudo et vale limite infero de lineas polygonale circumscripto.

Idea de limite supero es valore plus simplo inter differente sensu de vocabulo « limite », que nos decompone in l′, $l_{\prime}$, λ, Λ, δ Lm, lim.

✻ 13.

·1 $f \varepsilon \, \mathrm{qfr} : y,z \varepsilon \mathrm{r} \, . \supset_{y,z} . \; f(y+z) = fy+fz : x \varepsilon \mathrm{r} : \supset . \; fx = (f1)x$

[$\mathrm{Hp} . \supset . \; fx = f(x+0) = fx+f0 . \supset . \; f0 = 0$ (1)

$\mathrm{Hp} . \supset . \; fx+f-x = f[x+(-x)] = f0 = 0 . \supset . \; f-x = -fx$ (2)

$\mathrm{Hp} . n \varepsilon \mathrm{N}_1 . f(nx) = nfx . \supset . \; f[(n+1)x] = f(nx)+fx = nfx+fx = (n+1)fx$ (3)

$\mathrm{Hp} . n \varepsilon \mathrm{N}_1 . (3) . \mathrm{Induct} . \supset . \; f(nx) = nfx$ (4)

$\mathrm{Hp} . m \varepsilon \mathrm{n} . (1) . (2) . (4) . \supset . \; f(mx) = mfx$ (5)

$\mathrm{Hp} . n \varepsilon \mathrm{N}_1 . (4) . \supset . \; fx = f[n(x/n)] = nf(x/n) . \supset . \; f(x/n) = (fx)/n$ (6)

$\mathrm{Hp} . m \varepsilon \mathrm{n} . n \varepsilon \mathrm{N}_1 . (5) . (6) . \supset . \; f[(m/n)x] = f[m(x/n)] = (m/n)fx$ (7)

» » $. (1 \mid x)\mathrm{P}(7) . \supset . \; f(m/n) = (m/n)(f1)$ (8)

$(8) . \supset : \mathrm{P}$]

·2 $f \varepsilon \mathrm{qfq} : y,z \varepsilon \mathrm{q} \,.\supset_{y,z} .\, f(y+z) = fy + fz : x \varepsilon \mathrm{q{-}r} \,.\, fx - xf1 \neq 0$
$:\supset.\ \mathrm{l}'f`\theta = \infty$

[$g = (fz - xf1) \mid z \,.\, m,n \varepsilon \mathrm{r} \,.\supset.\, g(mx+n) = mgx$
« $.\, m \varepsilon \mathrm{r} \,.\supset.\, \exists\, \theta \frown (mx + \mathrm{r}) \,.\supset.\, mgx \,\varepsilon\, g`\theta$
« $.\supset.\, \mathrm{r} \times gx \supset g`\theta \,.\, gx \neq 0 \,.\supset.\, \mathrm{l}'(\mathrm{r} \times gx) = \infty$
$.\supset.\, \mathrm{l}'g`\theta = \infty \,.\supset.\, \mathrm{P}$]

·3 $f \varepsilon \mathrm{qfq} : y,z \varepsilon \mathrm{q} \,.\supset_{y,z}.\, f(y+z) = fy + fz : \mathrm{l}'f`\theta\, \varepsilon \mathrm{q} : x \varepsilon \mathrm{q} :\supset.$
$fx = (f1) \times x$
[P ·2 $.\supset.$ P]
{ Euler, vide §lim P31 ; Darboux MA. a.1880 t.17 p.55 }

Si f, functio reale de variabile reale, es distributivo ad $+$, id es satisfac ad aequatio $f(y+z) = fy + fz$, pro omni valore de y et de z, et si limite supero de valores de f in aliquo intervallo finito, p. ex. in intervallo θ, es finito, tunc functio coincide cum multiplicatio per numero, id es, pro omni valore de x, fx vale $(f1) \times x$.

·4 $f \varepsilon \mathrm{qfq} : y,z \varepsilon \mathrm{q} \,.\supset_{y,z}.\, f(y+z) = fy \times fz : \mathrm{l}'f`\theta\, \varepsilon \mathrm{q} : f1 \neq 0 :$
$x \varepsilon \mathrm{q} :\supset.\, f1 \,\varepsilon\, \mathrm{Q} \,.\, fx = (f1) \uparrow x$
{ Cauchy a.1821 p.103 }

$- \quad -\!\dashv \quad \vdash\!- \quad \vdash\!\dashv$

❋ 14. $a,b \varepsilon \mathrm{q} \cup \iota \pm \infty \,.\supset.$

·0 $a \,\bar{}\, b = \mathrm{q} \frown x \ni (a < x < b \,.\cup.\, a > x > b)$ Df

·1 $a \vdash\!\dashv b = a \,\bar{}\, b \cup \iota a \cup \iota b$ Df

·2 $a \vdash\!- b = a \,\bar{}\, b \cup \iota a \quad .\quad a -\!\dashv b = a \,\bar{}\, b \cup \iota b$ Df

·3 $b \,\bar{}\, a = a \,\bar{}\, b \,.\, b \vdash\!\dashv a = a \vdash\!\dashv b \,.\, a \vdash\!\dashv a = \iota a$

·4 $-\infty \,\bar{}\, \infty = \mathrm{q} \quad .\quad 0 \,\bar{}\, \infty = \mathrm{Q} \quad .\quad -\infty \vdash\!\dashv \infty = \mathrm{q} \cup \iota\infty \cup \iota{-\infty}$

·5 $\theta = 0 \,\bar{}\, 1 \,.\, \Theta = 0 \vdash\!\dashv 1$ Dfp

·6 $a,b \varepsilon \mathrm{q} \,.\supset.\, a \vdash\!\dashv b = a + \Theta(b-a)$ Dfp

·7 » $.\, a \neq b \,.\supset.\, a \,\bar{}\, b = a + \theta(b-a)$ Dfp

Notationes de intervallo sine extremo, vel cum ambo extremo, vel cum uno, adoptato in §cont, §D, §S, §vct.

§14 Log

❋ 1. $a, b \varepsilon \mathrm{Q}{-}\iota 1 \,.\, x, y \varepsilon \mathrm{Q} \,.\, m \varepsilon \mathrm{q} \;.\supset.$

·0 $^a\mathrm{Log}x = \imath\, \mathrm{q} \frown z \ni (a \uparrow z = x)$ Df

$=$ logarithmo in basi a de x.

·1 $^a\mathrm{Log}x \;\varepsilon \mathrm{q}$ ·2 $a \uparrow (^a\mathrm{Log}x) = x$

·3 $^a\mathrm{Log}(a \uparrow m) = m$ ·4 $^a\mathrm{Log}1 = 0$ ·5 $^a\mathrm{Log}a = 1$

·6 $^a\mathrm{Log}(x \times y) = {}^a\mathrm{Log}x + {}^a\mathrm{Log}y$

[P·2 $.\supset.$ $^a\mathrm{Log}(xy) = {}^a\mathrm{Log}[a \uparrow (^a\mathrm{Log}x) \times a \uparrow (^a\mathrm{Log}y)]$

§q 5·2 » $= {}^a\mathrm{Log}[a \uparrow (^a\mathrm{Log}x + {}^a\mathrm{Log}y)]$

P·3 » $= {}^a\mathrm{Log}x + {}^a\mathrm{Log}y$]

·7 $^a\mathrm{Log}/x = - {}^a\mathrm{Log}x$

[$^a\mathrm{Log}/x = {}^a\mathrm{Log}/[a \uparrow (^a\mathrm{Log}x)] = {}^a\mathrm{Log}[a \uparrow (- {}^a\mathrm{Log}x)] = - {}^a\mathrm{Log}x$]

·8 $^a\mathrm{Log}\, x^m = m \times {}^a\mathrm{Log}x$

$^a\mathrm{Log}(x^m) = {}^a\mathrm{Log}[(a \uparrow {}^a\mathrm{Log}x)^m] = {}^a\mathrm{Log}[a \uparrow (m \times {}^a\mathrm{Log}x)] = m \times {}^a\mathrm{Log}x$]

·9 $^a\mathrm{Log}x = {}^b\mathrm{Log}x \times {}^a\mathrm{Log}b$

[$x = b \uparrow (^b\mathrm{Log}x)$. Oper $^a\mathrm{Log}$ $.\supset.$ P] { NEPER, a.1614 {

❋ 2·0 $x \varepsilon \mathrm{Q} \;.\supset.\; \mathrm{Log}x = {}^{\mathrm{X}}\mathrm{Log}x$ Df

·1 » $\mathrm{E\,Log}x = \mathrm{ord}\, x$

[$\mathrm{E\,Log}x = \imath\, \mathrm{n} \frown y \ni (y \leqq \mathrm{Log}x < y+1) = \imath\, \mathrm{n} \frown y \ni [\mathrm{X} \uparrow y \leqq x < \mathrm{X} \uparrow (y+1)] = \mathrm{ord}x$]

·2 $x \varepsilon \mathrm{Q} \,.\, n \varepsilon \mathrm{N}_1 \;.\supset.\; \mathrm{Log}(\mathrm{X}^n x) = n + \mathrm{Log}x$

TABULA DE LOG. AD TRES DECIMALE

	10	*20*	*30*	*40*	*50*	*60*	*70*	*80*	*90*
0	000	301	477	602	698	778	845	903	954
1	041	322	491	612	707	785	851	908	959
2	079	342	505	623	716	792	857	913	963
3	113	361	518	633	724	799	863	919	968
4	146	380	531	643	732	806	869	924	973
5	176	397	544	653	740	812	875	929	977
6	204	414	556	662	748	819	880	934	982
7	230	431	568	672	755	826	886	939	986
8	255	447	579	681	763	832	892	944	991
9	278	462	591	690	770	838	897	949	995

Exemplo : Log 47 = 1 (per P2·1) + ·672 (ex tabula, in verticale cum titulo 40 et in horizontale cum titulo 7).

§15 Σ

✻ 1·0 $m\varepsilon N_0 . f\varepsilon \text{ q f } 0\cdots(m+1) .\supset. \Sigma(f, 0\cdots 0) = f0 .$
$\Sigma[f, 0\cdots(m+1)] = \Sigma(f, 0\cdots m) + f(m+1)$ Df

·1 $a\varepsilon N_0 . b\varepsilon a+N_0 . f\varepsilon \text{ q f } a\cdots b .\supset.$
$\Sigma(f, a\cdots b) = \Sigma[f(a+r)|r, 0\cdots(b-a)]$ Df

·2 $u\varepsilon \text{Cls'q} . r\varepsilon N_1 .\supset. \min_1 u = \min u .$
$\min_{r+1} u = \min[u\frown(\min_r u + Q)]$ Df

·21 $u\varepsilon \text{Cls'q} . \text{num}u\, \varepsilon N_1 .\supset. \min_{\text{num}u} u = \max u$

·22 $n\varepsilon N_1 .\supset. \min_n N_1 = n$

·3 $u\varepsilon \text{Cls'q} . \text{num}u\, \varepsilon N_1 .\supset. \Sigma u = \Sigma(\min_r u|r, 1\cdots \text{num}u)$ Df

·4 $u\varepsilon \text{Cls} . \text{num}u\, \varepsilon N_1 . f\varepsilon \text{qf}u .\supset. \Sigma(f,u) =$
$\imath x \ni [g\varepsilon (u \text{f } 1\cdots\text{num}u)\text{rcp} .\supset_g . x = \Sigma(fg, 1\cdots\text{num}u)]$ Df

·5 $u\varepsilon \text{Cls} . f\varepsilon \text{qf}u . \text{num}[u\frown x\ni(fx -= 0)]\, \varepsilon N_1 .\supset.$
$\Sigma(f,u) = \Sigma[f, u\frown x\ni(fx -= 0)]$ Df

·6 $u\varepsilon \text{Cls} . \text{num}u\, \varepsilon N_1 . f\varepsilon (\text{qf}u)\text{sim} .\supset. \Sigma(f,u) = \Sigma(f\text{‘}u)$

Si u es classe, et si f es quantitate functione des u, tunc $\Sigma(f, a\cdots b)$ vale « summa de valores de f, quando variabile varia in u ».

Illo es definito (P·0) si classe u habe forma $0\cdots m$, vel (P·1) forma $a\cdots b$.

P·3 defini summa de numeros de uno classe, que es summa de numeros disposito in ordine crescente. Ce ordine es indicato per operatione $\min_r u$, lege « minimo de loco r des u » (P2). Minimo de loco uno vale minimo; minimo de loco numero des u vale maximo des u (P·21).

P·4 defini summa de valores de functione f, quando variabile varia in classe arbitrario u. Nos dispone in ordine individuos de classe u. Occurre in P15.

Signo Σ occurre in Lagrange a.1772 t.3 p.451. Cauchy scribe:

$\sum_m^n {}_r\, fr$, ubi signo Σ porta tres indice m,n,r.

✻ 2.

·1 $n\varepsilon N_0 . f,g\varepsilon \text{ q f } 0\cdots n .\supset: \Sigma[(fr+gr)|r, 0\cdots n] =$
$\Sigma(f,0\cdots n) + \Sigma(g,0\cdots n)$ Distrib(Σ, +)

·2 $n\varepsilon N_0 . a\varepsilon q . f\varepsilon\, q f 0\cdots n .\supset. \Sigma[(a\times fr)|r, 0\cdots n] = a\Sigma(f, 0\cdots n)$
Comm($a\times,\Sigma$)

·3 $n\varepsilon N_0 . f\varepsilon\, q f 0\cdots n .\supset. \Sigma(f, 0\cdots n) = \Sigma[f(n-r)|r, 0\cdots n]$

·4 $a,b,c\varepsilon N_0 . a<b<c . f\varepsilon\, q f a\cdots c .\supset.$
$\Sigma(f, a\cdots c) = \Sigma(f, a\cdots b) + \Sigma[f, (b+1)\cdots c]$

·5 $n\varepsilon N_0 . f\varepsilon\, q f 0\cdots(n+1) .\supset.$
$\Sigma\{[f(r+1)-fr]|r, 0\cdots n\} = f(n+1)-f0$

❋ 3. $m\varepsilon N_1 .\supset.$

·1 $\Sigma(1\cdots m) = m(m+1)/2$

[$\Sigma(1\cdots m) = \Sigma(r|r, 0\cdots m) = \Sigma[(m-r)|r, 0\cdots m]$ (1)
$\Sigma(r|r, 0\cdots m) + \Sigma[(m-r)|r, 0\cdots m] = \Sigma(m|r, 0\cdots m) = m(m+1)$ (2)
(1) . (2) .⊃. P]

·2 $\Sigma[(2r+1)|r, 0\cdots m] = (m+1)^2$

·3 $a,b\varepsilon q . n\varepsilon N_1 .\supset. \Sigma\{[a+r(b-a)/n]|r, 0\cdots n\} = (n+1)(a+b)/2$

[$\Sigma\{[a+r(b-a)/n]|r, 0\cdots n\} = \Sigma\{[a+(n-r)(b-a)/n]|r, 0\cdots n\}$
$= \Sigma\{[b-r(b-a)/n]|r, 0\cdots n\}$ (1)
$\Sigma\{a+r(b-a)/n]|r, 0\cdots n\} + \Sigma\{b-r(b-a)/n|r, 0\cdots n\}$
$= \Sigma\{(a+b)|r, 0\cdots n\} = (n+1)(a+b)$ (2)
(1) . (2) .⊃. P]

❋ 4·1 $n\varepsilon N_1 . x\varepsilon\, q-\iota 1 .\supset. \Sigma(x^r|r, 0\cdots n) = (x^{n+1}-1)/(x-1)$

[P2·5 .⊃. $x^{n+1}-1 = \Sigma[(x^{r+1}-x^r)|r, 0\cdots n] = \Sigma[x^r(x-1) \mid r, 0\cdots n] =$
$(x-1)\Sigma[x^r|r, 0\cdots n]$. Oper $/(x-1)$.⊃. P]

·2 $a,b\varepsilon q . a-=b . n\varepsilon N_1 .\supset.$
$\Sigma(a^{n-r}b^r|r, 0\cdots n) = (b^{n+1}-a^{n+1})/(b-a)$ { Euclide ix P35 }

[$\Sigma(a^{n-r}b^r|r, 0\cdots n) = a^n\Sigma[(b/a)^r|r, 0\cdots n]$
$= a^n[(b/a)^{n+1}-1]/[(b/a)-1] = \ldots$]

❋ 5. $m\varepsilon N_1 .\supset.$

·1 $\Sigma[r(r+1)/2\, |r, 1\cdots m] = m(m+1)(m+2)/6$

[$3\Sigma[r(r+1)\,|r, 1\cdots m] = 3\Sigma[(r+1)(r+2)\,|r, 0\cdots(m-1)]$
$= \Sigma\{[(r+1)(r+2)(r+3) - r(r+1)(r+2)]\,|r, 0\cdots(m-1)\}$.⊃. P]

{ Aryabhata a.500 P21 }

·2 $\Sigma[r(r+1)(r+2)/6\,|r, 1\cdots m] = m(m+1)(m+2)(m+3)/24$
{ Alqâchânî a.1589 p.247 }

·3 $\Sigma\,(1\cdots m)^2 = m(m+1)(2m+1)/6$

$[\ \Sigma(r^2|r, 1\cdots m) = \Sigma\{[r(r+1)-r]\,|r, 1\cdots m\} = m(m+1)(m+2)/3 - m(m+1)/2\ ]$

$[\ (n+1)^3 = \Sigma\{[(r+1)^3-r^3]|r, 0\cdots n\} = \Sigma[(3r^2+3r+1)|r, 0\cdots n] = 3\Sigma(r^2|r,0\cdots n) +$
$\Sigma[(3r+1)|r,0\cdots n] = 3\Sigma(r^2|r,0\cdots n)+(3n+2)(n+1)/2 \,.\supset.\, \Sigma(r^2|r,0\cdots n) =$
$[(n+1)^3-(3n+2)(n+1)/2]/3 = \ldots\ ]$

{ Archimede *Περὶ Ἑλικῶν* P10 }

·4 $\Sigma\,(1\cdots m)^3 = [m(m+1)/2]^2$

$[\ r\varepsilon N_1 \,.\supset.\, r^3 = (r-1)r(r+1)+r\ ]$

·5 $\Sigma[(2r+1)^2|r, 0\cdots m] = (m+1)(2m+1)(2m+3)/3$

·6 $\Sigma[(2r+1)^3|r, 0\cdots m] = (m+1)^2(2m^2+4m+1)$

{ Ibn Albanna a.1369 p.5,6 }

✻ 6. $m,n\varepsilon N_1 \,.\, s_m = \Sigma\,(1\cdots n)^m \,.\supset.$

·1 $s_4 = n(n+1)(2n+1)(3n^2+3n-1)/30$

{ Alqâchânî p.247; Fermat a.1636 t.2 p.69:

« Exponantur quotlibet numeri in progressione naturali ab unitate; si a quadruplo ultimi, binario aucto et in quadratum trianguli numerorum ducto, demas summam quadratorum a singulis, fiet quintuplum quadratoquadratorum a singulis. » }

$s_5 = n^2(n+1)^2(2n^2+2n-1)/12$

$s_6 = n(n+1)(2n+1)[3n^2(n+1)^2-(3n^2+3n-1)]/42$

$s_7 = n^2(n+1)^2[3n^2(n+1)^2-2(2n^2+2n-1)]/24$

{ Wallis a.1655 t.1 p.381 }

$s_8 = n(n+1)(2n+1)[5n^3(n+1)^3-10n^2(n+1)^2+3(3n^2+3n-1)]/90$

$s_9 = n^2(n+1)^2[2n^3(n+1)^3-5n^2(n+1)^2+3(2n^2+2n-1)]/20$

$s_{10} = n(n+1)(2n+1)[3n^4(n+1)^4-10n^3(n+1)^3+17n^2(n+1)^2 -$
$5(3n^2+3n-1)]/66$

$s_{11} = n^2(n+1)^2[2n^4(n+1)^4-8n^3(n+1)^3+17n^2(n+1)^2-$
$10(2n^2+2n-1)]/24$

{ Jac. Bernoulli a.1713 p.97 }

·2 $s_3 = s_1^2$ $\quad 5s_4 = s_2(6s_1-1)$ $\quad 3s_5 = 4(s_1)^3 - s_3$

$7s_6 = 12s_2s_3 - 5s_4$ $\quad s_5+s_7 = 2s_1^4$

$9s_2^2 = s_1^2(8s_1+1)$ $\quad 3s_5 = s_1^2(4s_1-1)$ $\quad s_5 = 4s_1^3 - 3s_2^2$

$12(s_2)^3 = 16s_6 - 5s_4 + s_2$

$2s_7 = 4s_3^2 - 3s_2^2 + s_1^2$ $\quad 2s_5 = 3s_2^2 - s_1^2$

$3s_7 = s_1^2(6s_1^2-4s_1+1)$ $\quad s_7 = 2s_1^3(s_1-2)+3s_2^2$

$$3s_8 = 8s_1^2 s_2(s_1-1)+3s_4 \qquad 5s_9 = 4s_1^2[4s_1^2(s_1-1)+4s_1-1]$$

$$11s_{10} = 4s_1^2 s_2(12s_1^2-20s_1+17)-25s_4 = s_2(48s_1^4-80s_1^3+68s_1^2-30s_1+5)$$

$$33s_{10} = 4s_1^2 s_2(36s_1^2-10s_1+1)-75s_8$$

$$3s_{11} = 16s_1^6-32s_1^5+34s_1^4-15s_5 = s_1^2(16s_1^4-32s_1^3+34s_1^2-20s_1+5)$$

❋ 7. $n\varepsilon \mathrm{N}_1 \,.\, m\varepsilon \mathrm{Q} \;.\supset.$

·1 $\Sigma\,(1\cdots n)^m > n^{m+1}/(m+1)$

[§Q32·3 . $r\varepsilon\mathrm{N}_1 \;.\supset.\; (r+1)\uparrow(m+1) - r\uparrow(m+1) < (m+1)(r+1)\uparrow m$ (1)
(1) . $\mathrm{Oper}[\Sigma, 0\cdots(n-1)] \;.\supset.\; n\uparrow(m+1) < (m+1)\Sigma\,(1\cdots n)\uparrow m$

·2 $\Sigma\,(1\cdots n)^m < (n+1)^{m+1}/(m+1)$

[§Q32·1 . $r\varepsilon\mathrm{N}_1 \;.\supset.\; (r+1)\uparrow(m+1) - r\uparrow(m+1) > (m+1)\,r\uparrow m$ (1)
$\mathrm{Oper}(\Sigma, 1\cdots n) \;.\supset.$ P]

❋ 8. $m\varepsilon\, \mathrm{N}_1+1 \,.\, x,y,a\varepsilon\, \mathrm{q\,f}\,1\cdots m \;.\supset.$

·1 $\Sigma(x_r^2|r, 1\cdots m)\times\Sigma(y_r^2|r, 1\cdots m) - [\Sigma(x_r y_r|r, 1\cdots m)]^2 =$
$\Sigma\}\Sigma[(x_r y_s - x_s y_r)^2|\, s, (r+1)\cdots m]\,|r, 1\cdots(m-1)\{$

·2 $\Sigma(a_r x_r^2|r, 1\cdots m)\times\Sigma(a_r y_r^2|r, 1\cdots m) - [\Sigma(a_r x_r y_r|r, 1\cdots m)]^2 =$
$\Sigma\}\Sigma[a_r a_s (x_r y_s - x_s y_r)^2\,|s, (r+1)\cdots m\,]\,|r, 0\cdots(m-1)\{$

·3 $m,n\varepsilon\, \mathrm{N}_1 \,.\, a\varepsilon\, \mathrm{Q\,f}\,1\cdots n \;.\supset.\; \Sigma(a_r^m|r, 1\cdots n)/n \gtreqless [\Sigma(a, 1\cdots n)/n]^m$

❋ 10.

·1 $n\varepsilon\mathrm{N}_1 \,.\, a\varepsilon\, \mathrm{n\,f}\,1\cdots n \;.\supset.\; \mathrm{r}\frown x\ni[x^n+\Sigma(a_r x^{n-r}|r, 1\cdots n)=0]\supset\mathrm{n}$

❋ 11. $a,b\varepsilon\mathrm{N}_1 \;.\supset.$

·0 $n\varepsilon\mathrm{N}_1 \,.\, f\varepsilon\, \mathrm{N}_1\mathrm{f}1\cdots n \;.\supset.$
$\mathrm{rest}[\Sigma(f,1\cdots n), a] = \mathrm{rest}\}\Sigma[\mathrm{rest}(fr, a)\,|r, 1\cdots n], a\{$
[§rest 4·6 $.\supset.$ P]

·1 $a = \Sigma(\mathrm{X}^r\mathrm{Cfr}_r a\,|r, 0\cdots \mathrm{ord}a)$

·2 $\mathrm{rest}(a,b) = \mathrm{rest}\}\Sigma[\mathrm{Cfr}_r a \times \mathrm{rest}(\mathrm{X}^r,b)\,|r, 0\cdots\mathrm{ord}a], b\{$

·3 $a\varepsilon\, b\times\mathrm{N}_1 \;.=.\; \Sigma[(\mathrm{Cfr}_r a)\times\mathrm{rest}(\mathrm{X}^r,b)\,|r, 0\cdots\mathrm{ord}a]\;\varepsilon\; b\times\mathrm{N}_0$
} PASCAL a.1654 t.3 p.312 {

·4 $a\varepsilon\, 3\mathrm{N}_0 \;.=.\; \Sigma[(\mathrm{Cfr}_r a)\,|r, 0\cdots\mathrm{ord}a]\;\varepsilon\; 3\mathrm{N}_0$

·5 » 9 » » » 9 »

·6 $a\varepsilon\, 11\mathrm{N}_0 \;.=.\; \Sigma[(-1)^r\mathrm{Cfr}_r a\,|r, 0\cdots\mathrm{ord}a]\;\varepsilon\; 11\mathrm{n}$

✻ 12·0 $f \varepsilon \mathrm{Q f N_0} \,.\supset.\, \Sigma(f, \mathrm{N_0}) = \mathrm{l}'[\Sigma(f, 0\cdots n) \mid n \text{‘} \mathrm{N_0}]$ Df

·1 $x \varepsilon \theta \,.\supset.\, \Sigma(x^r \mid r, \mathrm{N_0}) = 1/(1-x)$

$$\begin{aligned}[\ \Sigma(x^r \mid r, \mathrm{N_0}) &= \mathrm{l}'[\Sigma(x^r \mid r, 0\cdots n) \mid n\text{‘}\mathrm{N_0}] \\ &= \mathrm{l}'[(1-x^{n+1})/(1-x) \mid n\text{‘}\mathrm{N_0}] \\ &= \mathrm{l}'\{[1/(1-x) - x^{n+1}/(1-x)] \mid n\text{‘}\mathrm{N_0}\} \\ &= 1/(1-x) - \mathrm{l}_{,}[x^{n+1}/(1-x) \mid n\text{‘}\mathrm{N_0}] \\ &= 1/(1-x) - \mathrm{l}_{,}[x^{n+1} \mid n\text{‘}\mathrm{N_0}]/(1-x) \\ &= \quad \text{»} \quad - 0/(1-x) = 1/(1-x)\ ]\end{aligned}$$

·2 $x \varepsilon \theta \,.\, a \varepsilon \mathrm{Q} \,.\supset.\, \Sigma(a x^r \mid r, \mathrm{N_0}) = a/(1-x)$

·3 $a \varepsilon \mathrm{Q} \,.\, \mathrm{ord}\, a = 0 \,.\supset.\, a = \Sigma(\mathrm{X}^{-r} \mathrm{Cfr}_{-r} a \mid r, \mathrm{N_0})$

·4 $a \varepsilon \mathrm{Q} \,.\, m \varepsilon \mathrm{n} \,.\, n \varepsilon \mathrm{N_1} : p \varepsilon\, m + \mathrm{N_1} \,.\supset_p.\, \mathrm{Cfr}_{-p} a = \mathrm{Cfr}_{-p-n} a \,:\supset.$
$a = (\mathrm{E\, X}^{m+n} a - \mathrm{E X}^m a]/[\mathrm{X}^m(\mathrm{X}^n - 1)]$

·5 $x \varepsilon \mathrm{R} \,.\supset.$

$\exists (m,n) \ni (m \varepsilon \mathrm{n} \,.\, n \varepsilon \mathrm{N_1} : p \varepsilon\, m + \mathrm{N_1} \,.\supset_p.\, \mathrm{Cfr}_{-p} x = \mathrm{Cfr}_{-p-n} x)$

[$a, b \varepsilon \mathrm{N_1} \,.\, x = a/b \,.\supset.\, \mathrm{rest}(\mathrm{X}^m a, b) \mid m\text{‘}0\cdots b \supset 0\cdots(b-1) \,.\supset.$
$\exists(m,n) \ni [m \varepsilon\, 0\cdots b \,.\, n \varepsilon \mathrm{N_1} \,.\, m+n \leqq b \,.\, \mathrm{rest}(\mathrm{X}^m a, b) = \mathrm{rest}(\mathrm{X}^n a, b)$ (1)
$\mathrm{Hp}(1) \,.\, m \varepsilon\, 0\cdots b \,.\, n \varepsilon \mathrm{N_1} \,.\, m+n \leqq b \,.\, \mathrm{rest}(\mathrm{X}^m a, b) = \mathrm{rest}(\mathrm{X}^{m+n} a, b) \,.$
$p \varepsilon \mathrm{N_1} \,.\supset.\, \mathrm{rest}(\mathrm{X}^{m+p} a, b) = \mathrm{rest}(\mathrm{X}^{m+n+p} a, b) \,.\supset.\, \mathrm{Cfr}_{-m-p}(a/b) =$
$\mathrm{Cfr}_{-m-p-n}(a/b)$]

{ WALLIS a.1685, Opera t.2 p.364 :

« ... post processum (Reductionis Fractionum vulgarium ad Decimales) aliquatenus continuatum, redeunt iidem numeri, et eodem ordine circulantur quo prius... semper, si non citius, post tot locos uno minus quot sunt in Divisore unitates ». }

C ✻ 13. $m, n \varepsilon \mathrm{N_1} \,.\supset.$

·1 $\Sigma[\mathrm{C}(m,r) \mid r, 0\cdots m] = \Sigma(\mathrm{C}, \iota m \,\text{;}\, 0\cdots m) = 2^m$

{ HERIGONE a.1644 t.2 p.122 }

·2 $\Sigma\{\mathrm{C}(m,s) \mid s, (0\cdots m) \frown (2\mathrm{N_0})\} = \Sigma\{\mathrm{C}(m,s) \mid s, (0\cdots m) \frown (2\mathrm{N_0}+1)\} = 2^{m-1}$

{ JAC. BERNOULLI a.1713 p.104 }

·3 $m \varepsilon\, 6\mathrm{N_0} \,.\supset.\, \Sigma[\mathrm{C}(m, r) \mid r, (0\cdots m) \frown 3\mathrm{N_0}] = (2^m + 2)/3$

$m \varepsilon\, 6\mathrm{N_0}+1 \smile 6\mathrm{N_0}+5 \,.\supset.$ » $= (2^m + 1)/3$

$m \varepsilon\, 6\mathrm{N_0}+2 \smile 6\mathrm{N_0}+4 \,.\supset.$ » $= (2^m - 1)/3$

$m \varepsilon\, 6\mathrm{N_0}+3 \,.\supset.$ » $= (2^m - 2)/3$

·4 $\Sigma\{(-1)^r (2m+1-2r) \mathrm{C}(2m+1, r) \mid r, 0\cdots m\} = 0$

※ 14. $a,b\varepsilon q\,.\,m,n\varepsilon N_0\,.\supset .$

·0 $C(a,0)=1$ Df

·01 $C(a,n+1)=C(a,n)\times(a-n)/(n+1)$ Df

·02 $C(a,1)=a\,.\;C(a,2)=a(a-1)/2$

·1 $C(a+1,n+1)=C(a,n)\times(a+1)/(n+1)$

·2 $C(a+1,n+1)=C(a,n+1)+C(a,n)$

·3 $C(-a,n)=(-1)^nC(a+n-1,n)$ {Euler PetrA. a.1784 p.86}

·4 $a,b\varepsilon q\,.\,n\varepsilon N_0\,.\supset .\;C(a+b,n)=\Sigma[C(a,n-s)\,C(b,s)\mid s,0\cdots n]$
{Vandermonde ParisM. a.1772}

[$a,b\varepsilon q\,.\supset .\;C(a+b,0)=C(a,0)\,C(b,0)=1$ (1)

$a,b\varepsilon q\,.\,n\varepsilon N_0\,.\,C(a+b,n)=\Sigma[C(a,n-s)\,C(b,s)\mid s,0\cdots n]\,.\supset .$

$C(a+b,n+1)=\Sigma[C(a,n-s)\,C(b,s)\,(a+b-n)/(n+1)\mid s,0\cdots n]$

$=\Sigma\{C(a,n-s)\,C(b,s)\,[a-(n-s)+(b-s)]/(n+1)\mid s,0\cdots n\}$

$=\Sigma\{C(a,n-s+1)\,C(b,s)\,(n-s+1)/(n+1)\mid s,0\cdots n\}$

$+\Sigma\{C(a,n-s)\,C(b,s+1)\,(s+1)/(n+1)\mid s,0\cdots n\}$

$=C(a,n+1)\,C(b,0)+\Sigma\{C(a,n-s+1)\,C(b,s)\,(n-s+1)/(n+1)\mid s,1\cdots n\}$

$+\Sigma\{C(a,n-s+1)\,C(b,s)\,s/(n+1)\mid s,1\cdots n\}+C(a,0)\,C(b,n+1)$

$C(a+b,n+1)=\Sigma[C(a,n+1-s)\,C(b,s)\mid s,0\cdots(n+1)]$ (2)

(1). (2) . Induct $.\supset .$ P]

{Dm. Cayley, t. 8 p.462 a.1869}

·5 $\Sigma[C(a+r,n)\mid r,0\cdots p]=C(a+p+1,n+1)-C(a,n+1)$

·6 $a\varepsilon q\,.\,n\varepsilon N_1\,.\supset .\;\Sigma\{(-1)^rC(a,r)\mid r,0\cdots n\}=(-1)^nC(a-1,n-1)$

※ 15. $a,b\varepsilon q\,.\,n\varepsilon N_0\,.\supset .$

·1 $(a+b)^n=\Sigma\{[C(n,r)a^{n-r}b^r]\mid r,0\cdots n\}$

Exprime « formula de binomio ».

{Tschu Schi Kih a.1303; Stifel *Arithmetica integra* a.1544 liber 1 c. 5; Tartaglia a.1556 p.73; B. Pascal a.1665}

·2 $(a+b)^{2n+1}=a[\Sigma(-1)^r\,C(2n+1,2r)\,a^{n-r}b^r\mid r,0\cdots n]^2+$
$b[\Sigma(-1)^r\,C(2n+1,2r)\,a^rb^{n-r}\mid r,0\cdots n]^2$

·3 $(a+b)^{2n}=[\Sigma(-1)^r\,C(2n,2r)a^{n-r}b^r\mid r,0\cdots n]^2+$
$ab[\Sigma(-1)^r\,C(2n,2r+1)\,a^{n-r-1}b^r\mid r,0\cdots(n-1)]^2$

$a,b,c,x\varepsilon q\,.\,m,n\varepsilon N_1\,.\supset .$

·4 $(a+b)^n=\Sigma\{[C(n,r)\,(a+rc)^{n-r}b(b-rc)^{r-1}]\mid r,0\cdots n\}$
{Abel a.1826 t.1 p.102}

·5 $m < n \,.\supset.\, \Sigma\{(-1)^r \mathrm{C}(n,r)(a+rb)^m \mid r, 0\cdots n\} = 0$
} Euler a.1743 CorrM. t.1 p.264 {

·6 $(1+x+x^2)^m = \Sigma\{(x^{m-r}+x^{m+r}) \times \Sigma[\mathrm{C}(m, r+2s) \times \mathrm{C}(r+2s, s)$
$\mid s, 0\cdots m] \mid r, 1\cdots m\} + x^m \times \Sigma\{\mathrm{C}(m, 2s) \times \mathrm{C}(2s, s) \mid s, 0\cdots m\}$
} Euler a.1778 PetrNA. a.1794 t.12 p.47 {

·7 $a,b\varepsilon \mathrm{q} \,.\, n\varepsilon \mathrm{N}_1 \,.\supset.\, (a+b)^{2n+1} - a^{2n+1} - b^{2n+1} =$
$(2n+1)ab(a+b)\Sigma\{(a^2+ab+b^2)^{n-1-3r}[ab(a+b)]^{2r}\mathrm{C}(n-r-1,2r)/$
$(2r+1) \mid r, 0\cdots \mathrm{E}[(n-1)/3]\}$ } Muir QJ. a.1879 t.16 {

❋ 16. $m,n\varepsilon \mathrm{N}_1 \,.\supset.$

·1 $(m+1)\Sigma(1\cdots n)^m =$
$(n+1)^{m+1} - 1 - \Sigma\{\mathrm{C}(m+1,r) \times \Sigma(1\cdots n)^r \mid r, 0\cdots(m-1)\}$
[$\Sigma\{\mathrm{C}(m+1,r) \times \Sigma(1\cdots n)^r \mid r, 0\cdots(m-1)\} =$
$\Sigma\{\Sigma[\mathrm{C}(m+1,r)s^r \mid r, 0\cdots(m-1)] \mid s, 1\cdots n\} =$
$\Sigma\{[(1+s)^{m+1} - (m+1)s^m - s^{m+1}] \mid s, 1\cdots n\} =$
$(n+1)^{m+1} - 1 - (m+1)\Sigma(1\cdots n)^m$]
} Pascal a.1655 t.3 p.309 {

·2 $\Sigma\{[\mathrm{C}(n,r)]^2 \mid r, 0\cdots n\} = \mathrm{C}(2n,n)$ }Lagrange a.1770 t.2 p.182{
[P14·4 .⊃. P]

·3 $\Sigma\{(-1)^r[\mathrm{C}(2n,r)]^2 \mid r, 0\cdots 2n\} = (-1)^n\mathrm{C}(2n,n)$
}Cesàro a.1884 Mathesis t.4 p.231{

·4 $\Sigma\{(-1)^r[\mathrm{C}(2n, r)]^3 \mid r, 0\cdots 2n\} = (-1)^n\mathrm{C}(3n,n) \times \mathrm{C}(2n,n)$
} Dixon Mm. a.1890 t.20 p.79 {

·5 $n! = \Sigma\{(-1)^r\mathrm{C}(n,r)\,(n-r)^n \mid r, 0\cdots n\}$
} Euler PetrNC. a.1768 t.13 p.28 {

·6 $\Sigma/(1\cdots n) = \Sigma[(-1)^{r+1}\mathrm{C}(n,r)/r \mid r, 1\cdots n]$
} Joh. Bernoulli a.1740 CorrM. t.2 p.35 {

·7 $\Sigma\{(-1)^{r+1}\mathrm{C}(n,r)/(n+r) \mid r, 1\cdots n\} = /\,[(2n+1)\,\mathrm{C}(2n,n)]$
}Wallis a.1655 p.425:

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{2\,.\,3}\,,\ \frac{1}{3} - \frac{2}{4} + \frac{1}{5} = \frac{1\,.\,2}{3\,.\,4\,.\,5}\,,\ \frac{1}{4} - \frac{3}{5} + \frac{3}{6} - \frac{1}{7} = \frac{1\,.\,2\,.\,3}{4\,.\,5\,.\,6\,.\,7}$$

... et sic deinceps ... {

·1 $n\varepsilon \mathrm{N}_1 \,.\supset.$
$n^n = \Sigma\{(-1)^r\mathrm{C}(n,r)\mathrm{C}[n(n-r),n] \mid r, 0\cdots n\}$
} N. J. Hatzidakis IdM. t.8 a.1901 {

✻ 17·1 $\text{num}[(N_0 \text{ F } 1\cdots m) \frown x\ni(\Sigma x = n)] = C(m+n-1, n)$
{ Frénicle ParisM. a.1693 t.5 {

Numero de « combinationes cum repetitione de objectos $1\cdots m$, ad n ».

·2 $a\varepsilon N_1+1 \,.\!\supset\!.\, \text{num}\,(x;y)\ni[x,y\varepsilon N_1 \,.\, a = \Sigma x\cdots(x+y)] =$
$\text{num}(2N_1+1)\frown(a/N_1)$ { Sylvester CR. t.96 {

Np ✻ 20·1 $m\varepsilon \text{Np} \,.\, n\varepsilon N_1 \,.\, m > n+1 \,.\!\supset\!.\, \Sigma\,(1\cdots m)^n \;\varepsilon\; N_1\times m$
{ Lionnet ; voir Catalan BelgiqueM. t.46 a.1886 p.14 {

mp ✻ 21·1 $p\varepsilon \text{Np} \,.\, b\varepsilon 0\cdots(p-1) \text{ F } 0\cdots n \,.\, a = \Sigma(b_r p^r \,|\, r, 0\cdots n)$
$.\!\supset\!.\, \text{mp}(p,a!) = (a-\Sigma b)/(p-1)$
{ Kummer a.1852 JfM. t.44 p.115 {

·2 $p\varepsilon \text{Np} \,.\, a\varepsilon N_1 \,.\!\supset\!.\, \text{mp}(p, a!) = \Sigma\{[\,\text{E}(a/p^r)] \,|\, r, N_1\}$
{ Legendre a.1830 t.1 p.11 {

E ✻ 22·0 $x\varepsilon Q \,.\, a\varepsilon N_1 \,.\!\supset\!.\, \Sigma\{[\text{E}(x+r/a)] \,|\, r, 0\cdots(a-1)\} = \text{E}\,ax$
{ Bertrand *Arithmétique* a.1851 p.109 {

·1 $m,n\varepsilon N_1 \,.\!\supset\!.\, m = \Sigma[\text{E}(m+r)/n \,|\, r, \; 0\cdots(n-1)]$
$[\;(m/n, n) \,|\, (x, a) \;\text{P·0} \,.\!\supset\!.\; \text{P}\;]$

·2 $x\varepsilon q \,.\, a\varepsilon N_1 \,.\, -\exists\,[x\times(1\cdots a)]\frown n \,.\!\supset\!.$
$\Sigma\{(\text{E}\,rx) \,|\, r, 1\cdots a\} + \Sigma\{[\text{E}(r/x)] \,|\, r, 1\cdots \text{E}\,ax\} = a\,\text{E}\,ax$
{ Gauss a.1808 t.2 p.7 {

·3 $x\varepsilon q \,.\!\supset\!.\, \text{E}x = \Sigma\{[\text{E}(x/2^r + /2)] \,|\, r, \; N_1\}$
$= \text{E}(x/2+/2)+\text{E}(x/4+/2)+\ldots$
{ Cesàro *Excursions Arithm.* a.1885 p.36 {

·4 $a,b\varepsilon N_1 \,.\, \text{Dvr}(2a+1, 2b+1) = 1 \,.\!\supset\!.$
$\Sigma\{[\text{E}r(2a+1)/(2b+1)] \,|\, r,1\cdots b\} + \Sigma\{[\text{E}r(2b+1)/(2a+1)] \,|\, r,1\cdots a\} = ab$
{ Gauss a.1808 t.2 p.7 {

·5 $a,b\varepsilon N_1 \,.\!\supset\!.\, \text{D}(a,b) = b + \Sigma[\text{E}(ah/b) \,|\, h,1\cdots b] + \Sigma[\text{E}(-ah/b) \,|\, h,1\cdots b]$

nt ✻ 23. $p\varepsilon \text{Np} \,.\, p>3 \,.\!\supset\!.$
·1 $\text{nt}\;\Sigma/[1\cdots(p-1)] \;\varepsilon\; p^2\times N_1$ { Osborn a.1892 Mm. t.22 p.51 {
·2 $\text{nt}\;\Sigma/[1\cdots(p-1)]^2 \;\varepsilon\; p\times N_1$ { Glaisher a.1900 QJ. t.31 p.337 {

Φ ✻ 24·1 $\Sigma(\Phi, N_1\frown a/N_1) = a$ { Gauss a.1801 t.1 p.31 {

·2 $n\varepsilon N_1 \,.\!\supset\!.\, \Sigma[(\Phi r)\text{E}(n/r) \,|\, r,1\cdots n] = n(n+1)/2$
{ L. Carlini LazzeriP. a.1902 p329 {

§16 Π

❋ 1·0 $m\varepsilon \mathrm{N}_0 . f\varepsilon\, \mathrm{q\,f}\, 0\cdots(m+1) .\supset. \Pi(f, 0\cdots 0) = f0 .$
$\Pi(f, 0\cdots(m+1)) = \Pi(f, 0\cdots m)\times f(m+1)$ Df

$(\Pi \mid \Sigma)$ §Σ 1·1

·2 $m\varepsilon \mathrm{N}_1 . f\varepsilon\, \mathrm{q\,f}\, 1\cdots m .\supset: \Pi(f, 1\cdots m) = 0 .=. 0\varepsilon f\text{`}\, 1\cdots m$

$(\Pi \mid \Sigma)$ §Σ 2

Definitione de $\Pi(f,u)$, « producto de valores de fonctione f, quando variabile sume omni valore in classe u », analogo ad Df de $\Sigma(f,u)$.

❋ 2. $m,n\varepsilon \mathrm{N}_1 .\supset.$

·1 $(n+2)\Sigma\{\Pi[(r+s)|s, 0\cdots n]\,|r, 1\cdots m\} = \Pi[(m+s)|s, 0\cdots(n+1)]$
{Fermat t.1 p.341:

« In progressione naturali, quae ab unitate sumit exordium, quilibet numerus [m] in proxime majorem [$\times(m+1)$] facit duplum sui trianguli [$=2\times(1+2+\ldots+m)$]; in triangulum proxime majoris, facit triplum suae pyramidis; in pyramidem proxime majoris facit quadruplum sui triangulo-trianguli; et sic uniformi et generali in infinitum methodo. » }

·2 $a\varepsilon \mathrm{q} .\supset. m\Sigma/\Pi[(a+r+s)|s, 0\cdots m)\,|r, 1\cdots n] =$
$/\Pi[(a+s)|s, 1\cdots m] - /\Pi[(a+n+s)|s, 1\cdots m]$

·3 $u\varepsilon \mathrm{Q\,f}\, 1\cdots n .\supset.$
$1 = \Sigma\{u_r/\Pi[(1+u_s)\,|s, 1\cdots r]\,|r, 1\cdots n\} + /[\Pi(1+u_s)\,|s, 1\cdots n]$
} Nicole ParisM. a.1727 p.257 {

❋ 3.

·1 $n\varepsilon \mathrm{N}_1+1 . x\varepsilon \mathrm{Q\,f}1\cdots n .\supset. \Pi[(1+x_r)|r, 1\cdots n] > 1+\Sigma(x, 1\cdots n)$

[$n=2 .\supset. (1+x_1)(1+x_2) = 1+x_1+x_2+x_1x_2 > 1+(x_1+x_2)$ (1)

$n\varepsilon \mathrm{N}_1 . \Pi[(1+x_r)|r,1\cdots n] > 1+\Sigma(x,1\cdots n) .\supset.$
$\Pi[(1+x_r)|r,1\cdots(n+1)] = \Pi[(1+x_r)|r,1\cdots n]\times(1+x_{n+1}) >$
$[1+\Sigma(x,1\cdots n)](1+x_{n+1}) = 1+\Sigma[x,1\cdots(n+1)]+x_{n+1}\Sigma(x,1\cdots n) >$
$1+\Sigma[x,1\cdots(n+1)]$ (2)

(1) . (2) . Induct .⊃. P]

·2 $n\varepsilon \mathrm{N}_1+1 . x\varepsilon\, \theta\, \mathrm{f}\, 1\cdots n .\supset. \Pi[(1-x_r)|r, 1\cdots n] > 1-\Sigma(x,1\cdots n)$

·3 $n\varepsilon \mathrm{N}_1 . a\varepsilon \mathrm{Q\,f}\, 1\cdots n .\supset. \sqrt[n]{\Pi(a, 1\cdots n)} \leq \Sigma(a, 1\cdots n)/n$

·4 $n \varepsilon N_1 . a, m \varepsilon Qf1 \cdots n .\supset.$

$\Pi(a_r \upharpoonright m_r | r, 1 \cdots n) \upharpoonright / \Sigma(m, 1 \cdots n) \leqq \Sigma(m_r a_r | r, 1 \cdots n) / \Sigma(m, 1 \cdots n)$

[§Q30·0 $.\supset. (a_1 \upharpoonright m_1)(a_2 \upharpoonright m_2) \leqq [(m_1 a_1 + m_2 a_2)/(m_1 + m_2)] \upharpoonright (m_1 + m_2)$ (1)

$n \varepsilon N_1 . \Pi(a_r \upharpoonright m_r | r, 1 \cdots n) \leqq [\Sigma(m_r a_r | r, 1 \cdots n)/\Sigma(m, 1 \cdots n)] \upharpoonright$

$\Sigma(m, 1 \cdots n)$. Oper $\times a_{n+1} .\supset. \Pi[a_r \upharpoonright m_r | r, 1 \cdots (n+1)] \leqq$

$[\Sigma(m_r a_r | r, 1 \cdots n)/\Sigma(m, 1 \cdots n)] \upharpoonright \Sigma(m, 1 \cdots n) \times a_{m+1} \upharpoonright m_{n+1} .\supset.$

$\Pi[a_r \upharpoonright m_r | r, 1 \cdots (n+1)] \leqq \{\Sigma[m_r a_r | r, 1 \cdots (n+1)]/\Sigma[m, 1 \cdots (n+1)]\} \upharpoonright$

$\Sigma[m, 1 \cdots (n+1)]$ (2)

(1) . (2) . Induct $.\supset.$ P]

✻ 4·0 $m \varepsilon N_1 . a \varepsilon N_1 F(1 \cdots m) .\supset. (\Sigma a)! \varepsilon N_1 \times \Pi(a!)$

·1 $m, n \varepsilon N_1 . a \varepsilon qF1 \cdots m .\supset. (\Sigma a)^n =$

$\Sigma\{[n!/\Pi(u!, 1 \cdots m)] \times \Pi(a_r \upharpoonright u_r | r, 1 \cdots m) | u, (N_0 F 1 \cdots m) \frown u \ni (\Sigma u = n)\}$

{ LEIBNIZ a.1678; Mss. *Math.* III A 3 fol.16;

BERNOULLI Joh. a.1695, (LEIBNIZ *MathS.* t.3 p.181):

« Esto... polynomium quodcunque $s+x+y+z$ etc. elevandum ad potentiam quamcunque r; Dico coefficientem termini $s^a\ x^b\ y^c\ z^e$ etc. fore

$$\frac{r \,.\, r-1 \,.\, r-2 \,.\, r-3 \,.\, r-4 \,.\,.\,.\,.\, 1}{1.2.3 \ldots a \times 1.2.3 \ldots b \times 1.2.3 \ldots c \times 1.2.3 \ldots e\ \&}$$ » }

·2 $n \varepsilon N_1 . a \varepsilon qF1 \cdots n :\supset.$

$\Sigma\{(\Pi r) \times \Sigma[(r_s a_s)^n | s, 1 \cdots n] | r, (\iota 1 \cup \iota{-1}) F1 \cdots n\} = 2^n \times n! \times \Pi a$

{ LAGRANGE Mss. in-4° t.6 a.1782 }

✻ 5·1 $a \varepsilon N_1 .\supset. a = \Pi\{[x \upharpoonright mp(x,a)] | x, Np\}$

·2 $a \varepsilon N_1 .\supset. num(N_1 \frown a/N_1) = \Pi\{[mp(x,a)+1] | x, Np\}$

{ WALLIS a.1685; Opera t.2 p.498 }

·3 $u \varepsilon Cls'N_1 . \exists u .\supset. Du = \Pi\{[x \upharpoonright \min mp(x,u)] | x, Np\}$

·4 ——. num$u \varepsilon N_1 .\supset.$ mu = —— max ————

·5 $a \varepsilon N_1 .\supset. \Sigma(N_1 \frown a/N_1) = \Pi(\{[x \upharpoonright [mp(x,a)+1] - 1]/(x-1)\} | x, Np)$

$= \Pi\{\Sigma[x^r | r, 0 \cdots mp(x,a)] | x, Np\}$

{ WALLIS a.1658; Opera t.2 p.814:

« Si duorum pluriumve numerorum primorum potestates quaelibet invicem ducantur, factus partibus suis aliquotis auctus, aequatur facto ex componentibus partium suarum aliquotarum additione auctis ». }

·6 $a \varepsilon N_1 + 1 .\supset. \Phi a = \Pi\{[x \upharpoonright (mp(x,a)-1) \times (x-1)] | x, Np \frown a/N_1\}$

{ EULER PetrNC. a.1760-61 t.8 p.85-103 }

·7 $a \varepsilon R .\supset. mp(p,a) = mp(p, nt\,a) - mp(p, dt\,a)$ Df

·8 $a \varepsilon R^m .=: x \varepsilon Np .\supset_x. mp(x,a) \varepsilon n \times m$ Ths·1·3·4

❋ 6·1 $n \varepsilon \mathrm{N}_1 \,.\, a \varepsilon (\mathrm{qf}1\cdots n)\mathrm{sim} \,.\, x \varepsilon \mathrm{q}-(a\text{‘}1\cdots n) \,.\, m \varepsilon 0\cdots(n-1) \;.\supset.$
$x^m/\Pi[(x-a_r)|r,1\cdots n] =$
$\Sigma\{a_r{}^m/[(x-a_r)\Pi[(a_r-a_s)|s,1\cdots n-\iota r]]|r,1\cdots n\}$

Decompositione de fractione, que habe ut denominatore producto de factores in x, in summa de fractiones simplice.

·2 $n \varepsilon \mathrm{N}_1 \,.\, m \varepsilon \mathrm{q}-(-n\cdots 0) \;.\supset.$
$\Sigma[(-1)^r\, \mathrm{C}(n,r)/(m+r)|r,0\cdots n] = n!/\Pi(m+0\cdots n)$

§17 Δ (differentia)

❋ 1. $a,b \varepsilon \mathrm{n} \,.\, a<b \,.\, f,g \varepsilon \mathrm{qf}\,a\cdots b \;.\supset.$

·0 $(\Delta f)a = \Delta fa = \Delta(f,a) = f(a+1) - fa$ Df Δ

Si f es quantitate functione de numeros inter a et b, tunc differentia $f(a+1)-fa$ indicare per $\Delta(f,a)$, vel per $(\Delta f)a$, vel, in modo plus simplice, per Δfa. Ce differentia non vale $\Delta(fa)$, id es non es functio de quantitate fa, sed es functio de functio f, et de numero a.

Signo Δ es introducto per Euler, Calc. diff. a.1755.

·1 $\Delta(f+g,\, a) = \Delta fa + \Delta ga$

·2 $c \varepsilon \mathrm{q} \;.\supset.\; \Delta(a\times f,\, a) = c\times\Delta fa$

·3 $\Delta(fx\times gx\,|x,\, a) = f(a+1)\Delta ga + ga\Delta fa$

·4 $a \varepsilon \mathrm{N}_1 \;.\supset.\; \Delta(/,\, a) = -1/[a(a+1)$

·5 $g \varepsilon \mathrm{Qf}a\cdots b \;.\supset.\; \Delta(fx/gx|x,\, a) = (ga\Delta fa - ga\Delta fa)/[ga\times g(a+1)]$

·6 $\Delta[f(-x)|x,\, -b] = -\Delta f(b-1)$

❋ 2·1 $c \varepsilon \mathrm{Q} \,.\, a \varepsilon \mathrm{n} \;.\supset.\; \Delta(c^x|x,\, a) = c^a(c-1)$

·2 $a \varepsilon \mathrm{n} \,.\, n \varepsilon \mathrm{N}_1 \;.\supset.\; \Delta[\mathrm{C}(a,n)|a,\, a] = \mathrm{C}(a,n-1)$

·3 » $\Delta[\mathrm{C}(-a,n)|a,a] = -\mathrm{C}(-a-1,n-1)$

·4 $x,y \varepsilon \mathrm{N}_1 \;.\supset.\; \Delta[\Pi(x+0\cdots y)\,|x,\, x] = (y+1)\Pi(x+1\cdots y)$

·5 $x,y \varepsilon \mathrm{N}_1 \;.\supset.\; \Delta\,/\Pi[(x+0\cdots y)\,|y,y] = -(y+1)/\Pi[x+0\cdots(y+1)]$

·6 $n \varepsilon \mathrm{N}_1 \,.\, a \varepsilon \mathrm{qf}0\cdots n \;.\supset.\; \Delta\{\Sigma[a_r x^{n-r}/(n-r)!|r,0\cdots n]|x,x\} =$
$\Sigma\{\Sigma[a_s/(r-s)!|s,0\cdots(r-1)]x^{n-r}/(n-r)!|r,1\cdots n\}$

✻ 3. $a,b \varepsilon \mathrm{n} \,.\, a<b \,.\, f \varepsilon \mathrm{qf}\, a \cdots b \,.\supset.$

·1 $\Sigma(\Delta f, a \cdots b-1) = fb - fa$

·2 $\Sigma(f, a \cdots b-1) = (b-a)fa + \Sigma[(b-x-1)\Delta fx | x, a \cdots b-2]$

·3 $n \varepsilon \mathrm{N}_1 \,.\supset.\, \Sigma[\mathrm{C}(b-z-1, n)fz | z, a \cdots (b-n-1)] =$
$\mathrm{C}(b-a, n+1)fa + \Sigma[\mathrm{C}(b-z-1, n+1)\Delta fz | z, a \cdots (b-n-2)]$

·4 $c \varepsilon \mathrm{Q} \,.\supset.\, \Sigma[c^x fx | x, a \cdots (b-1)] = (c^b fb - c^a fa)/(c-1)$
$-\Sigma(c^x \Delta fx | x, a \cdots b-1)c/(c-1)$

[Hp $.\, x \varepsilon a \cdots b-1 \,.\supset.\, \Delta c^x fx = c^{x+1}\Delta fx + c^x (c-1)fx$. OperΣ $.\supset.$
$c^b fb - c^a fa = c\Sigma(c^x \Delta fx | x, a \cdots b-1) + (c-1)\Sigma(c^x fx | x, a \cdots b-1) \,.\supset.$ P]

✻ 4. $a \varepsilon \mathrm{n} \,.\, n \varepsilon \mathrm{N}_1 \,.\, f \varepsilon \mathrm{qf}(a+0 \cdots n) \,.\supset.$

·0 $\Delta^n(f,a) = \Delta^n fa = (\Delta^n f)a$ Df

·1 $f(a+n) = \Sigma[\mathrm{C}(n,r)\Delta^r fa | r, 0 \cdots n]$

·2 $\Delta^n fa = \Sigma[(-1)^r \mathrm{C}(n,r) f(a+n-r) | r, 0 \cdots n]$

{ ·1-·2 MERCATOR a.1668 p.12 }

·3 $a,b \varepsilon \mathrm{n} \,.\, n \varepsilon \mathrm{N}_1 \,.\, b > a+n \,.\, f \varepsilon \mathrm{qf}\, a \cdots b \,.\supset.\, fb =$
$\Sigma[\mathrm{C}(b-a,r)\Delta^r fa | r, 0 \cdots n] + \Sigma[\mathrm{C}(b-x-1,n)\Delta^{n+1} fx | x, a \cdots (b-n-1)]$

[Hp $.\, g = fb - \Sigma[\mathrm{C}(b-x,r)\Delta^r fx | r, 0 \cdots n] | x \,.\supset.\, gb=0$ (1)
$\Delta g = -\mathrm{C}(b-x-1,n)\Delta^{n+1} fx$ (2)
$gb - ga = \Sigma[\Delta g, a \cdots (b-1)]$ (3)
(1) . (2) . (3) $.\supset.$ P]

·4 $a,b \varepsilon \mathrm{n} \,.\, a > b+1 \,.\, f \varepsilon \mathrm{qf} a \cdots b \,.\supset.\, \Sigma(f, a \cdots b) =$
$(b-a+1)(fa+fb)/2 - \Sigma[(x-a)(b-x)\Delta^2 f(x-1) | x, (a+1) \cdots (b-1)]$

[Hp $.\, g = (x-a)(x-b)|x \,.\, x \varepsilon a \cdots b-2 \,.\supset.\, \Delta[(fx\Delta gx - gx\Delta fx)|x, x] =$
$f(x+1)\Delta^2 gx - g(x+1)\Delta^2 fx \,.\, \Delta gx = 2x+1-a-b \,.\, \Delta^2 gx = 2 \,.\, ga = gb = 0$ (1)
(1) . Oper$\Sigma(, a \cdots b-1) \,.\supset.\, fb(b-a+1) + fa(b-a-1) = 2\Sigma(f, a+1 \cdots b) -$
$\Sigma[(x+1-a)(x+1-b)\Delta^2 fx | x, a \cdots b-1]$ (2)
(2) $.\supset.$ P]

§18 B (numeros de Bernoulli), A, Bern

✻ 1·0 $\mathrm{A} = \iota\, (\mathrm{qfN}_0) \cap a \ni \{ a_0 = 1 \,.\, n \varepsilon \mathrm{N}_1 + 1 \,.\supset_n.$
$\Sigma[a_r/(n-r)! | r, 0 \cdots (n-1)] = 0 \}$ Df

Nos voca A successione de quantitates que satisfac ad conditiones:
$\mathrm{A}_0 = 1, \quad \mathrm{A}_0/2! + \mathrm{A}_1 = 0, \quad \mathrm{A}_0/3! + \mathrm{A}_1/2! + \mathrm{A}_2 = 0,$ etc.

·1 $A_0 = 1$. $A_1 = -1/2$. $A_2 = 1/12$. $A_3 = 0$. $A_4 = -1/720$ . ...

·2 $n \varepsilon N_1 . \supset . A_{2n+1} = 0$

·3 $n \varepsilon N_1 . \supset . B_n = (-1)^{n-1}(2n)! A_{2n}$ Df

·4 $B_1 = /6$. $B_2 = /30$. $B_3 = /42$. $B_4 = /30$.
$B_5 = 5/66$. $B_6 = 691/2730$. $B_7 = 7/6$. $B_8 = 3617/510$.
$B_9 = 43867 / 798$. $B_{10} = 1\,74611 / 330$. $B_{11} = 8\,54513 / 138$.
$B_{12} = 2363\,64091 / 2730$. $B_{13} = 85\,53103 / 6$.
$B_{14} = 2\,37494\,61029 / 870$. $B_{15} = 861\,58412\,76005 / 14322$.
$B_{16} = 770\,93210\,41217 / 510$. $B_{17} = 257\,76878\,58367 / 6$.
$B_{18} = 26315\,27155\,30534\,77373 / 19\,19190$
$B_{19} = 2\,92999\,39138\,41559 / 6$
$B_{20} = 2\,61082\,71849\,64491\,22051 / 13530$
$B_{21} = 15\,20097\,64391\,80708\,02691 / 1806$
$B_{22} = 278\,33269\,57930\,10242\,35023 / 690$
$B_{23} = 5964\,51111\,59391\,21632\,77961 / 282$
$B_{24} = 560\,94033\,68997\,81768\,62491\,27547 / 46410$
$B_{25} = 49\,50572\,05241\,07964\,82124\,77525 / 66$
$B_{26} = 80116\,57181\,35489\,95734\,79249\,91853 / 1590$
$B_{27} = 29\,14996\,36348\,84862\,42141\,81238\,12691 / 798$
$B_{28} = 2479\,39292\,93132\,26753\,68541\,57396\,63229 / 870$
$B_{29} = 84483\,61334\,88800\,41862\,04677\,59940\,36021 / 354$
$B_{30} = 121\,52331\,40483\,75557\,20403\,04994\,07982\,02460\,41491 / 567\,86730$

B_n es n^{esimo} de numeros que «ab inventore Jacobo Bernoulli vocari solent Bernoulliani» (Euler *Calc. diff.* t.2 §122).

Euler indica illos per B_1, B_3, B_5 ...; notatio B_1, B_2, B_3,... que nos seque, es adoptato per Binet, JP. a.1839 t.16 cah.27 p.240, per Ohm, Markoff.

Bernoulli a.1713 calcula B_5 ; Euler B_{15}; Rothe (communicatos per Ohm in JfM. a.1840 t.20 p.11) B_{31} ; Adams (JfM. a.1878 t.85 p.269) B_{62}.
Bibliographia de ce numeros in AJ. t.5 a.1882 p.228.

·5 $n \varepsilon N_1 . \supset . B_n \, \varepsilon R$

·6 $n \varepsilon N_1 . \supset . 2(2^{2n}-1)B_n \; \varepsilon N_1$
{ Genocchi Ann. di Mat. a.1852 t.3 p.399 }

·7 $n \varepsilon N_1 . \supset . \beta[(-1)^n B_n] = \Sigma \, /Np \frown z \ni [2n \, \varepsilon \, (z-1) \times N_1]$
{ Staudt JfM. a.1840 t.21 ; Clausen, AstronNachr. t.17 }

·8 $n \varepsilon Np \frown (N_1 + 3) . \supset . \text{nt} \, B_n \, \varepsilon \, n \times N_1$
{ Adams a.1878 JfM. t.85 p.269 }

✱ 2. $x \varepsilon \mathrm{q} \,.\, n \varepsilon \mathrm{N}_1 \,.\supset.$

·0 $\mathrm{Bern}(x,n) = \Sigma[\mathrm{A}_r x^{n-r}/(n-r)! \mid r, 0\cdots(n-1)]$ Df

$= x^n/n! - x^{n-1}/[2(n-1)!] +$

$\Sigma\{(-1)^{r-1}\mathrm{B}_r x^{n-2r}/[(2r-1)!(n-2r)!] \mid r, 1\cdots \mathrm{E}[(n-1)/2]\}$

$\mathrm{B}(x,n)$ = functione de Bernoulli, de ordine n (Raabe a.1848).

·1 $\mathrm{Bern}(x,1) = x$. $\mathrm{Bern}(x,2) = x^2/2 - x/2$.
$\mathrm{Bern}(x,3) = x^3/6 - x^2/4 + x/12$ . ...

·2 $\mathrm{Bern}(0,n) = 0$

·3 $\mathrm{Bern}(x+1,n) - \mathrm{Bern}(x,n) = x^{n-1}/(n-1)!$

·4 $n,x \varepsilon \mathrm{N}_1 \,.\supset.\, \Sigma(1\cdots x)^n = n!\, \mathrm{Bern}(x+1,n+1)$

. { Jac. BERNOULLI a.1713 p.97:

« $\int n^c \propto \frac{1}{c+1} n^{c+1} + \frac{1}{2} n^c + \frac{c}{2} \mathrm{A} n^{c-1} + \frac{c \,.\, c-1 \,.\, c-2}{2 \,.\, 3 \,.\, 4} \mathrm{B} n^{c-3} + \ldots$

et ita deinceps, exponentem potestatis ipsius n continue minuendo binario, quousque perveniatur ad n vel n^2. Literæ capitales A, B, C, D, etc., ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$ etc. nempe $\mathrm{A} \propto 1/6$, $\mathrm{B} \propto -1/30$, $\mathrm{C} \propto 1/42$, $\mathrm{D} \propto -1/30$ ». }

·5 $\mathrm{Bern}(x,n) - (-1)^n\, \mathrm{Bern}(-x,n) = -x^{n-1}/(n-1)!$

·6 $\mathrm{Bern}(x,n) + (-1)^{n-1}\, \mathrm{Bern}(1-x,n) = 0$

·7 $\mathrm{Bern}(1,n) = 0$

·8 $n \varepsilon 2\mathrm{N}_1 + 1 \,.\supset.\, \mathrm{Bern}(1/2,n) = 0$

§19 Med

✱ 1. $u,v \varepsilon \mathrm{Cls}'\mathrm{q} \,.\, a,b \varepsilon \mathrm{q} \,.\supset.$

·0 $\mathrm{Med}\,u = \mathrm{q} \frown x \ni (\mathrm{l}_{\prime} u \leqq x \leqq \mathrm{l}' u)$ Df

Si u es classe de numeros, nos voca medio des u omni numero inter limite supero et infero des u.

Cauchy a.1821 p.29 introduce signo «medio» sub forma «M».

·1 $\mathrm{Med}\,\mathrm{Med}\,u = \mathrm{Med}\,u$ ·2 $v \supset u \,.\supset.\, \mathrm{Med}\,v \supset \mathrm{Med}\,u$

·3 $\mathrm{Med}(\mathrm{Med}\,u \frown \mathrm{Med}\,v) = \mathrm{Med}\,u \frown \mathrm{Med}\,v$

·4 $\mathrm{Med}(a+u) = a + \mathrm{Med}\,u$. $\mathrm{Med}\,au = a\,\mathrm{Med}\,u$
$\mathrm{Med}(-u) = -\mathrm{Med}\,u$. $\mathrm{Med}(\iota a \smile \iota b) = a \frown b$

·5 $u \varepsilon$ Cls'Q $.\supset.$ Med$/u = /$Med$u$
$u \varepsilon$ Cls'Q $.\supset.$ Med$(u \mathbin{\text{⋔}} a) = (Medu) \mathbin{\text{⋔}} a$
$a \varepsilon$ Q $.\supset.$ Med$(a \mathbin{\text{⋔}} u) = a \mathbin{\text{⋔}} (Medu)$
$a \varepsilon$ Q$-\iota 1$. $u \varepsilon$ Cls'Q $.\supset.$ Med ^{a}Log$u = {}^a$Log Med$u$

·6 λ Med$u =$ Med λu

·7 l'mod$u =$ l'mod Med$u$

* 2. $n \varepsilon N_1$. $x \varepsilon$ qf $1 \cdots n$. $a,b \varepsilon$ Qf $1 \cdots n$ $.\supset.$

·1 $[\Sigma(x, 1 \cdots n)]/n \ \varepsilon$ Med x‘$(1 \cdots n)$
$=$ « valore medio arithmetico des x ».

·2 $\Sigma(a_r \times x_r | r, 1 \cdots n) / \Sigma(a, 1 \cdots n) \ \varepsilon$ Med x‘$(1 \cdots n)$
$=$ « id. cum coefficientes a ».

·3 $\sqrt{[\Sigma(a^2, 1 \cdots n)/n]} \ \varepsilon$ Med a‘$(1 \cdots n)$
$=$ « medio quadratico ».

·4 $\sqrt[n]{\Pi(a, 1 \cdots n)} \ \varepsilon$ Med$a$‘$1 \cdots n$
$=$ « medio geometrico des a ».

·5 $[\Pi(a_r \mathbin{\text{⋔}} b_r | r, 1 \cdots n)] \mathbin{\text{⋔}} / \Sigma(b, 1 \cdots n) \ \varepsilon$ Med$a$‘$1 \cdots n$
$=$ « id. cum pondere b ».

* 3. $a,b \varepsilon$ n . $n \varepsilon N_1$. $b > a+n$. $f \varepsilon$ qf$a \cdots b$ $.\supset.$

·1 $fb - fa \ \varepsilon \ (b-a)Med\Delta f$‘$[a \cdots (b-1)]$
[P2·1. §ΔP3·1 $.\supset.$ P]

·2 $g \varepsilon$ qf$a \cdots b$. $\Delta g \ \varepsilon$ Qf$a \cdots b-1$ $.\supset.$
$(fb - fa)/(gb - ga) \ \varepsilon$ Med$[\Delta f x / \Delta g x | x$‘$a \cdots (b-1)]$
[P2·2 $.\supset.$ P]

·3 $x \varepsilon \ a \cdots b$ $.\supset.$ $fx - fa - (x-a)(fb - fa)/(b-a) \ \varepsilon$
$(x-a)(x-b)/2$ Med$\Delta^2 f$‘$a \cdots (b-2)$

·4 $fb - \Sigma[C(b-a,r)\Delta^r fa | r, 0 \cdots n] \ \varepsilon$
$C(b-a, n+1)$Med$\Delta^{n+1} f$‘$a \cdots (b-n-1)$
[§ΔP4·3 $.\supset.$ P]

·5 $\Sigma(f, a \cdots b) - (b-a+1)(fa+fb)/2$
$\varepsilon \ -C(b-a+1, 3)Med\Delta^2 f$‘$(a \cdots b-2)/2$
[§ΔP4·4 $.\supset.$ P]

·6 $\Sigma[C(b-z-1, n) fz | z, a \cdots (b-n-1)] \ \varepsilon$
$C(b-a, n+1)fa + C(b-a-1, n+1)(fb - fa) / (n+2)$
$+ (n+1)/2 \ C(b-a+1, n+3)Med\Delta^2 f$‘$[a \cdots (b-2)]$

§20 Num infn

✻ 1. $a,b\varepsilon$ Cls $.\supset:$

·0 $\mathrm{Num}a = \mathrm{Num}b .=. \exists\, (bfa)\mathrm{rcp}$ Df

Si a et b es classe, tunc nos dice que numero des a aequa numero des b, quando existe correspondentia reciproco inter a et b.

Signo « num » de II §10 es casu particulare de signo « Num », et primo, quando existe, vel quando classe es finito, coincide cum secundo.

Num'Cls significa « numero de aliquo classe ». G. Cantor voca illos « numero cardinale », et in loco de « Numero » dice « Mächtigkeit » « puissance »,

Df ·0 es expresso per solo signo de logica I et III §1-§4. Hic Arithmetica pote incipe, et nos defini, in modo directo, signos $>$, 0, 1, $+$, $\times$, $\uparrow$, N_0.

·1 $\mathrm{Num}\, a \leqq \mathrm{Num}\, b .=. \exists\, \mathrm{Cls}'b \cap x\ni(\mathrm{Num}a = \mathrm{Num}x)$ Df

·2 $a \supset b .\supset. \mathrm{Num}\, a \leqq \mathrm{Num}\, b$

✻ 2. $x,y,z\varepsilon$ Num'Cls $.\supset\therefore$

·4 $x < y .=. x \leqq y . x -= y$

·5 $x \leqq y . y \leqq z .\supset. x \leqq z$

·6 $x \leqq y . y \leqq x .\supset. x = y$

{ G. Cantor RdM. a.1895 p.135; Dm. Bernstein, publiée par Borel *Th. des Fonctions* a.1898 p.104 }

0 ✻ 3·0 $0 = \mathrm{Num}\Lambda$ Df

·1 $a\varepsilon \mathrm{Cls} .\supset: \mathrm{Num}a = 0 .=. a = \Lambda$

Numero de a es 0, quando classe a es nullo.

1 ✻ 4·0 $1 = \imath\, \mathrm{Num}'[\mathrm{Cls} \cap a\ni(\exists a : x,y\varepsilon a .\supset_{x,y}. x = y)]$ Df

·1 $a\varepsilon \mathrm{Cls} .\supset\therefore \mathrm{Num}\, a = 1 .=: \exists a : x,y\varepsilon a .\supset_{x,y}. x = y$

+ ✻ 5·0 $x,y\varepsilon \mathrm{Num'Cls} .\supset. x+y = \imath\, z\ni[\, a,b\varepsilon \mathrm{Cls} . \mathrm{Num}\, a = x . \mathrm{Num}\, b = y . a\cap b = \Lambda .\supset_{a,b}. z = \mathrm{Num}(a\cup b)\,]$ Df

·1 $a,b \varepsilon$ Cls . $a \frown b = \Lambda$.⊃. Num$(a \smile b)$ = Num a + Num b

·2 $x,y,z \varepsilon$ Num'Cls .⊃. $x+y=y+x$. $x+(y+z)=(x+y)+z$. $0+x=x$. $x \leqq x+y$

·1 Si duo classe a et b habe nullo elemento commune, tunc numero de $a \smile b$ vale numero de a plus numero de b. Es transformabile in P·0, definitione de summa de duo numero cardinale.

× ✻ 6·0 $x,y \varepsilon$ Num'Cls .⊃. $x \times y = \imath\, z \ni [\, a,b \varepsilon$ Cls . Num a = x . Num$b = y$.⊃$_{a,b}$. z = Num$(a \!:\! b)$] Df

·1 $a,b \varepsilon$ Cls .⊃. Num$(a \!:\! b)$ = Num$a \times$ Num$b$

·2 $x,y,z \varepsilon$ Num'Cls .⊃. $xy = yx$. $x(yz) = (xy)z$. $x(y+z) = xy + xz$

Numero de individuo constituente systema $(a \!:\! b)$ vale producto de numero de a per numero de b.

↑ ✻ 7·0 $x,y \varepsilon$ Num'Cls .⊃. $x \uparrow y = \imath\, z \ni [\, a,b \varepsilon$ Cls . Num$a$ = x . Num$b = y$.⊃$_{a,b}$. z = Num$(a$F$b)$] Df

·1 $a,b \varepsilon$ Cls .⊃. Num$(a$F$b)$ = Num$a \uparrow$ Num$b$

Numero de functione definito que ad omni b fac corresponde aliquo a vale numero de a elevato ad numero de b.

infn ✻ 8·0 infn = Num'{Cls ⌢ $a \ni [\exists$ Cls ⌢ $u \ni (u \supset a$. $u - = a$. Num$u$ = Num$a)]$} Df

·1 $a \varepsilon$ Cls .⊃:
Num a ε infn .=. ∃ » » » » » »

·2 $a \varepsilon$ Cls .⊃: Num$a$ ε infn .=. 1+Num$a$ = Num$a$ Dfp

Nos dice que numero de a es uno infinito, si existe aliquo classe u parte de a, et tale que numero de parte u aequa numero de classe totale a.

Existe plure infinito, de valore differente. Vide P11·2.

N_0 ✻ 9·0 N_0 = (Num'Cls) − infn Df

·1 $a \varepsilon$ Cls .⊃. Num a ε $N_0 \smile$ infn

·2 $a \varepsilon$ Cls .⊃:: Num$a$ ε N_0 .=.˙. −∃a .∪: $x \varepsilon a$.⊃$_x$. Num$a$ −= Num$(a - \iota x)$

·3 Num N_1 = Num N_0 [+ ε $(N_1$f$N_0)$rep]

·4 Num N_0 ε infn [$N_1 \supset N_0$. $N_1 - = N_0$. Num N_1 = Num N_0]

·5 $x\varepsilon$ Num'Cls . $y\varepsilon N_0$. $x \leqq y$.⊃. $x\varepsilon N_0$

·6 » » . $x \leqq$ Num N_0 . $x\varepsilon$ infn .⊃. $x =$ Num N_0

{ G. Cantor JfM. a.1877, AM. t.2 p.313

N_0 (jam considerato in Arithmetica) indica omni numero non infinito, vel vale « numero finito ». Es suo definitione in praesente novo theoria.

Numero de N_1, de numero pari, de numero impari aequa numero de N_0; Nam existe corrispondentia reciproco inter ces classe. Ergo, numero de N_0 es infinito.

✻ 10·1 $m\varepsilon N_1$.⊃. $m +$ Num $N_0 =$ Num N_0

[$N_1 = (1\cdots m) \cup (m+N_1)$. Num$(1\cdots m) = m$. Num$(m+N_1) =$ NumN_0]

·2 Num N_0 + Num N_0 = Num N_0

[$N_0 = (2N_0) \cup (2N_0+1)$. Num$(2N_0)$ = Num$(2N_0+1)$ = NumN_0]

·3 Num n = Num N_0

[[mod$(2x-1/2) - 1/2$] | x ε $(N_0$fn)rcp]

[(-1)↑rest$(x+1, 2) \times$ quot$(x+1, 2)$ | x ε (n f N_0)rcp]

Numero de n « numero integro positivo et negativo », aequa numero de N_0, vel de N_1 « numero naturale ». In vero si ad numero n:

n)	...	−3	−2	−1	0	+1	+2	+3 ...

nos fac corresponde:

N_0)	...	6	4	2	0	1	3	5 ...

resulta corrispondentia reciproco inter n et N_0. Si nos scribe in primo loco numero N_0, idem corrispondentia sume forma:

N_0)	0	1	2	3	4	5	6	...
n)	0	+1	−1	+2	−2	+3	−3	...

In demonstratione symbolico, nos exprime N_0 per n, et n per N_0.

·4 $m\varepsilon N_0$.⊃. $m\times$ Num N_0 = Num N_0

·5 Num$(N_0 \,\vdots\, N_0)$ = Num N_0

[[$(x+y)(x+y+1)/2+y$] | $(x;y)$ ε N_0f$(N_0\vdots N_0)$rcp]

{ G. Cantor, RdM. a.1895 t.5 p.144 {

In vero, figura fac corresponde numero scripto ad numero de horizontale et ad numero de verticale.

0	1	3	6	10	...
2	4	7	11	...	
5	8	12	...		
9	13	...			
14	...				
...					

·6 Num $N_0 \times$ Num N_0 = Num N_0 [=P·5]

·7 Num R = Num N_0

[R=N_1/N_1 .⊃. NumR ≦ Num$N_1 \times$NumN_1 =NumN_1 . NumR ε infn . P9·6 .⊃P]

·8 $m\varepsilon N_1$.⊃. $(\text{Num}N_0)^m$ = Num N_0

·9 $\mathrm{Num}\{(\mathrm{N}_0\mathrm{F}0\cdots n)|n`\mathrm{N}_0\} = \mathrm{Num}\mathrm{N}_0$

} CANTOR AM. t.2 p.306:

« On peut faire correspondre un à un les nombres algébriques aux nombres appartenants à la série des entiers positifs ». {

Classe *u* vocare « numerabile », « de primo potestate » si $\mathrm{Num}u = \mathrm{Num}\mathrm{N}_0$. n, R, r es classe numerabile (P·3·7).

Si ad classe numerabile nos adde numero finito de elemento, resulta classe numerabile (P·1). Classe que consta de duo classe numerabile (P·2), vel de *m* classe numerabile (P·4) es numerabile. Classe formato per systema de 2, 3,... *m*,... individuo de classe numerabile es numerabile (P·8). Et classe de systema de numero finito, sed arbitrario, de individuo de classe numerabile, es numerabile (P·9). Per ex. numero de functiones algebrico de gradu finito, cum coefficientes integro, vel rationale, vel in aliquo classe numerabile, vale $\mathrm{Num}\mathrm{N}_0$.

* 11·1 $f \varepsilon\, \Theta\,\mathrm{f}\,\mathrm{N}_0 \,.\supset.\, \exists\, \Theta - f`\mathrm{N}_0$

[$\Sigma[10^{-n}\,\mathrm{rest}(\mathrm{Cfr}_{-n} fn + 5,\ 10) \mid n,\ \mathrm{N}_1] \ \varepsilon\ \Theta - f`\mathrm{N}_0$]

Si *f* es serie de quantitates, in aliquo intervallo finito, p. ex. Θ, semper existe in ce intervallo, aliquo numero que non pertine ad serie.

In vero, me evolve in fractione decimale omni numero de serie. Me forma numero decimale, que habe:

cifra decimale de ordine 1 differente de cifra decimale de ordine 1 de primo numero de serie,

cifra decimale de ordine 2 differente de cifra decimale de ordine 2 de secundo numero de serie,

et ita porro. Me forma cifra differente de cifra dato, cum additione d 5, et substractione de 10, si es necesse.

Numero que resulta es differente de omni numero de serie.

} CANTOR JfM. a.1874 p.258; AM. p.308 {

·2 $\mathrm{Num}\,\Theta > \mathrm{Num}\,\mathrm{N}_0$ [P·1 $\supset$ R]

Ergo numeros reale inter 0 et 1 non es ordinabile in serie.

$\mathrm{Num}\Theta$ vocare « secundo potestate » vel « potestate de continuo ».

·3 $\mathrm{Num}\,\mathrm{Q} = \mathrm{Num}\,\mathrm{q} = \mathrm{Num}\,\Theta = \mathrm{Num}\,(\Theta - \mathrm{R})$

} CANTOR JfM. a.1877 p.242 ; AM. a.1883 p.316 {

·4 $2 \uparrow \mathrm{Num}\,\mathrm{N}_0 = \mathrm{Num}\,\mathrm{N}_0 \uparrow \mathrm{Num}\,\mathrm{N}_0 = \mathrm{Num}\,(\mathrm{Cls}`\mathrm{N}_0) = \mathrm{Num}\,\mathrm{Q}$

·5 $\mathrm{Num}(\mathrm{q}:\mathrm{q}) = \mathrm{Num}\mathrm{Q}$

·6 $n \varepsilon \mathrm{N}_1 \,.\supset.\, \mathrm{Num}(\mathrm{q}\ \mathrm{F}\ 1\cdots n) = \mathrm{Num}\,\mathrm{Q}$

} CANTOR AM. t.2 p.315 :

« On peut faire correspondre d'une façon complète et à sens unique un ensemble continu à *n* dimensions à un ensemble continu d'une seule dimension ». {

§21 $\Lambda \quad \lambda \quad \nabla \quad \delta$

* 1. $u, v \varepsilon \text{Cls'q} - \iota\Lambda \,.\supset.$

·0 $\Lambda' u = 1'\text{‘}(\text{Cls'}u - \iota\Lambda) \quad . \quad \Lambda_{\prime} u = 1_{\prime}\text{‘}(\text{Cls'}u - \iota\Lambda)$ Df

·01 $\Lambda u = \Lambda' u \cup \Lambda_{\prime} u$ Df Λ

·02 $\lambda u = \text{q} \cap \Lambda u = \text{q} \cap x \ni [\, 1_{\prime} \operatorname{mod}(u - x) = 0 \,]$ Df λ

Si u es classe non nullo de quantitate:

$\Lambda' u =$ limites supero de classes non nullo in u.

$\Lambda_{\prime} u =$ » infero » » .

$\Lambda u =$ « limites de u » vel « classe limite » es classe composito de duo classe praecedente.

Nos pote lege « $x \,\varepsilon\, \Lambda u$ » per « x es prope (proximo) ad x ».

λu indica limites finito de u; id es classe composito per quantitates x tale que limite infero de valores absoluto de differentias inter omni u et x es nullo.

$1_{\prime} \operatorname{mod}(u - x)$, in Geometria, es dicto « distantia de figura u ad puncto x », et coincide cum distantia minimo, quando minimo existe. Ergo λu es classe de punctos x, que habe distantia nullo de figura u.

·03 $x \varepsilon \lambda u \,.=:\, x \varepsilon \text{q} : h \varepsilon \text{Q} \,.\supset_h .\, \exists\, u \cap y \ni [\, \operatorname{mod}(y - x) < h \,]$ Dfp

[P·02 . Oper $x\varepsilon$. Df $1_{\prime}$ $.\supset.$ P]

·1 $u \supset \lambda u \cap \Lambda' u \cap \Lambda_{\prime} u$

[$x \varepsilon u \,.\supset.\, 0 \varepsilon\, u - x \,.\supset.\, 0 \varepsilon \operatorname{mod}(u - x) \,.\supset.\, 1_{\prime} \operatorname{mod}(u - x) = 0 \,.\supset.\, x \varepsilon \lambda u$]

·2 $\lambda\lambda u = \lambda u$

[$(\lambda u \mid u)$P·1 $.\supset.\, \lambda u \supset \lambda\lambda u$ (1)

$x \varepsilon \lambda\lambda u \,.\, h \varepsilon \text{Q} \,.$ P·01 $.\supset.\, \exists\, \lambda u \cap y \ni [\, \operatorname{mod}(y - x) < h/2 \,]$ (2)

Hp(2) $.\, y \varepsilon \lambda u \,.\supset.\, \exists\, u \cap z \ni [\, \operatorname{mod}(z - y) < h/2 \,]$ (3)

Hp(3) $.\, \operatorname{mod}(y - x) < h/2 \,.\, z \varepsilon u \,.\, \operatorname{mod}(z - y) < h/2 \,.\supset.\, \operatorname{mod}(z - x) < h$ (4)

Hp(4) $.\supset.\, \exists\, u \cap z \ni [\, \operatorname{mod}(z - x) < h \,]$ (5)

$x \varepsilon \lambda\lambda u \,.\, h \varepsilon \text{Q} \,.$ (5) Elim(y, z) . (2).(3) $.\supset.\, \exists\, u \cap z \ni [\, \operatorname{mod}(z - x) < h \,]$ (6)

$x \varepsilon \lambda\lambda u$. (6) Export $.\supset.\, x \varepsilon \lambda u$ (7)

(1).(7) $.\supset.$ P]

·3 $\Lambda(u \cup v) = \Lambda u \cup \Lambda v$ Distrib(Λ,$\cup$)

[Df λ $.\supset.$ $\lambda(u \cup v) = q \cap x \ni \{1, \text{mod}[(u \cup v) - x] = 0\}$
Distrib(‘,$\cup$) $.\supset.$ » $= q \cap x \ni \{1,[\text{mod}(u-x) \cup \text{mod}(v-x)] = 0\}$
§Q 13·8 $.\supset.$ » $= q \cap x \ni \{1,\text{mod}(u-x) = 0 .\cup. 1, \text{mod}(v-x) = 0\}$
Df λ $.\supset.$ » $= \lambda u \cup \lambda v$]

·31 $u \supset v .\supset. \Lambda u \supset \Lambda v$
[Hp $.\supset.$ $v = u \cup v$. P·3 $.\supset.$ $\Lambda v = \Lambda u \cup \Lambda v$ $.\supset.$ Ths]

·4 $\Lambda(u \cap v) \supset \Lambda u \cap \Lambda v$
[$u \cap v \supset u$. $u \cap v \supset v$. P·31 $.\supset.$ $\lambda(u \cap v) \supset \lambda u$. $\lambda(u \cap v) \supset \lambda v$. Cmp $.\supset.$ P]

·41 $\lambda u = u$. $\lambda v = v .\supset. \lambda(u \cap v) = u \cap v$
[Hp . P·4 . P·1 $.\supset.$ $\lambda(u \cap v) \supset u \cap v$. $u \cap v \supset \lambda(u \cap v)$ $.\supset.$ Ths]

·5 $l'u = l' \lambda u = \max \Lambda u$: $l_{,}u = l_{,} \lambda u = \min \Lambda u$

·6 $l'u \,\varepsilon\, q .\supset. l'u = \max \lambda u$: $l_{,}u \,\varepsilon\, q .\supset. l_{,}u = \min \lambda u$

·7 $\Lambda u = \lambda u \cup \iota l'u \cup \iota l_{,}u$ Dfp Λ

Classe Λu es formato de classe λu, et de ∞ et de $-\infty$, quando illos es limite supero, aut infero, des u.

·8 $l'u, l_{,}u \,\varepsilon q .\supset. \Lambda u = \lambda u$

❋ 2. $u,v \varepsilon$ Cls'q $- \iota\Lambda$. $a \varepsilon q .\supset$:

·1 $\lambda(a+u) = a + \lambda u$
[$x \varepsilon \lambda(a+u) .=. 1, \text{mod}(a+u-x) = 0 .=. 1, \text{m}[u-(a-x)] = 0 .=.$
$x - a \,\varepsilon\, \lambda u .=. x \varepsilon\, a + \lambda u$]

·11 $\lambda u + \lambda v \supset \lambda(u+v)$
[$a = b + c$. $b \varepsilon \lambda u$. $c \varepsilon \lambda v$ $.\supset.$ $1, \text{m}(u-b) = 0$. $1, \text{m}(v-c) = 0$ $.\supset.$
$1, \text{mod}(u+v-b-c) = 0$ $.\supset.$ $1, \text{mod}(u+v-a) = 0$ $.\supset.$ $a \varepsilon \lambda(u+v)$ (1)
(1) . Elim(b;c) $.\supset.$ P]

·12 $\lambda(u+v) = \lambda(\lambda u + \lambda v)$
[P2·11 . P1·1 . Oper λ $.\supset.$ $\lambda(\lambda u + \lambda v) \supset \lambda(u+v)$. $u+v \supset \lambda u + \lambda v$ $.\supset.$ P]

·13 $l' \text{mod}\, u, l' \text{mod}\, v \,\varepsilon\, Q .\supset. \lambda(u+v) = \lambda u + \lambda v$ Distrib(λ,+)

·2 $\lambda(a \times u) = a \times \lambda u$ ·21 $\lambda u \times \lambda v \supset \lambda(u \times v)$

·3 $m \varepsilon N_1 .\supset. \lambda(u^m) = (\lambda u)^m$ Comm(λ,↑)

·31 $m \varepsilon q$. $u \varepsilon$ Cls'Q $.\supset.$ Ths·3

·32 $a \varepsilon Q .\supset. \lambda(a \uparrow u) = a \uparrow (\lambda u)$

·4 $\Lambda(-u) = -\Lambda u$ Comm(Λ,$-$)

·5 $0 -\varepsilon \lambda u$. l'mod$u$ εQ .⊃ $\lambda/u = /\lambda u$ Comm(λ,/)

·6 $\Lambda N_0 = N_0 \cup \iota\infty$. Λn = n$\cup\iota\infty\cup\iota-\infty$.
$\Lambda R = Q \cup \iota 0 \cup \iota\infty$. Λr = q $\cup\iota\infty\cup\iota-\infty$.
$\lambda/N_1 = /N_1 \cup \iota 0$. $\lambda(/N_1 + /N_1) = (/N_1 + /N_1) \cup \iota 0$

✻ 3. $u\varepsilon$ Cls'q $-\iota\Lambda$.⊃: ·1 Num$u$ εN_1 .⊃. $\Lambda u = u$

·2 $a,b\varepsilon$q . $a<b$.⊃. $\Lambda(a^- b) = a^{\vdash} b$

·3 $a\varepsilon$ Q$-\iota$1 .⊃. λ "Log$u$ = "Log λu

✻ 4. $u,v\varepsilon$ Cls'q $-\iota\Lambda$.⊃.

·0 $\nabla u = x\ni[x\varepsilon\ \Lambda(u-\iota x)]$ Df ∇

·01 δu = q$\frown\nabla u$ = q$\frown x\ni${l, mod[$(u-\iota x)-x$] =0} Df δ

∇u, «classe derivata de u» es classe de x que es prope alios u, id es de u differente de x. δu indica valores finito in ∇u.

·1 u⊃v .⊃. ∇u ⊃ ∇v ·2 $\nabla(u\cup v) = \nabla u \cup \nabla v$

·3 $\delta\delta u$ ⊃ δu ·4 $m\varepsilon N_1$.⊃. $\delta^m u$ ⊃ δu

Classe derivata de derivata de u continere in derivata de u. Nota suo differentia de P1·2.

·5 ∇n $= \pm\infty$. $\nabla/N_1 = \iota 0$
$\nabla(/N_1 + /N_1) = \iota 0 \cup /N_1$. $\nabla(/N_1 - /N_1) = /N_1 \cup -/N_1 \cup \iota 0$

·6 $a\varepsilon$q .⊃. $\delta(a+u) = a + \delta u$

·7 $(u+\delta v)\cup(v+\delta u)\cup(\delta u+\delta v)$ ⊃ $\delta(u+v)$

·8 l'mod$u$, l'mod$v$ εQ .⊃. $\delta(u+v) = (u+\delta v)\cup(v+\delta u)\cup(\delta u+\delta v)$

·9 $\Lambda u = u \cup \nabla u$ Dfp

✻ 5. Hp P4 .⊃.

·1 Num u ε infn .⊃. ∃ ∇u : Num u εN_1 .⊃. $\nabla u = \Lambda$

·2 δu⊃u .⊃. Num$u$ ε $N_0 \cup \iota$NumN$_0 \cup \iota$Num Q

·3 $\delta u = u$.⊃. Num$u$ = NumQ

·4 $u \frown \delta u = \Lambda$.⊃. Num$u$ ε $N_0 \cup \iota$ Num N_0

·5 Num δu = Num N_0 .⊃. Num$u$ = Num N_0

{ G. Cantor MA. a.1882 p.51; a.1884 p.485; AM. a.1883 p.374 }

Bibliographia: Vivanti Formulario 1895 p.71.

§ 22 Intv in ex am

❋ 1·0 $\text{Intv} = x \ni \exists(a,b) \ni (a,b \varepsilon \text{q} \,.\, a < b \,.\, x = a \frown b)$ Df

« Intv », lege « intervallo » es omni objecto reductibile ad forma $a \frown b$, ubi a et b es quantitate determinato et finito, et a es minore de b. Notatione utile in calculo integrale.

·1 $u \varepsilon \text{Intv} \,.\supset.\, u = \text{l}_{\prime} u \frown \text{l}' u$

·2 $u \varepsilon \text{Intv} \,.\supset.\, \text{Long}\, u = \text{l}' u - \text{l}_{\prime} u$ Df

= amplitudo, longitudo, long-ore de u.

❋ 2. $u,v \varepsilon \text{Cls'q} \,.\supset.$ ·0 $\text{in}u = \text{q} - \lambda(\text{q} - u) = u - \delta(\text{q} - u)$. $\text{ex}u = \text{in}(\text{q} - u) = \text{q} - \lambda u$. $\text{am}u = \text{q} - \text{in}u - \text{ex}u = \lambda u \cap \lambda(\text{q} - u)$ Df

·1 $\text{in}u \supset u$. $\text{ex}u \supset -u$. $u \supset \text{in}u \cup \text{am}u$. $\text{q} - u \supset \text{ex}u \cup \text{am}u$

·2 $\text{in}u \cap \text{ex}u = \Lambda$. $\text{in}u \cap \text{am}u = \Lambda$. $\text{ex}u \cap \text{am}u = \Lambda$. $\text{q} = \text{in}u \cup \text{ex}u \cup \text{am}u$

·3 $\text{in in}\, u = \text{in}u$. $\text{in ex}u = \text{ex}u$. $\text{am}u = \text{in am}u \cup \text{am am}u$. $\text{am am}u = \text{am in}u \cup \text{am ex}u$. $\text{in}(u \cap \text{am}u) = \Lambda$. $\text{ex am}u = \text{in}u \cup \text{ex}u$. $\text{ex in}u = \text{q} - (\text{in}u \cup \text{am in}u)$. $\text{ex ex}u = \text{q} - (\text{ex}u \cup \text{am ex}u)$. $\text{in am in}u = \Lambda$. $\text{in am ex}u = \Lambda$. $\text{in am am}u = \Lambda$. $\text{am am am}u = \text{am am}u$. $\text{am am in}u = \text{am in}u$. $\text{am am ex}u = \text{am ex}u$. $\text{am in am}u \supset \text{am am}u$

·4 $\text{in}(u \cap v) = \text{in}u \cap \text{in}v$. $\text{ex}(u \cup v) = \text{ex}u \cap \text{ex}v$. $\text{in}u \cup \text{in}v \supset \text{in}(u \cup v) \supset \text{in}u \cup \text{in}v \cup \text{am}u\,\text{am}v$. $\text{exu} \cup \text{ex}v \supset \text{ex}(u \cap v) \supset \text{ex}u \cup \text{ex}v \cup \text{am}u\,\text{am}v$. $\text{in}u\,\text{am}v \cup \text{in}v\,\text{am}u \supset \text{am}(u \cap v) \supset \text{in}u\,\text{am}v \cup \text{in}v\,\text{am}u \cup \text{am}u\,\text{am}v$. $\text{ex}u\,\text{am}v \cup \text{ex}v\,\text{am}u \supset \text{am}(u \cup v) \supset \text{ex}u\,\text{am}v \cup \text{ex}v\,\text{am}u \cup \text{am}u\,\text{am}v$. $\text{in}\,(\text{in}u \cup \text{in}v) = \text{in}u \cup \text{in}v$. $\text{in}(\text{am am}u \cup \text{am am}v) = \Lambda$

| Peano, Arithmetices principia, a.1889 |

in$u$ es classe de numeros reale, vel de punctos, in u, (inter u, interno ad u, interiore ad u).

ex$u$ indica punctos ex (extra, externo ad, exteriore ad) u.

am$u$ indica punctos am (circum), « frontière » (Jordan a.1893).

am es vocabulo de antiquo latino (Catone, ...). ⊃ L am-bo HI, am-plo A, am-puta A, amb-iguo A, amb-itione DR.

‖ G amphi ⊃ amphi-theatro ADFHIR, amphi-bio, ...

‖ D um. ‖ ? S abhi, R oba.

§23 prob (probabilitate)

$a,b,s\varepsilon$ Cls . Num$s$ εN_1 .⊃.

·0 prob(a,s) = Num$(a \frown s)$ / Num$s$ Df

a et s es classe; numero des s es finito. Tunc probabilitate de casu a inter casu s es ratione de numero des s que es a ad numero totale des s.

Objectos de classe s dicere « casus possibile » ; objectos de classe $a \frown s$ « casus favorabile ».

In applicationes de probabilitate ad praxi, casus s es supposito possibile in gradu aequale.

Per exemplo, probabilite que pila extracto ex urna, que contine m pila albo et n nigro, es albo, vale $m/(m+n)$, Nos suppone que pilas habe idem volumen pondere, forma...

·01 a ⊃ s .⊃. prob(a,s) = Num$a$ / Num$s$ Dfp

Si classe a de P·0 continere in s, Df sume forma plus simplice ·01.

·1 prob(Λ,s) = 0 ·11 prob(s,s) = 1

·2 prob$(-a, s)$ = 1 − prob(a,s)

·3 $a \frown b = \Lambda$.⊃. prob$(a \smile b, s)$ = prob(a,s) + prob(b,s)

[Hp .⊃. Num$(a \smile b) \frown s$ = Num$(a \frown s)$ + Num$(b \frown s)$. Oper Num$s$.⊃. P]

Si nos parti casus favorabile $a \smile b$ in duo classe a et b, probabilitate de $a \smile b$ es summa de probabilitates de a et b. Ce P appellare « theorema de probabilitate totale ».

·4 prob$(a \frown b,s)$ = prob(a,s) × prob$(b, a \frown s)$

Probabilitate, dicto composito per $\frown$, que inter casus s se praesenta in idem tempore casus a et b, es producto de probabilitate que inter s occurre a, per probabilitate de casu b inter s que es a, id es, per probabilitate de b, quando nos sci que a es facto.

·5 $u,v\varepsilon$Cls . Num$u$,Num$v \varepsilon N_1$. $a,b\varepsilon$Cls .⊃. prob$(a{:}b, u{:}v)$ = prob(a,u) × prob(b,v)

Probabilitate composito per ⁝.

✻ 2·1 prob$(2N_1, 1 \cdots 10)$ = 1/2 . prob$(3N_1, 1 \cdots 10)$ = 3/10
prob$(3N_1, 1 \cdots 10 \frown 2N_1)$ = 1/5 . prob$(6N_1, 1 \cdots 10)$ = 1/10

Probabilitate que uno numero inter numeros de 1 ad 10 es pari vale 1/2. Probabilitate que illo es multiplo de 3 vale 3/10. Probabilitate que illo es multiplo de 6 vale producto de probabilitate que illo es pari, per probabilitate que numero pari inter 1 et 10 es multiplo de 3.

·2 $n \varepsilon N_1 . \supset .$
$\}\mathrm{prob}[(1 \cdots n F 1 \cdots n)\mathrm{rcp} \cap f \ni (x \varepsilon 1 \cdots m . \supset_x . fx -= x), (1 \cdots n F 1 \cdots n)\mathrm{rcp}]\{$
$= \Sigma[\{(-1)^n / n! \, | n, 0 \cdots n\}]$

} Moivre *Misc. Anal.* a.1730 p.185 {

Probabilitate que in extractione de numeros $1 \cdots n$, nullo se praesenta in suo ordine.

·3 $m,n \varepsilon N_1 . p \varepsilon 1 \cdots m . q \varepsilon 1 \cdots n . a,b \varepsilon \mathrm{Cls} . \mathrm{Num}a = m . \mathrm{Num}b = n$
$a \cap b = \Lambda . \supset . \mathrm{prob}[\mathrm{Cls}'(a \cup b) \cap u \ni (\mathrm{Num}u \cap a = p . \mathrm{Num}u \cap b = q),$
$\mathrm{NumCls}'(a \cup b) \cap u \ni (\mathrm{Num}u = p+q)]$
$= C(m,p) \times C(n,q) / C(m+n, p+q)$
$= m!n!(p+q)!(m+n-p-q)! / [(m+n)!p!q!(m-p)!(n-q)!]$

Urna contine pilas de specie a, lege albo, in numero de m, et pilas b, lege nigro, in numero de n. Nos extrahe combinatione de $p+q$ pila. Formula da probabilitate que combinatione extracto contine p pila albo et q nigro.

·4 $s \varepsilon \mathrm{Cls} . \mathrm{Num}s \varepsilon N_1 . a \varepsilon \mathrm{Cls} . m \varepsilon N_1 . n \varepsilon 1 \cdots m . \supset . \mathrm{prob}[s F 1 \cdots m \cap$
$u \ni [\mathrm{Num} 1 \cdots m \cap x \ni (ux \varepsilon a) = n], s F 1 \cdots m] =$
$C(m,n) \mathrm{prob}(a,s)^n \mathrm{prob}(-a,s)^{m-n}$

Si s es casus possibile, a es casus favorabile, formula da probabilitate que in repetitione de m vice de experimento, n vice illo es favorabile.

§24 Cx

✻ 1·0 $n \varepsilon N_1 . \supset . \mathrm{Cx}\, n = q \, F \, 1 \cdots n$ Df

Nos voca « numero complexo de ordine n » et indica per Cxn, complexo de n numero reale.

$n \varepsilon N_1 . a,b,c \varepsilon \mathrm{Cx}n . h,k \varepsilon q . \supset \therefore$

·1 $a = b . =: r \varepsilon 1 \cdots n . \supset_r . a_r = b_r$ [III §4 F P1·6 $. \supset .$ P]

Duo complexo es aequale quando habe elementos correspondenté æquale.

·2 $a+b = \iota \, \mathrm{Cx}n \cap x \ni (r \varepsilon 1 \cdots n . \supset_r . x_r = a_r + b_r)$
$= [(a_r + b_r) | r, 1 \cdots n]$ Df

Summa de duo complexo es complexo que habe ut elementos summa de elementos correspondente.

·3 $0 = (\iota 0 \,\vdots\, 1 \cdots n)$ Df

Df de complexo nullo.

·4 $-a = \imath\, \mathrm{Cx}n \frown x\ni(a+x=0) = (-a_r|r, 1\cdots n)$ Df

·5 $a-b = a+(-b)$ Df

·6 $ha = \imath\, \mathrm{Cx}n \frown x\ni(r\varepsilon 1\cdots n .\supset_r. x_r = ha_r)$
$= (ha_r|r, 1\cdots n)$ Df

·7 $a+b = b+a$. $a+(b+c) = (a+b)+c = a+b+c$.
$a+0 = a$. $-0 = 0$. $a-a = 0$.
$ha\, \varepsilon\, \mathrm{Cx}n$. $1a = a$. $h(a+b) = ha+hb$.
$(h+k)a = ha+ka$. $h(ka) = (hk)a = hka$

❋ 2. unit

$n\varepsilon \mathrm{N}_1 . r\varepsilon 1\cdots n .\supset.$

·0 $\mathrm{unit}(n,r) = \imath\, \mathrm{Cx}n \frown x\ni(x_r = 1 : s\varepsilon (1\cdots n) - \iota r .\supset_s. x_s = 0)$ Df

$a,b\varepsilon\, \mathrm{Cx}n . h\varepsilon \mathrm{q} .\supset.$

·01 $\mathrm{unit}(n,r) = \{\mathrm{E}[(r+s+1)/n] - \mathrm{E}[(r+s)/n]|s, 1\cdots n\}$

·1 $a = \Sigma[a_s\, \mathrm{unit}(n,s)\, |s, 1\cdots n]$

·2 $a+b = \Sigma[(a_s+b_s)\, \mathrm{unit}(n,s)\, |s, 1\cdots n]$

·3 $ha = \Sigma[(ha_s)\, \mathrm{unit}(n,s)\, |s, 1\cdots n]$

« unit (n,r) », « unitate de ordine n et de loco r », « Haupteinheite » es complexo, que habe 1 ut elemento de loco r, et omni alio nullo.

mod ❋ 3. $n\varepsilon \mathrm{N}_1 . x,y\varepsilon\, \mathrm{Cx}n . a\varepsilon \mathrm{q} .\supset.$

·0 $\mathrm{mod}x = \sqrt{[\Sigma(x_r^2|r,1\cdots n)]}$ Df

·1 $\mathrm{mod}x\ \varepsilon \mathrm{Q}_0$

·2 $\mathrm{mod}(x+y) \leqq \mathrm{mod}x + \mathrm{mod}y$

[§Σ 9·1 $.\supset. (x_1^2+x_2^2+\ldots)(y_1^2+y_2^2+\ldots) \geqq (x_1y_1+x_2y_2+\ldots)^2$.
P·0 $.\supset.$ $\mathrm{mod}x\, \mathrm{mod}y \geqq x_1y_1+x_2y_2+\ldots$
$.\supset.$ $(\mathrm{mod}x)^2+(\mathrm{mod}y)^2+2\, \mathrm{mod}x\, \mathrm{mod}y \geqq (x_1+y_1)^2+(x_2+y_2)^2+\ldots$
$.\supset.$ $(\mathrm{mod}x+\mathrm{mod}y)^2 \geqq [\mathrm{mod}(x+y)]^2 .\supset.$ Ths]

·3 $\mathrm{mod}\,(ax) = \mathrm{mod}a\, \mathrm{mod}x$ ·4 $\mathrm{mod}x = 0 .=. x = 0$

$\lambda\ \delta$ ❋ 4. $(\mathrm{Cx}n \mid \mathrm{q})$ §λ §δ

·0 $u\varepsilon\, \mathrm{Cls}'\mathrm{Cx}n .\supset. \Lambda u = \lambda u \cup (\iota\infty \frown \iota\, \mathrm{l}'\mathrm{mod}u)$ Df
$\Delta u = \delta u \cup (\iota\infty \frown \iota\, \mathrm{l}'\mathrm{mod}u)$ Df

Med ❋ 5·0 $n\varepsilon \mathrm{N}_1 . u\varepsilon\, \mathrm{Cls}'\mathrm{Cx}n .\supset. \mathrm{Med}\, u = \mathrm{Cx}n \frown x\ni$
$\{a\varepsilon\, \mathrm{Cx}n .\supset_a. \Sigma(a_rx_r\, |r, 1\cdots n)\ \varepsilon\ \mathrm{Med}[\Sigma(a_rz_r\, |r, 1\cdots n)\, |z'u]\}$ Df

§Med P1·1·2·3·4·6

Nos dice que classe u es *convexo*, si $\mathrm{Med}u = u$.

§25 Dtrm (determinante)

* 1·0 $m\varepsilon N_1 . u\varepsilon(1\cdots m F 1\cdots m)\text{rcp} .\supset.$
$\text{sgn}u = (-1)\upharpoonright \text{Num}[(x;y)\ni(x,y\varepsilon 1\cdots m . x<y . ux>uy)]$ Df

·01 $m\varepsilon N_1 . u,v\varepsilon(1\cdots m F 1\cdots m)\text{rcp} .\supset. \text{sgn}\, uv = (\text{sgn}u)\times(\text{sgn}v)$

Si u es correspondentia reciproco, vel permutatione, de numeros $1\cdots m$, sgn$u$ indica unitate positivo aut negativo, secundo que numero de ambos $(x;y)$ que forma inversione, es pari aut impari.

·1 $m\varepsilon N_1 . a\varepsilon \text{qF}(1\cdots m \,\dot{:}\, 1\cdots m) .\supset.$
$\text{Dtrm}a = \Sigma\{\text{sgn}u\, \Pi[a(r,u_r) \mid r, 1\cdots m] \mid u, (1\cdots m F 1\cdots m)\text{rcp}\}$ Df
{ Leibniz a.1678 MathS. t.7 p.5 :

« Inveni Canonem pro tollendis incognitis quotcunque aequationes non nisi simplici gradu ingredientibus... Fiant omnes combinationes possibiles literarum coefficientium, ita ut nunquam concurrant plures coefficientes ejusdem incognitae et ejusdem aequationis. Hae combinationes affectae signis, ut mox sequetur, componantur simul... Lex signorum haec est. Uni ex combinationibus assignetur signum pro arbitrio, et caeterae combinationes quae ab hac differunt coefficientibus duabus, quatuor, sex etc., habebunt signum oppositum ipsius signo ; quae vero ab hac differunt coefficientibus tribus, quinque, septem etc., habebunt signum idem cum ipsius signo» . }

m es numero, et a es litera, que cum duo indice integro inter 1 et m, repraesenta quantitate.

Plure A. scribe valores de a super m linea horizontale et m verticale, et appella « matrice » ce figura.

Si u es permutatione de numeros $1\cdots m$, nos considera producto de valores $a(r,u_r)$, ubi r varia de 1 ad m, et multiplica illo per sgn$u$, id es per unitate positivo aut negativo, secundo que permutatione u habe numero pari aut impari de inversiones. Summa de productos, que resulta, si nos substitue u per omni permutatione de numeros de 1 ad m, vocare « determinante des a » (Gauss), que nos abbrevia in Dtrm$a$.

* 2. $m\varepsilon N_1 . a\varepsilon \text{qF}(1\cdots m \,\dot{:}\, 1\cdots m) .\supset:$

·1 $\text{Dtrm}[a(r,s) \mid (s,r)] = \text{Dtrm}a$

·2 $u\varepsilon\ (1\cdots m\ F\ 1\cdots m)\text{rcp} .\supset.$
$\text{Dtrm}\, a(r,u_s)|(r,s) = \text{sgn}u \times \text{Dtrm}a$

·3 $n\varepsilon N_1 . u,v\varepsilon \text{Cls}'N_1 . \text{Num}u = \text{Num}v = n . a\varepsilon \text{qf}(u\dot{:}v) .\supset.$
$\text{Dtrm}(a, u\dot{:}v) = \text{Dtrm}\{a(\min_r u, \min_s v) \mid (r,s), 1\cdots n\dot{:}1\cdots n\}$ Df

Dtrm$(a, u\dot{:}v)$, es dicto « subdeterminante, determinante partiale, complementare ».

·4 $n \varepsilon N_1 \,.\, a \varepsilon \mathrm{qF}(1 \cdots n : 1 \cdots n) \,.\, r \varepsilon 1 \cdots n \,.\supset.$

$\mathrm{Dtrm}a = \Sigma[(-1)^{r+s} a_{r,s}\, \mathrm{Dtrm}(a, 1 \cdots n - \iota r : 1 \cdots n - \iota s) \,|\, s, 1 \cdots n]$

{ CRAMER a.1750 p.656 }

·5 $n \varepsilon N_1 \,.\, a \varepsilon \mathrm{qF}(1 \cdots n : 1 \cdots n) \,.\, u \varepsilon \mathrm{Cls}\text{'}1 \cdots n \,.\, \exists u \,.\supset.$

$\mathrm{Dtrm}a = \Sigma\{ (-1)\uparrow(\Sigma u + \Sigma v) \times \mathrm{Dtrm}(a, u:v) \times \mathrm{Dtrm}(a, 1 \cdots n - u : 1 \cdots n - v) \,|\, v, (\mathrm{Cls}\text{'}1 \cdots n) \cap v \ni (\mathrm{Num}v = \mathrm{Num}u)\}$

{ LAPLACE ParisM. a.1772 p.267 }

·6 $\mathrm{Hp}\cdot 5 \,.\supset.\, \mathrm{Dtrm}[(-1)^{r+s}\, \mathrm{Dtrm}(a, 1 \cdots m - \iota r : 1 \cdots m - \iota s) \,|\, (r,s), 1 \cdots m : 1 \cdots m] = (\mathrm{Dtrm}a)\uparrow(m-1)$

{ CAUCHY JP. a.1812 p.82 }

·7 $n \varepsilon N_1 \,.\, a \varepsilon \mathrm{qf}(1 \cdots n : 1 \cdots n) \,.\supset.$

$\mathrm{mod}\, \mathrm{Dtrm}a \leqq \sqrt{\Pi[\Sigma(a_{r,s}^2 | s, 1 \cdots n) | r, 1 \cdots n]}$

·8 $\mathrm{mod}\, \mathrm{Dtrm}(a, 1 \cdots n : 1 \cdots n) \leqq [\max \mathrm{mod}\, a\text{'}(1 \cdots n : 1 \cdots n)]^n\, n\uparrow(n/2)$

❋ 3·1 $m \varepsilon N_1 \,.\, a,b \varepsilon \mathrm{qF}(1 \cdots m : 1 \cdots m) \,.\supset.$

$\mathrm{Dtrm}a \times \mathrm{Dtrm}b = \mathrm{Dtrm}\{[\Sigma(a_{r,t}\, b_{s,t} \,|\, t, 1 \cdots m) \,|\, (r,s), 1 \cdots m : 1 \cdots m]\}$

·2 $m,n \varepsilon N_1 \,.\, m > n \,.\, a,b \varepsilon \mathrm{qf}(1 \cdots m : 1 \cdots n) \,.\supset.$

$\mathrm{Dtrm}[\Sigma(a_{r,t}\, b_{s,t} \,|\, t, 1 \cdots m) \,|\, (r,s), 1 \cdots n : 1 \cdots n] =$

$\Sigma\{\mathrm{Dtrm}(a, v : 1 \cdots n) \times \mathrm{Dtrm}(b, v : 1 \cdots n) \,|\, v, (\mathrm{Cls}\text{'}1 \cdots m) \cap v \ni (\mathrm{Num}v = n)\}$

{ BINET a.1813 p.287 :

« ... Avec des x', x'', x''', &c., y', y'', y''', &c., z', z'', z''', &c., ayant formé $n \frac{n-1}{2} \frac{n-2}{3}$ résultantes à trois lettres, et aussi d'autres résultantes avec des ξ, v, ζ, semblablement accentués, on trouve que la somme des produits des résultantes correspondantes...

$$\Sigma(x,y',z'')(\xi,v',\zeta'') = \Sigma x\xi \Sigma yv \Sigma z\zeta + \Sigma y\xi \Sigma zv \Sigma x\zeta + \Sigma z\xi \Sigma xv \Sigma y\zeta - \Sigma x\xi \Sigma yv \Sigma z\zeta - \Sigma y\xi \Sigma xv \Sigma z\zeta - \Sigma z\xi \Sigma yv \Sigma z\zeta$$

Ce dernier membre est de la forme (x,y',z''); on en peut donc conclure que le produit d'un nombre quelconque de fonctions, telles que $\Sigma(x,y',z'')(\xi,v',\zeta'')$ est de la forme (x,y',z''). » }

·3 $m,n \varepsilon N_1 \,.\, m < n \,.\, a,b \varepsilon \mathrm{qf}(1 \cdots m : 1 \cdots n) \,.\supset.$

$\mathrm{Dtrm}[\Sigma(a_{r,t}\, b_{s,t} \,|\, t, 1 \cdots m) \,|\, (r,s), 1 \cdots n : 1 \cdots n] = 0$

❋ 4·1 $m \varepsilon N_1 + 1 \,.\, a \varepsilon \mathrm{qf}\, 1 \cdots m \,.\supset.$

$\mathrm{Dtrm}[a_r^{m-s} | (r,s), 1 \cdots m : 1 \cdots m] = \Pi\{\Pi[(a_r - a_s) | s, 1 \cdots (r-1)] | r, 2 \cdots m\}$

{ VANDERMONDE ParisM. a.1772 p.518 ; CAUCHY a.1821 p.426 }

·2 $m \varepsilon N_1+1 . \supset . \mathrm{Dtrm}[r^s|(r,s), 1\cdots m : 1\cdots m] = \Pi(r!\,|r, 1\cdots m)$

·3 $\mathrm{Hp}·1 . \supset . \mathrm{Dtrm}\{[\Sigma(a, N_1 \frown r/N_1 \frown s/N_1)]\,|(r,s), 1\cdots m : 1\cdots m\}$
$= \Pi(a, 1\cdots m)$ { MANSION *CorrN.* t.4 a.1878 p.109 }

Dvr mlt Φ ❊ 5. $m \varepsilon N_1+1 . \supset .$

·1 $\mathrm{Dtrm}(\mathrm{Dvr}, 1\cdots m : 1\cdots m) = \Pi(\Phi, 1\cdots m)$
———————— $= m!\, \Pi\{(1-/p) \upharpoonright \mathrm{E}(m/p)\,|p, \mathrm{Np} \frown 1\cdots m\}$
{ SMITH a.1876 t.2 p.161:

«Let (m,n) denote the greatest common divisor of the integral numbers m and n; and let $\psi(m)$ be the number of numbers not surpassing m and prime to m; the symmetrical determinant...

$$\Sigma \pm (1,1)\,(2,2) \ldots (m,m)$$

s equal to $\psi 1 \times \psi 2 \times \ldots \times \psi(m)$. » }

·2 $\mathrm{Dtrm}\{\mathrm{mlt}, 1\cdots m : 1\cdots m\} = m!\, \Pi\{[(1-p) \upharpoonright \mathrm{E}(m/p)]\,|\,p, \mathrm{Np} \frown 1\cdots m\}$
{ SMITH a.1876 t.2 p.163 }

Vide alios propositione super Dtrm, in Formul. t. 4, p. 207.

§26 lin Subst Sb

❊ 1. $m,n \varepsilon N_1 . \supset .$ ·0 $(\mathrm{Cx}m \mathrm{F} \mathrm{Cx}n)\mathrm{lin} =$
$(\mathrm{Cx}m \mathrm{F} \mathrm{Cx}n) \frown f\ni[x,y \varepsilon \mathrm{Cx}n . \supset_{x,y} . f(x+y) = fx + fy :$
$\mathrm{l'mod}f\text{‘}[\mathrm{Cx}n \frown u\ni(\mathrm{mod}u<1)] \varepsilon \mathrm{Q}$ Df

·01 $\mathrm{SubstCx}n = (\mathrm{Cx}n \mathrm{F} \mathrm{Cx}n)\mathrm{lin}$ Df

Nos dic que aliquo complexo de ordine m, functione de complexos de ordine n, es functione l i n e a r e, si functione de summa es summa de functiones. Nos adde conditione que limite supero de valores absoluto de functione in campo finito es finito, pro deduce P·11.

« Substitutione de complexos de ordine n », breviato in « SubstCxn », vale Cxn functione lineare des Cxn.

Functiones lineare, et substitutiones occurre in:

G a u s s a. 1801 t.1 p. 302

S e r v o i s Ann. a.1815 p.93, sub nomen : functiones d i s t r i b u t i v o.

B o o l e, Cambridge Math. Journal a. 1841 t.3 p.1, 106.

G r a s s m a n n a.1844, Werke t.2 p.241.

C a y l e y a. 1845, Math. Papers t,1

L a g u e r r e a, 1867, OEuvres t.1 p. 221-266.

P e i r c e, John Hopkins Circular, Baltimore a.1882 N.22.

$f,g,h \varepsilon$ (Cxm F Cxn)lin $. \supset :$

·1 $x,y \varepsilon$ Cxn $. \supset .$ $f(x+y) = fx+fy$

·11 $k \varepsilon$ q $. \supset .$ $fkx = kfx$ [§q(III §13) P13·3 $. \supset .$ P]

·2 $f+g = [(fx+gx)|x$, Cx$n]$ Df

·21 $f+g\ \varepsilon$ (Cxm F Cxn)lin

[Hp . $x,y \varepsilon$ Cxm $. \supset .$ $(f+g)(x+y) = f(x+y) + g(x+y) = fx+fy+gx+gy$ $= fx+gx+fy+gy = (f+g)x+(f+g)y$]

·22 $f+g = g+f$ ·23 $(f+g)+h = f+(g+h) = f+g+h$

Summa de functiones lineare es commutativo et associativo.

$m,n,p \varepsilon N_1$. $f,f' \varepsilon$ (Cxm F Cxn)lin . $g,g' \varepsilon$ (Cxp F Cxm)lin . $k \varepsilon$ q $. \supset .$

·3 $gf = [gfx|x$, Cx$n]$ Df

·31 $gf\ \varepsilon$ (Cxp F Cxn)lin

[Hp . $x,y \varepsilon$ Cxn . §f P2·2 $. \supset .$ $(gf)(x+y) = g[f(x+y)] = g(fx+fy) = g(fx)$ $+g(fy) = (gf)x+(gf)y$]

·32 $g(f+f') = gf+gf'$. $(g+g')f = gf+g'f$

Producto de functiones lineare es distributivo et associativo, sed non commutativo in generale.

·4 $k = (k\times$, Cx$n)$ Df ·41 $k \varepsilon$ Subst Cx n . $fk = kf$

Multiplicatione per numero reale es functione lineare.

·5 $a \varepsilon$ Subst$n$ $. \supset .$ $a^m\ \varepsilon$ Subst$n$

·6 $p \varepsilon N_1$. $a \varepsilon$ Cxn F $1 \cdots p$. $-\exists$ Cx$p - \iota 0 \frown x \ni [\Sigma(x_r\, a_r\, |r, 1 \cdots n) = 0]$. $b \varepsilon$ Cxm F $1 \cdots p$ $. \supset .$ $b/a = \imath$ [Cxm F Σ (q$a, 1 \cdots p$)] lin $\frown$ $f \ni [r \varepsilon 1 \cdots p\ . \supset_r .\ fa_r = b_r]$ Df

Si a et b es successione de p complexo de respectivo ordine n et m, tunc b/a indica illo correspondentia lineare inter complexos de ordine m et complexos de systema qa_1+qa_2+...+qa_p , tale que ad $a_1, a_2, \ldots a_p$ responde $b_1, b_2, \ldots b_p$. Nos suppone que complexos a es independente, id es que non existe successione de numeros $x_1, x_2, \ldots x_p$, non omni nullo, que redde $x_1 a_1 + x_2 a_2 + \ldots + x_p\, a_p = 0$.

mod Subst

* 2. $n \varepsilon N_1$. $a,b \varepsilon$ SubstCxn . $x \varepsilon$ Cxn . $k \varepsilon$ Q $. \supset .$

·0 mod$a$ = max{ [(mod ax)/(mod$x$)] | x '(Cx$n - \iota 0$) } Df

·1 $\mathrm{mod}\,a\ \varepsilon\mathrm{Q}_0$

[$x\varepsilon\ \mathrm{Cx}n - \iota 0\ .\ y = x/\mathrm{mod}x\ .\supset.$
$y\varepsilon\,\mathrm{Cx}n\ .\ \mathrm{mod}y = 1\ .\ (\mathrm{mod}\,ax)/\mathrm{mod}x = (\mathrm{mod}\,ay)/\mathrm{mod}y\ .\supset.$
$(\mathrm{mod}\,ax)/\mathrm{mod}x\ |x\ \text{‘}(\mathrm{Cx}n - \iota 0) = (\mathrm{mod}\,ax)/\mathrm{mod}x\ |x\ \text{‘}[\mathrm{Cx}n \frown y\ni(\mathrm{mod}y=1)$ (1)
$[(\mathrm{mod}\,ax)/\mathrm{mod}x\,]|x\ \varepsilon\ \mathrm{Q}_0\mathrm{f}[\mathrm{Cx}n \frown y\ni(\mathrm{mod}y=1)]\mathrm{cont}$ (2)
(2) . §cont 1·3 . (1) . P·0 .⊃. P]

·11 $\mathrm{mod}\,ax \leqq \mathrm{mod}a\ \mathrm{mod}x$ [P·0 . P·1 .⊃. P]

·12 $\mathrm{mod}a = 0\ .=.\ a = 0$

·2 $\mathrm{mod}(a+b) \leqq \mathrm{mod}a + \mathrm{mod}b$

[Hp . $x\varepsilon\ \mathrm{Cx}n$. P2·1 .⊃. $\mathrm{mod}\,(a+b)x = \mathrm{mod}(ax+bx)$
———— . §cx 3·2 .⊃. ———— $\leqq \mathrm{mod}\,ax + \mathrm{mod}\,bx$
———— . P·11 .⊃. ———— $\leqq \mathrm{mod}a\ \mathrm{mod}x + \mathrm{mod}b\ \mathrm{mod}x$
$\leqq (\mathrm{mod}a + \mathrm{mod}b)\mathrm{mod}x$ (1)
Hp . (1) .⊃: $x\varepsilon\mathrm{Cx}n\ .\supset.\ [\mathrm{mod}\,(a+b)x]/\mathrm{mod}x \leqq \mathrm{mod}a + \mathrm{mod}b$ (2)
(2) . P·0 .⊃. P]

·21 $\mathrm{mod}(ka) = k\,\mathrm{mod}a$

·3 $\mathrm{mod}(ab) \leqq \mathrm{mod}a\ \mathrm{mod}b$

[Hp . $x\varepsilon\ \mathrm{Cx}n$. P·1 .⊃.
$\mathrm{mod}[(ab)x] = \mathrm{mod}[a(bx)] \leqq \mathrm{mod}a\ \mathrm{mod}\,bx \leqq \mathrm{mod}a\ \mathrm{mod}b\ \mathrm{mod}x$
Hp : $x\varepsilon\mathrm{Cx}n\ .\supset.\ [\mathrm{mod}\,(ab)x]/\mathrm{mod}x \leqq \mathrm{mod}a\ \mathrm{mod}b$: P·0 :⊃. P]

·4 $m\varepsilon\mathrm{N}_1\ .\supset.\ \mathrm{mod}(a^m) \leqq (\mathrm{mod}a)^m$ [P·3 .⊃. P]

Sb ✻ 3. $n\varepsilon\mathrm{N}_1\ .\ u,v\varepsilon\ \mathrm{qF}(1\cdots n : 1\cdots n)\ .\supset.$

·0 $\mathrm{Sb}u = \{\,[\,\Sigma(u_{r,s}x_s\,|s,\,1\cdots n\,)|r,\,1\cdots n\,]|x,\,\mathrm{Cx}n\,\}$ Df

Si u es matrice quadro de ordine n, Sbu (substitutione repraesentato per matrice u) indica operatione que, ad complexo x fac corresponde complexo que habe, ut elemento de loco r, $\Sigma\,(u_{r,s}x_s\,|s,\,1\cdots n)$.

·01 $x\varepsilon\ \mathrm{Cx}n\ .\supset.\ (\mathrm{Sb}u)x = \{[\Sigma(u_{r,s}x_s\,|s,\,1\cdots n)]|r,\,1\cdots n\}$

·1 $\mathrm{Sb}u\ \varepsilon\ \mathrm{SubstCx}n$

·2 $\mathrm{Sb}u + \mathrm{Sb}v = \mathrm{Sb}(u+v)$

·3 $(\mathrm{Sb}v)(\mathrm{Sb}u) = \mathrm{Sb}[\,\Sigma(v_{rq}u_{qs}\,|q,\,1\cdots n)\,|(r,s),\,1\cdots n : 1\cdots n\,]$

·4 $a\varepsilon\ \mathrm{SubstCx}n\ .\supset.\ a = \mathrm{Sb}\{\,[a\ \mathrm{unit}(n,s)]_r|(r,s),\,1\cdots n : 1\cdots n\,\}$

Dtrm ✻ 4. $n\varepsilon\mathrm{N}_1\ .\ a,b\varepsilon\ \mathrm{SubstCx}n\ .\ u\varepsilon\ \mathrm{qF}(1\cdots n : 1\cdots n)\ .\supset.$

·0 $\mathrm{Dtrm}a = \mathrm{Dtrm}\{\,[a\ \mathrm{unit}(n,s)]_r|(r,s),\,1\cdots n : 1\cdots n\,\}$ Df

·01 $\mathrm{Dtrm}\, a \mathbin{-}= 0 \;.\supset.\; a\,\varepsilon\,(\mathrm{SubstCx}n)\mathrm{rcp}$

·02 $x\varepsilon\,\mathrm{Cx}n - \iota 0 \,.\, ax = 0 \;.\supset.\; \mathrm{Dtrm}a = 0$

·03 $\mathrm{Dtrm}a = 0 \;.=.\; \exists\, \mathrm{Cx}n - \iota 0 \cap x\ni(ax = 0)$

·04 $h\varepsilon\mathrm{q} \,.\, x\varepsilon\,\mathrm{Cx}n - \iota 0 \,.\, ax = hx \;.\supset.\; \mathrm{Dtrm}(a-h) = 0$

·1 $\mathrm{Dtrm}(ab) = \mathrm{Dtrm}a\; \mathrm{Dtrm}b$

·11 $m\varepsilon\mathrm{N}_1 \;.\supset.\; \mathrm{Dtrm}(a^m) = (\mathrm{Dtrm}a)^m$

·2 $\mathrm{Dtrm}a \mathbin{-}= 0 \;.\supset.\; a^{-1} = /a = \imath\,\mathrm{SubstCx}n \cap z\ni(az = 1)$ Df

·3 $\mathrm{Dtrm}\,\mathrm{Sb}u = \mathrm{Dtrm}u$

·4 $(\mathrm{Sb}u)^{-1} =$

$\mathrm{Sb}\{\, (-1)^{r+s}\mathrm{Dtrm}[\, u,\, 1\cdots n - \iota r : 1\cdots n - \iota s\,] \,|(r,s),\; 1\cdots n : 1\cdots n\,\} / \mathrm{Dtrm}\,u$

·5 $\mathrm{Dtrm}u \mathbin{-}= 0 \,.\, y\varepsilon\,\mathrm{Cx}n \;.\supset:\; x\varepsilon\,\mathrm{Cx}n \,.\, (\mathrm{Sb}u)x = y \;.=.\; x = (\mathrm{Sb}u)^{-1}y$

* 5. $n\varepsilon\mathrm{N}_1 \,.\, a,b\varepsilon\,\mathrm{SubstCx}n \;.\supset:$

·0 $r\varepsilon\, 1\cdots n \;.\supset.$

$\mathrm{Invar}_r a = \Sigma[\mathrm{Dtrm}(a,\, u{:}u)\,|u,\, (\mathrm{Cls}'1\cdots n)\cap u\ni(\mathrm{Num}u = r)]$ Df

« Invariante de gradu r de substitutione a ».

·01 $\mathrm{Invar}a = \mathrm{Invar}_1 a$

Nos tace indice 1.

·1 $\mathrm{Invar}_n a = \mathrm{Dtrm}a$

·2 $h\varepsilon\mathrm{q} \;.\supset.\; \mathrm{Invar}_r(ha) = h^r\mathrm{Invar}_r a$.

Dice que Invar_r de substitutione es functio homogeneo de gradu r de coefficientes de a.

·3 $\mathrm{Invar}(a+b) = \mathrm{Invar}a + \mathrm{Invar}b$

·4 $a^n + \Sigma\{(-1)^r(\mathrm{Inv}_r\, a)a^{n-r}\,|r,\, 1\cdots n\} = 0$

{ Cayley LondonT. a.1858 ; Papers t.2 p.475 }

Dem : Laguerre JP. t.25 a.1867 p.215, Frobenius JfM. t.84 a.1878 p.1, Berlin Ber. a.1896 p.601.

Aequatio algebrico ad que satisfac Subst a dicere aequatio characteristico (Cauchy, Frobenius), aequatio latente (Sylvester).

·41 $h\varepsilon\mathrm{q} \;.\supset.\; \mathrm{Dtrm}(a+h) = h^n + \Sigma[(\mathrm{Invar}_r a)h^{n-r}\,|r,\, 1\cdots n]$

·5 $u\varepsilon\,\mathrm{qF}(1\cdots n : 1\cdots n) : r,s\varepsilon\, 1\cdots n \;.\supset_{r,s}.\; u_{r,s} = u_{s,r} \;:\supset.$

$\exists\,(\mathrm{q\,f}\; 1\cdots n) \cap x\ni\{h\varepsilon\mathrm{q} \;.\supset_h.\; \mathrm{Dtrm}(\mathrm{Sb}u - h) = \Pi[(x_r - h)|r,\, 1\cdots n\,]\}$

{ Lagrange BerlinM. a.1773 p.108, pro $n = 3$;

Dem. Cauchy *Exerc.* a.1829 t.4 p.140, Œuvres s.2 t.9 p.174. }

Si matrice u es « symmetrico », aequatio characteristico

$$\mathrm{Dtrm}(\mathrm{Sb}u - h) = 0$$

habe omni radice reale.

§27 i (unitate imaginario) q′ (numero imaginario)

* 1·0 $\mathrm{i} = \mathrm{Sb}[(0, -1), (1, 0)]$ Df

·01 $\mathrm{q}' = \mathrm{q} + \mathrm{iq}$ Df

·1 $u,v \varepsilon \mathrm{q} \;.\supset.\; \mathrm{i}(u,v) = (-v, u)$

·2 $\mathrm{i}^2 = -1$

·3 $\mathrm{mod}\,\mathrm{i} = 1 \;.\; \mathrm{Inv}_1 \mathrm{i} = 0 \;.\; \mathrm{Dtrm}\,\mathrm{i} = 1$

·4 $x,y,u,v \varepsilon \mathrm{q} \;.\supset.\; (x+\mathrm{i}y)(u,v) = (xu-yv, xv+yu)$

·5 $x,y \varepsilon \mathrm{q} \;.\supset.\; \mathrm{mod}(x+\mathrm{i}y) = \sqrt{(x^2+y^2)} \;.\; \mathrm{Inv}_1(x+\mathrm{i}y) = 2x$
$\mathrm{Dtrm}(x+\mathrm{i}y) = x^2+y^2$

·6 $x,y,x',y' \varepsilon \mathrm{q} \;.\; x+\mathrm{i}y = x'+\mathrm{i}y' \;.\supset.\; x = x' \;.\; y = y'$
[$x+\mathrm{i}y = x'+\mathrm{i}y' \;.\supset.\; x-x' = \mathrm{i}(y'-y) \;.\supset.\; (x-x')^2 = -(y'-y)^2 \;.\supset.$
$(x-x')^2+(y-y')^2 = 0 \;.\supset.\; x=x' \;.\; y=y'$]

(q′ | n) III §6 n P 5···26 (q′ | r) §9 r P 2···4

Unitate imaginario es substitutio repraesentato per matrice

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Unitate imaginario, « più di meno » de Bombelli a.1579, es indicato per « i ». (Euler in Memoria praesentato ad PetrA. a.1777, et publicato in Calc. Integr. a.1794 t.4 p.184).

q′ « numero imaginario » (Cauchy), es omni Substitutione de forma $x+\mathrm{i}y$, ubi $x,y\varepsilon\mathrm{q}$, et i habe valore explicato.

Gauss a.1831 t.2 p.102 muta nomen in « numero complexo »; quod produce ambiguitate cum Cx2.

Super alios definitione de numero imaginario, vide meo articulo : « Principio de permanentia » in RdM t.8, p. 84, et Form. t.4 p.219.

real imag K

* 2. $a,b \varepsilon \mathrm{q}' \;.\; x,y \varepsilon \mathrm{q} \;.\supset.$

·0 $\mathrm{real}\, a = \imath\, \mathrm{q} \frown x \ni (a - x \;\varepsilon\; \mathrm{i}\,\mathrm{q})$ Df

$\mathrm{imag}\, a = \imath\, \mathrm{q} \frown y \ni (a - \mathrm{i}y \;\varepsilon\; \mathrm{q}) = (a - \mathrm{real}\, a)/\mathrm{i}$ Df

$\mathrm{K} a = \mathrm{real}\, a - \mathrm{i}\; \mathrm{imag}\, a = 2\mathrm{real} a - a$ Df

« real » = parte reale, occurre in quaterniones de Hamilton sub forma S = scalare, et in Weierstrass sub forma R.

« imag » = coefficiente de parte imaginario.

« K » = conjugato (Cauchy a.1821).

= conjuncto (Gauss a.1831 t.2 p.102).

·1 $\text{real}(x+\text{i}y) = x \;.\; \text{imag}(x+\text{i}y) = y \;.\; \text{K}(x+\text{i}y) = x - \text{i}y$

·2 $a = \text{real}a + \text{i imag}a$

$\text{real}a = (a + \text{K}a)/2 \;.\; \text{imag}a = (a - \text{K}a)/(2\text{i})$ Dfp

·3 $\text{real}(a+b) = \text{real}a + \text{real}b \;.\; \text{imag}(a+b) = \text{imag}a + \text{imag}b$

$\text{K}(a+b) = \text{K}a + \text{K}b$

·4 $\text{real i}a = -\text{imag}a \;.\; \text{imag i}a = \text{real}a \;.\; \text{K i}a = -\text{i K}a$

·5 $\text{mod}a = \surd[(\text{real}a)^2 + (\text{imag}a)^2]$

·6 $\text{Inv}_1 a = \text{real}a \;.\; \text{Dtrm}a = \text{mod}a^2 = a \times \text{K}a$

·7 $\text{mod}\, ab = \text{mod}a\, \text{mod}b \;.\; \text{K}\, ab = \text{K}a\, \text{K}b$

·8 $m \varepsilon \text{N}_1 .\supset. \text{mod}(a^m) = (\text{mod}a)^m \;.\; \text{K}\, a^m = (\text{K}a)^m$

$\surd \; \surd^*$

✻ 3. $a \varepsilon \text{q}' \,.\, m,n \varepsilon \text{N}_1 .\supset.$ ·0 $^m\surd^* a = \text{q}' \frown x \ni (x^m = a)$ Df

·01 $a -\varepsilon -\text{Q} .\supset. {}^m\surd a = \iota\, ({}^m\surd^* a) \frown x \ni (\text{real}x = \text{max real}\; {}^m\surd^* a)$ Df

$^m\surd^* a$, id es $(^m\surd^*)a$, indica classe de radices algebrico de gradu m de a. $^m\surd a$ indica « radice principale », que habe parte reale maximo. Cauchy, Œuvres s.2 t.12 p.220.

·1 $^m\surd^* 0 = \iota 0$

·2 $a -= 0 .\supset. \text{Num}\, {}^m\surd^* a = m$

·3 $x \varepsilon\; {}^m\surd^* a .\supset. {}^m\surd^* a = x \times {}^m\surd^* 1$

·4 $x,y \varepsilon\; {}^m\surd^* 1 .\supset. x \times y,\, x/y,\, x^n,\, x^{-n},\, \text{K}x \;\varepsilon\; {}^m\surd^* 1$

·5 $x \varepsilon \text{q} \,.\, y \varepsilon \text{Q} .\supset.$

$\surd(x+\text{i}y) = \surd\{[\surd(x^2+y^2)+x]/2\} + \text{i}\surd\{[\surd(x^2+y^2)-x]/2\}$.

$\surd(x-\text{i}y) =$ » » $-\text{i}$ » »

·6 $\surd^* 1 = \iota 1 \cup \iota -1 \;.\; {}^3\surd^* 1 = \iota 1 \cup \iota(-1+\text{i}\surd 3)/2 \cup \iota(-1-\text{i}\surd 3)/2$

❋ 4.

·1 $n \varepsilon \mathrm{N}_1 \,.\, a \varepsilon\, \mathrm{q'f}(1 \cdots n) \;.\supset.\; \exists\, \mathrm{q'} \frown x \ni [\, x^n + \Sigma(a_r x^{n-r} | r, 1 \cdots n) = 0 \,]$

·2 $\mathrm{Hp}·1 \;.\supset.\; \exists (\mathrm{q'f}1 \cdots n) \frown z \ni \{ x \varepsilon \mathrm{q'} \;.\supset_x.\; x^n + \Sigma(a_r x^{n-r} | r, 1 \cdots n) = \Pi[(x - z_r) | r, 1 \cdots n] \}$

{ GIRARD a.1629 fol. E3:

« Toutes les equations d'algebre reçoivent autant de solutions, que la denomination de la plus haute quantité le demonstre ».}

Bibliographia: Loria RdM. a.1891 t.1 p.185, t.2 p.37.

VOCABULARIO III.

§1.

187. ***Algebra,*** (L.mathematico a. 1200) ADHIR algebra, F algèbre. $\subset$ Arabo: alg'ebr (Muhammed ben Musa a.820) = reunione.

188. ***functione,*** A function, D funktion, F fonction, H funcion, I funzione, R functsija. $\subset$ functo — -o + -ione (118).

189. functo $\supset$ (188), de-functo. $\subset$ funge- — -e + -to.
-g- + -t- $\supset$ -ct-: rege — -e + -to $\supset$ recto, lege — -e + -to $\supset$ lec-to.

190. funge-, I funge = habe, frue, fac. || S bhung'-ate.

191. ***operatione,*** AD operation, F opération, H operacion, I operazione, R operatsija. $\subset$ opera- (53) + -tione (12).

192. ***correspondentia*** (non L.), A correspondence, correspondency, D korrespondenz, F correspondance, H correspondencia, I corrispondenza, R correspondentsĭja. $\subset$ (193) — -e + -ia (143).

193. ***correspondente*** (non L.), A correspondent, correspond-ing, D korrespondent, F correspondant, H correspondiente, I corrispondente, R correspondent'. $\subset$ (194) + -nte (142).

194. ***corresponde*** (non L) FH, A correspond, D korrespondire, I corrisponde. $\subset$ con- (47) + responde.
con- + -r- $\supset$ corr-: con- + -rige $\supset$ corrige.

195. ***responde*** H, A respond, D respondire, F répond-re, I risponde. $\subset$ re- (186) + sponde.

196. sponde = promitte. $\supset$ (195), sponso. || GD spende = distribue.

197. ***simile*** AI, H simil. $\supset$ simil-itudo AFHI. || G homalo. $\subset$ sim- (198) + -ile (199).

198. sim-, sem- = uno, toto, omni. $\supset$ sem-per, sin-gulo, sem-el,... || G homo, hen = uno, A same, some, to seem, D zu-samm-en R s', so, sy-, sam', S sama.

199. -ile = (197) ⌒ ag-ile ⌒ fac-ile ⌒ ...
⊂ -i- (9) + -le (6). || -ulo (224).

200. ***reciproco*** HI, A reciproc-al, F réciproque, D rekiprosche.
= pro et retro.
⊂ re- (186) + -co (201) — -o + -i- (9) + pro (134) + -co (201).

201. -co ⊃ alter-co ⌒ pris-co ⌒ -i-co (35). || G -co = physi-co..., S -ca

§2.

202. ***inverso*** HI, AF inverse. ⊂ in (113) + verso (203).

203. ***verso*** HI, F vers. ⊃ verso (in poesia), vers-ione, di-verso...
⊂ verte — -e + -to (135).
-t- + -to ⊃ -so : ute- — -e + -to ⊃ uso
flecte — -e + -to ⊃ flexo.

204. ***verte*** HI. ⊃ vert-ice, verte-bra, con-verte,...
|| D werde = fi, (vor)-wärts, R vraz'da = adversione, S vart.

§4.

205. ***variabile*** I, AFH variable, D variabel. ⊂ varia + -bile.

206. ***varia*** HI, A vary, D variire, F varie. ⊃ (205), varia-tione.
⊂ vario (207) — -o + -a (4).

207. ***vario*** HI, A vari-ous, vari-ed, F vari-é.

208. -bile, AFH -ble, D -bel, I -bile.
⊃ (205) ⌒ mo-bile ⌒ sci-bile ⌒ sta-bile ⌒ ... = -ile (199).

§6.

209. ***relativo*** HI, A relative, F relatif. ⊂ relato — -o + -ivo.

210. ***relato*** H, A related. ⊃ (209), relat-ione AFHIR, relato-re.
⊂ re- (186) + lato.

211. lato ⊃ trans-lato, lato-re, ob-lat-ione, super-lat-ivo. || G tleto-s
⊂ tule (212) — -e + -to (135 per via non simplice).

212. tule (L. antiquo) = fer, eleva, aequa. ⊃ L tuli, tol-era, tol-le.
|| G tal-anto = pondo, A-tla-nte, A thole = D ge-dul-d = tolerantia
S tul, tula.

213. -ivo A -ive, D -iv, F -if, -ive, HI -ivo, R -iv'.
⊃ -t-ivo (44) ⌒ pass-ivo ⌒ noc-ivo... ⊂ -i- (9) + -vo (214).

214. -vo, -uo ⊃ (213) ⌒ ae-vo ⌒ ar-vo ⌒ sal-vo ⌒ noc-uo ⌒ vac-uo ⌒
assid-uo ⌒ individ-uo ⌒ contin-uo ⌒ perpet-uo.

215. ***positivo*** HI, A positive, D positiv, F positif, (R positiv', in sensu particulare). ⊂ posito — -o + -ivo (213).

216. ***posito***, ***posto*** (L. raro).
⊃ (215) ⌒ posit-ione ⌒ de-posito. ⊃ I.intn. posta. || D fest.
⊂ po- + -sito. = sito.

217. ***po-, pos*** (L. antiquo), ***post;*** F puis, H pues, Port. pos, I poi, do-po. ⊃ (216), po-li, post-ero, post-scripto, ADFHI. ‖ R po, po-, pos-, S pas'cad, paçca. (= ab-, G apo, secundo Brugmann).

218. ***sito*** HI. ⊃ sit-uato, sit-uatione. ‖ G heto ⊃ cat-heto, S ava-sita. ⊂ si- + -to (135).

219. si- = si-ne ∩ si-to ∩ de-si-nentia. = pone.
‖ G he, ἑ- = ἑ-τό-ς ∩ ἕ-ς ∩ ἕ-μενο-ς ∩ ..., S ava-sa.

220. ***negativo*** HI, A negative, D negativ, F négatif.
⊂ nega + -tivo (44).

221. ***nega*** HI, F nie. ⊃ (220) ∩ nega-tione.
⊂ ne- (62) + (-ga = ai = dic, secundo Van.).

§7.

222. ***modulo*** HI, AF module, F modèle, moule, D modell, R modul'.
⊂ modo — -o + -ulo.

223. ***modo*** HI (F mode) ⊃ (222), mod-ifica, mod-era, mod-esto ADFHI.
= mensura, regula. ‖ D mass.

224. -ulo, -lo ⊃ (222), capit-ulo, calc-ulo, ... cred-ulo, pend-ulo, temp-lo, exemp-lo. = -ile (199): fac-*ile*, fac-*ul*tate; sim-*ile*, sim-*ula*;...
‖ A -le = cripp-le ∩ spind-le ∩ litt-le. D -el
‖ G -lo, ido-lo, mega-lo; S -la; R -l', -la, -lo.

225. ***valore*** I, AH valor, F valeur. ⊂ vale — -e + -ore (56).

226. ***vale*** HI, F val-oir. ⊃ (225), val-uta AFIR. ‖ S bala = vi.

227. ***absoluto*** H, A absolute, D absolut, F absolu, I assoluto, R absoljut-nyj. ⊂ ab + soluto.

228. ***ab*** ⊃ (227), ab-dica, ab-scissa ADFHIR, ...
‖ G apo ⊃ apo-stolo ∩ apo-crypho ∩ ... ‖ D ab, A of, S apa.

229. ***soluto*** ⊃ solut-ione. ⊂ solve — -e + -to (135).

230. ***solve*** A, F ré-soudre, H sol-tar, I scioglie, ri-solve, D re-solv-ire.
⊂ se- + -lue. *e* de *se* fi *o* per influxu de *u* in *lue*.

231. ***se*** FH, I se, si.
‖ A se-lf, D si-ch, se-in, se-lbst, G he, R se-bja, -sja, S sva.
sed, se- ⊃ se-cessione, se-cerne, sed-itione, se-lecto, ...
= de se, ab se, ab. = I ma, F mais ⊂ L magis.

232. -lue ⊃ di-lue ∩ so-lve, so-lu-to
‖ G λύε, ana-ly-si, A lose, D los, S lu.

233. ***mole*** HI. = magnitudine. ⊃ mole-cula ADFHI. ‖ G molo AFHIR.
Vocabulo « mole » de Leibniz es plus conforme ad etymologia que « modulo » et « valore absoluto », sed hodie non es adoptato.

234. ***signo*** H, A sign, F signe, I segno. ⊃ sign-a D, sign-ale DR.
⊂ sig- (|| D sage, secundo Van., Fick, || L seca, secundo Henry) + -no (160).

§8.

235. ***rationale***. AD rational, F rationnel, H racional, I razionale, R ratsional-ṇyj. ⊂ ratione — -e + -ale.

236. ***ratione***, A ratio, ration, D ration, F ration, raison, H racion, razon, I razione, ragione, R ratsion'. ⊂ rato — -o + -ione (118).

237. ***rato*** HI. ⊃ (236), rat-ifica ADFHIR. ⊂ ra- + -to (135).

238. ra- = calcula, cogita. ⊃ (236). || G ar-ithmo, D re-de.

239. -ale = (235) ⌢ differenti-ale ⌢ integr-ale ⌢ radic-ale ⌢ ...
⊂ -a (4, vel alio desinentia -a) + -le (6).

§10.

240. ***integro*** HI, A integer, F entier, H entero, I intero.
⊃ integr-ale ADFHIR. ⊂ in- + tage — -e + -ro (107).

-a- de thema fi -e- in compositione, ante duo consonante: *ca*nde, ac-cende; f*a*cto, per-fecto; c*a*pto, ac-cepto; ...

241. in- = ne (62). ⊃ (240), in-ertia. || G a-, an-, AD un-.

242. tage (L.antiquo) ⊃ (240), ta-n-ge, tac-to, con-tag-io
|| G tag-, S tag'.

243. ***fractione***, AF fraction, H fraccion, I frazione.
⊂ fracto — -o + -ione (118).

244. ***fracto***, I fratto. ⊃ (243), fract-ura AdFHI.
⊂ frag- + -to (135). Vide nota (189).

245. frag- ⊃ (244), fra-n-ge, frag-ile, frag-mento ADFHI.
|| L freg-i, A break, D breche, G reg-, S bhrag'. ⊂ E bhrege.

246. ***denominatore*** I, A denominator, F denominateur, H denominador
⊂ denomina + -tore (138).

247. ***denomina*** HI, A denomina-te. ⊂ de (21) + nomina.

248. ***nomina*** HI, A nomina-te, F nomme. ⊃ nomina-le, nomina-tivo, ...
⊂ nomine — -e + -a (92).

249. ***nomen, nomine,*** F nom, H nombre, I nome.
⊃ nomen-clatura DR. || G o-noma ⊃ onoma-stico, syn-onymo.
|| AD name, R imja, imen-i, S naman.
⊂ gno- + -men (secundo Van.).

gn- initiale ⊃ n-: no-men, co-gno-men, co-gni-to; nato, co-gnato.

250. ***gno-*** = no-sce, co-gno-sce. ⊃ no-men, no-to, i-gno-to.
|| G gno-, gno-si, gno-mone.
|| A know, D kenne, kunst, R zna-tı, S g'na. ⊂ E gno.

Nota : E *g* ⊃ L *g* || G *g* || A *c*, *k* || D *k* || R *z* || S *g*, *g'*.
Exemplo : L *g*enu || G *g*onu || A *k*nee || D *k*nie || S *g'*anu;
L *g*rano || A *c*orn || D *k*orn || R *z*erno || S *g'*irna.

251. -men, -mine = (249) ⌢ flu-men ⌢ acu-men ⌢ li-men...
|| G -ma (60), D -me = na-me ⌢ blu-me, R -mja, -men-, bre-mja, vre-mja, sje-mja = L se-men = D sa-me, S -man.

252. ***numeratore*** I, A numerator, F numérateur, H numerador.
⊂ numero (105) — -o + -a (92) + -tore (138).

§12.

253. ***limite*** FHI, A limit.
⊂ (secundo Van.) lic- + -mo (125) — -o + -ite.
Nota : -c + m ⊃ -m-, et -c + n- ⊃ -n-. luc- + -na ⊃ luna.

254. lic- = es transverso. ⊃ (253), ob-liq-uo, li-mine, luc-satione.
|| G λέχ-ος, loc-sodromia.

255. -ite = (253) ⌢ equ-ite ⌢ mil-ite ⌢ superst-ite ⌢ ...

256. ***supero*** || D obere. ⊂ super + -o (182).

257. ***super, supra,*** F sur, H sobre, I sopra, ADFHI super-
|| A over, upper, D über, Gotico ufar, G hyper, S upari.
⊂ sup- (124) + -er (Brug. et Van. puta illo suffixo comparativo; Fick suffixo locativo).

258. ***superiore*** I, ADH superior, F supérieur.
⊂ supero (256) — -o + -iore (110).

259. ***infero*** ⊃ (261), infer-nale.
|| S adhara, A under, D unter (in- de infero = ni, secundo Fick).

260. ***ni*** (thema E) = infra, sub. S ni, ni-tara, a-dhara = L infero, a-dhama = L infimo, A un-der, D nie-der, un-ten, G ne-rthen, nei-othi, R ni-zco, ni-z'e, L in-fra, in-fero, in-fimo.

261. ***inferiore*** IAFH. ⊂ infero (259) — -o + -iore (110).

262. ***quantitate,*** A quantity, D quantität, F quantité, H cantidad, I quantità. ⊂ quanto — -o + -itate (8).

263. ***quanto*** I, F quant, H cuanto, LADF quantum.
⊂ quam (61) + -to (135). || G pant- = toto.

264. ***infinito*** HI, A infinite, F infini.
⊂ in- (241) + fini (82) + -to (135).

265. ***radice*** I, LA radix, F racine, H raiz.
|| A root, D wurz-el, G riza. ⊂ rad- (vide : rad-io) + -ice.

266. ***radicale*** (L mathematico) I, ADFHR radical. ⊂ radic- + -ale.

267. ***intervallo*** (§13 q P12) I, A interval, D intervall, F intervalle, H intervalo, R interval'. ⊂ inter + vallo.

268. ***inter, intra, intro,*** AFH entre, HI d-entro, I entro, ADFHIR inter-. ⊂ in (113) + -ter. || S antar, G enter-o, D unter = L inter, infra, D unter-scheide = L inter-scinde.

269. -ter, -tr- = in-ter, u-tr-o, dex-ter-o, sinis-tr-o, al-ter-o, de-ter-iore, ex-ter-no, magis-tr-o, minis-tr-o. || G -ter-o, S -tar-a, R (co)-tor-(yĭ). Indica comparativo = -iore.

270. vallo = palo || G helo, D wall. ⊃ (267).

§14.

271. ***logarithmo*** introducto per Neper a.1614.
⊂ logo (33) — -o + arithmo (103).
= numero (exponente) de ratione (basi).

In theoria de Logarithmo decimale:

E Log$x$ es vocato « characteristica ».
β Log$x$ » « mantissa ».

272. ***characteristica,*** G χαρακτηριστική, A characteristic, D charakteristik, F caractéristique, H caracteristica, I caratteristica, R characteristic'escĭj. ⊂ characteristico — -o — -a (37).

273. ***characteristico*** GADFHIR.
⊂ characteriza — -a + -tico = caractere — -e + -istico.

274. ***characteriza*** ADFHI. ⊂ charactere — -e + -iza.

275. ***charactere,*** G χαρακτήρ, A character, D charakter, F caractère, H caracter, I carattere, R character'. = signo.
⊂ char- (sculpe) + -ac- (|| L -ace = rap-ace ⌢ ten-ace) + -tere (|| L -tore 138).

276. -iza (L tardo) G -ίζε, A -ize, D -isire, F -ise, H -iza, I -izza.
⊃ (274), thesaur-iza.

277. -tico LG -τικό-ς. ⊃ G mathema-tico ⌢ analy-tico ⌢ ...
⊃ L rus-tico ⌢ silva-tico ⌢ ...
⊂ -to (135, vel -te, -ti, ...) — -o + -ico (35).

278. ***mantissa***, vocabulo etrusco per origine.
(Non existe in commune vocabulario ADFHIR).

§19.

279. ***medio*** HI, A medial, mean, D mediane, F moyen, mediane; ADFR medium. || A mid, midle, D mit, mittel, G messo, meso, R mes'du, mes'', S madhja.

§21.

280. ***derivata*** HI, F dérivée, D derivierte, A derivative.
⊂ deriva + -to (135) — -o + -a (126, suffixo de feminile; indica classe, functione).

281. ***deriva*** HI, A derive, D derivire, F dérive.
⊂ de- (21) + rivo — -o + -a (92).

282. ***rivo*** I, A river, F ri-vière, H rio. $\subset$ ri- + -vo (214).
ri- = flue. || G rhe-, rheu-ma, rhe-o-phoro,
A strea-m = D stro-m = R stru-ja, S ri, sru.
E -sr- $\supset$ ADR -str- :
E svesor, S svasar, L soror, A sister, D schwester, R sestra.

***

In multo libro es adoptato nomenclatura sequente, introducto per G Cantor:

« classe u es clauso » .=. $\delta u \supset u$.=. $u = \iota u$
« » u es condensato in se » .=. $u \supset \delta u$
« » u es perfecto » .=. $u = \delta u$
« » u es isolato » .=. $u \frown \delta u = \Lambda$.

283. ***clauso***, I chiuso. = F fermé, D abgeschlossen.
$\subset$ claude — -e + -to (135). Vide nota 174.

284. ***claude,*** I chiude. $\subset$ clau- + -de.
clau- || G cleie. $\supset$ L clave (F clef H llave I chiave), G clei-s, R cljuc''.
-de = pen-de $\frown$ ten-de $\frown$ ar-de. || -de (120).

285. ***condensato*** in se I, F condensé en soi. = D insichdicht.
$\subset$ con- (47) + denso — -o + -a (4) + -to (135).

286. ***denso*** HI, AF dense. || G dasy.

287. ***perfecto*** H, A perfect, F parfait, I perfetto.
$\subset$ per (57, in sensu de ultra) + fac (137, nota 240) + -to (135).

288. ***isolato*** (non L) I, A isolate, D (scientifico, non commune) isoliert, F isolé, H aislato. $\subset$ isola + -to (135).

289. ***isola*** (verbo, non L) I, A isolate, F isole, H aisla.
$\subset$ (290) — -a (126, suffixo de feminile) + -a (4, suffixo de verbo).

290. ***isola*** (nomen, non L) I, A isle, F île, H isla. $\subset$ insula.

291. ***insula*** $\supset$ (290), A insula-te, insula-r, ... || D insel.
$\subset$ in + sal (292, = mare) + -a (126 feminile, indica terra).

292. ***sal, sale,*** A sal, F sel, H sal, I sale.
|| A salt, D salz, G hal-s (= sale, mare), R solı.

§23.

293. ***probabilitate,*** A probability, F probabilité, H probabilidad, I probabilità. $\subset$ probabile — -e + -itate.

294. ***probabile*** AFHI. $\subset$ proba + -bile (208).

295. ***proba*** (verbo) H,F preuve, I prova, D prüfe.
» (nomen), F épreuve, H prueva, I prova, A proof, D probe.
$\subset$ probo — -o + -a (4).

296. ***probo*** FI, = bono, vero.

§24.

297. ***complexo*** H, AD complex, F complexe, I complesso.
⊃ complex-ione DR. ⊂ com- (47) + plexo.

298. ***plexo*** ⊂ plecte — -e + -to (135, vide nota 203).

299. ***plecte*** || D flecte, R plesti. = plica.
⊂ plec- (thema E = L plica, G plece) + -te.

300. -te = flec-te ⌢ plec-te ⌢ u-te- ⌢ verte ⌢ nec-te. || GD -te, S -t.á.
⊂ E -te, cum valore de præsente.

301. ***ordo, ordine*** I, A order, DHR orden, D ord-nung, F ordre.
⊃ ordin-a I, ordin-ata ADR, ordin-ale, ordin-ario D. ⊂ or- + -dine.

302. ***or-, ori-*** ⊃ or-do, or-igine, or-iente, or-tu.
|| G or-nymi, S ar.

303. -do, -dine = (299) ⌢ dulce-dine ⌢ cupi-dine.
-dine = -do — -o + -ine.

304. -ine = hom-ine (= homo) ⌢ vorag-ine (= vorago) ⌢ orig-ine (= origo) ... ⌢ consuetud-ine (= consuetudo).
-ine — -o in L transforma thema de nominativo singulare in thema de alio casu.
|| D -n = blume-n ⌢ zunge-n (transforma singulare in plurale). I uomo, uomini. || G -ne, -ν, -νι. Solo-ne ⌢ Strabo-ne ⌢ ..., S -na.

305. ***unitate,*** A unity, F unité, H unidad, I unità.
⊂ uno (114) — -o + -itate (8).

§25.

306. ***determinante*** ADFHI. ⊂ determina + -nte (142).

307. ***determina*** AFHI. ⊂ de- (21) + termina.

308. ***termina*** AFHI, = da termine. ⊂ termine — -e + -a (92).

309. ***termo, termine*** I, termino LH, AF terme, AL terminus, D termin.
|| G terma, termon, S tarman ⊂ tere — -e + -mine (251).

310. ***tere*** = fini, consuma. ⊃ tere-bra, tri-to, at-trito.
|| G tere-tro = L tere-bra.

§26.

311. ***lineare*** I, A linear, D lineal, L linejnyj.
⊂ linea — -a + -are (382).

312. ***linea*** HI, A line, D linie, F ligne, R linija.
⊂ lino — -o + -eo (418) — -o + -a (126) (secundo Bréal; secundo alios, de li- thema de li-ne, li-tera, li-mo,...).

313. ***lino*** (planta texile), D lein, F lin, HI lino, R len'.
⊃ F linge, A linen. || G lino-n.

314. li- E ⊃ L li-ne, po-li, li-tera, li-mo, A li-me = D lehm = L li-mo, G a-li-ne = L line, R li-tị = funde, S ri.

315. ***invariante*** ADFHI. ⊂ in (241) + variante.

316. ***variante*** D. ⊂ varia (206) + -nte (142).

§27.

317. ***imaginario*** H, A imaginary, D imaginär, F imaginaire, I immaginario. ⊂ imagine — -e + -ario = imagina + -rio.

318. ***imagina-*** AFHI. ⊂ imagine — -e + -a (4).

319. ***imago,*** imagine, AF image, H imagen, I immagine.
⊃ imaginatione AFHI. ⊂ im- + -ago.

320. im- ⊃ im-ita-re, im-ago, aem-ulo. || S jama = gemino.

321. -ario, A -ary, D -är, F -aire, -ier, HI -ario, R -er'.
⊃ agr-ario, imagin-ario, caten-ari-o.
⊂ -a (finale de plure vocabulo) + -rio.

322. -rio ⊃ -a-rio, catena-rio, imagina-rio, corolla-rio, suaso-rio, contra-rio.

323. -ago, -agine ⊃ im-ago, vor-ago, farr-ago. ⊂ -a (finale de plure vocabuio) + -gine (377).

324. ***reale*** (L. scientifico) I, ADH real, F réel, R real'. ⊂ re + -ale.

325. ***re*** = proprietate, F chose. ⊃ qua-re, F car; re-publica ADFHIR.
|| S ra. F rien ⊂ L re, cum valore negativo, ut F pas ⊂ L passu, F point ⊂ L puncto.

326. ***conjugato*** I, A conjugate, D conjugiert, F conjugué, H conjugado.
⊂ conjuga + -to (135).

327. ***conjuga*** HI, A conjugate, F conjugue.
⊂ con — (47) + jugo — -o + -a (92).

328. ***jugo,*** F joug, H juego, I giogo.
|| A yoke, D joch, G zygo, R igo, S juga. ⊂ juge — -e + -o.

329. juge (thema latino) = liga, junge. ⊃ ju-n-ge, jug-o, con-juge.
|| G zyg-, S jug'.

330. ***conjuncto,*** AF conjoint, H conjunto, I congiunto.
⊂ con + junge — -e + -to.

331. ***junge*** ⊂ jug- + -n- (341).

IV

GEOMETRIA

IV. GEOMETRIA.

§1 pnt (puncto) vct (vectore)

❋ 1.

« p » vel « pnt » significa « puncto ».

Nos non pote scribe aequalitate de forma:

p = (expressione composito per signos de Logica, de Arithmetica et de Algebra considerato in partes I, II, III).

Ergo « puncto » non es definibile in modo logico; idea de puncto resulta de consideratione de mundo physico.

·1 $\mathrm{p} \,\varepsilon\, \mathrm{Cls}$ ·2 $\exists\, \mathrm{p}$ Pp

·3 $a \varepsilon \mathrm{p} \,.\supset.\, \exists\, \mathrm{p} - \iota a$ Pp

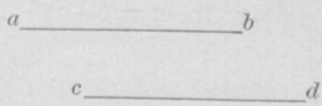
Punto es classe; existe aliquo puncto; dato puncto a, existe alio puncto differente de a, non coincidente cum a.

❋ 2.

Relatione inter quatuor puncto a, b, c, d:

$$a - b = c - d$$

(lege ut in Arithmetica) indica que puncto considerato habe inter se positione ut in figura:



Si nos ute nomenclatura de Geometria elementare, relatione considerato significa que segmento ab et cd es aequale, parallelo, et in idem sensu.

Si nos ute nomenclatura de Geometria de positione, relatione considerato, si puncto a, b, c, d non jace super uno recta, significa que recta ab es parallelo ad recta cd, et recta ac es parallelo ad recta bd.

Si nos ute nomenclatura de Mechanica, relatione considerato significa que nos pote fac coincide systema de duo puncto a, b cum duo puncto c, d, per motu de translatione.

Omni explicatione praecedente exprime relatione considerato per plure idea primitivo.

Nos considera relatione $a—b = c—d$ ut idea primitivo.

$a,b,c,d,e,f \, \varepsilon \, \mathrm{p} \, . \supset .$

·1 $a—b=a—b$ Pp

·2 $a—b=c—d \, . \supset . \, c—d=a—b$ Pp

·3 $a—b=c—d \, . \, c—d=e—f \, . \supset . \, a—b=e—f$ Pp

·4 $a—b=c—d \, . \supset . \, a—c=b—d$ Pp Altern

P ·1 ·2 ·3 dice que relatione $a—b = c—d$, que nos scribe sub forma de aequalitate, gaude de proprietates de aequalitate (I § 1).

P ·4 dice que relatione $a—b = c—d$, que nos scribe sub forma de aequidifferentia, gaude de proprietate fundamentale de aequidifferentia, id es, lice « alterna » elementos medio b et c.

P ·4 exprime propositione de Geometria elementare; « si segmento ab es aequale, parallelo et in idem sensu ad segmento cd, tunc segmento ac es aequale, parallelo et in idem sensu ad segmento bd »; EUCLIDES, I, P33:

Αἱ τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιξευγνύουσαι εὐθεῖαι καὶ αὐταὶ ἴσαι τε καὶ παράλληλοί εἰσιν.

·5 $a—c=b—c \, . \supset . \, a=b$ Pp

·6 $\exists \, \mathrm{p} \frown x \ni (x—a=b—c)$ Pp

Independentia de propositiones primitivo.

Ut nos proba que Pp es independente, nos adduce interpretationes de idea primitivo « puncto, P1 », et « relatione inter quatuor puncto, P2 », que satisfac ad aliquo Pp et non ad omni.

·1. Si ad relatione nos tribue sensu: « a,b,c,d es elemento coincidente », tunc es vero Pp·2, que significa « si a,b,c,d es coincidente, et c,d,a,b es coincidente », et Pp·3, ·4, ·5, sed non Pp·1, nam non semper « duo elemento a,b arbitrario es coincidente ».

·2. Si ad relatione nos tribue sensu « distantia $ab \leqq$ distantia cd », tunc es satisfacto Pp·1, ·3, et non Pp·2.

·3. Si ad relatione nos tribue sensu « a,b,c,d es puncto complanare », tunc es satisfacto Pp·1, que significa « duo puncto semper jace in uno

plano », et P·2, que significa « si a,b,c,d es complanare, et c,d,a,b es complanare », sed non es satisfacto Pp·3 ; nam si a,b,c,d es complanare, et c,d,e,f es complanare, non semper a,b,e,f es complanare.

·4. Si ad relatione nos tribue valore « distantia ab = distantia cd », tunc es satisfacto Pp·1·2·3, que exprime proprietates de aequalitate, sed non Pp·4, que exprime proprietate de aequidifferentia.

·5. Si ad relatione fundamentale nos tribue sensu « relatione $a'-b' = c'-d'$ subsiste inter projectiones de a,b,c,d super plano fixo », nos satisfac ad Pp·1·2·3·4 et non ad Pp·5. Nam ex Hyp. seque solo que a et b habe identico projectione.

·6. Si ad signo p nos tribue valore N_1, et ad relatione fundamentale valore que habe in Arithmetica, nos satisfac Pp·1-·5, et non Pp·6.

Historia.

Relatione de P2, vel idea de vectore, occurre in Euclide (vide citatione post P2·4), et in omni libro de Geometria, sub forma complexo.

Wessell in a.1797, vide que ce relatione habe proprietates de aequalitate, et indica illo per $ab = cd$.

Bellavitis, a.1832, funda super idem idea «Calculo de aequipollentias».

H. Grassmann, a.1844 vide in relatione considerato proprietates de aequidifferentia ; Werke t.1 p.303 :

«... ich sage also, dass $B-A$ dann und nur dann gleich B_1-A_1 sei, wenn die geraden Linien von A nach B und von A_1 nach B_1 gleiche Länge und Richtung haben.»

Idem observatione occurre in W. Hamilton, a.1845, Cambridge J. t.1, p.47 :

«... the symbolic equation, D—C = B—A, may denote that the point D is ordinarily related (in space) to the point C as B is to A, and may in that view be expressed by writing the *ordinal analogy*, D..C :: B..A : which admits of *inversion* and *alternation*.»

Ce notatione es implicito, non explicito, in Möbius a.1827, et a.1844 p.608; vide P10.

Ad notatione de Wessell :

$$AB = -BA,\ AB + BC = AC\ ,\ \text{de}\ AB = CD \supset AC = BD$$

responde novo notatione :

$$A-B = -(B-A),\ (A-B)+(B-C)=A-C,\ A-B=C-D \supset A-C=B-D.$$

conforme ad regulas de Algebra, et ad principio de oeconomia.

Hamilton a.1845 p.56 introduce vocabulo « vectore ».

✻ 3.

« Vectore », que nos indica per « v » vel « vct » es differentia de duo puncto.

Vectore nullo, indicato per 0, es differentia de duo puncto coincidente.

·0 $\mathrm{v} = \mathrm{vct} = x \ni [\exists\, (a,b) \ni (a,b \varepsilon \mathrm{p} \,.\, x = b-a)]$ Df

·01 $\mathrm{v} = \mathrm{p}-\mathrm{p}$

·1 $0 = \iota\, x \ni (a \varepsilon \mathrm{p} \,.\supset_a .\, x = a-a)$ Df

·2 $0 \varepsilon \mathrm{v}$ ·3 $a \varepsilon \mathrm{p} \,.\supset.\, a-a = 0$

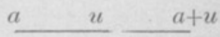
[$a,b \varepsilon \mathrm{p}$. P2·1 $.\supset.$ $b-a = b-a$. Altern $.\supset.$ $b-b = a-a$ (1)
$b \varepsilon \mathrm{p}$. (1) $.\supset:$ $a \varepsilon \mathrm{p} \,.\supset_a .\, b-b = a-a$ (2)
$b \varepsilon \mathrm{p}$. (2) $.\supset.$ $\exists\, x \ni (a \varepsilon \mathrm{p} \,.\supset_a .\, x = a-a)$ (3)
(3) Elim$b$. P1·2 $.\supset.$ ———————— ——————— (4)
$a \varepsilon \mathrm{p} \,.\supset.\, x = a-a \,.\, y = a-a$. P2·3 $.\supset.$ $x = y$ (5)
(4) . (5) . §ɿ P·2 $.\supset.$ P·2·3]

·4 $a,b \varepsilon \mathrm{p} \,.\supset:\, a = b \,.=.\, a-b = 0$
[Hp . $a=b$. P·3 $.\supset.$ $a-b = 0$ (1)
Hp . $a-b = 0$. P·3 $.\supset.$ $a-b = b-b$. P2·5 $.\supset.$ $a = b$ (2)
(1) . (2) $.\supset.$ P]

❋ 4. $a,b \varepsilon \mathrm{p} \,.\, u,v \varepsilon \mathrm{v} \,.\supset:$

·0 $a+u = \iota\, \mathrm{p} \frown x \ni (x-a = u)$ Df

Si a es puncto et u es vectore, $a+u$ indica illo puncto x tale que $x-a = u$.



·1 $a+u \,\varepsilon \mathrm{p}$ ·2 $(a+u)-a = u$
[P2·6 $.\supset.$ $\exists\, \mathrm{p} \frown x \ni (x-a = u)$ (1)
$x,y \varepsilon \mathrm{p} \,.\, x-a = u \,.\, y-a = u$. P2·3 $.\supset.$ $x-a = y-a$. P2·5 $.\supset.$ $x = y$ (2)
(1) . (2) . §ɿ P·2 $.\supset.$ P·1·2]

·21 $u = v \,.=.\, a+u = a+v$
[$u = v \,.=.\, (a+u)-a = (a+v)-a \,.=.\, a+u = a+v$]

·3 $a+(b-a) = b$
[P·2 $.\supset.$ $[a+(b-a)]-a = b-a$. P2·5 $.\supset$ P]

·31 $a+0 = a$ [$(b|a)$P·3 $.\supset$ P]

·4 $b-a = u \,.=.\, b = a+u$ Oper $+a$
[P·21 $.\supset.$ $b-a = u \,.=.\, a+(b-a) = a+u \,.=.\, b = a+u$]

·5 $(a+u)-(b+u) = a-b$
[Hp . P·2 $.\supset.$ $(a+u)-a = (b+u)-b$. Altern $.\supset.$ P]

·51 $a+u = b+u \,.=.\, a = b$
[$a+u = b+u \,.=.\, (a+u)-(b+u) = 0 \,.=.\, a-b = 0 \,.=.\, a = b$

·6 $a+u+v=(a+u)+v$ Df

·7 $a+u+v=a+v+u$
$[(a+u+v)-(a+v)=(a+u)-a=u\,.\,\mathrm{Oper}+(a+v)\,.\supset.\ \mathrm{P}\ ]$

·8 $(a+u+v)-a=(b+u+v)-b$
$[\ \mathrm{P}{\cdot}5\ .\supset.\ (a+u+v)-(b+u+v)=(a+u)-(b+u)=a-b\,.\,\mathrm{Altern}\,.\supset.\ \mathrm{P}\ ]$

✻ 5.

·0 $u,v\varepsilon\mathrm{v}\ .\supset.\ u+v=\iota\, x\ni[\,a\varepsilon\mathrm{p}\ .\supset_a.\ x=(a+u+v)-a\,]$ Df

Dato duo vectore u et v, sume puncto arbitrario a; construe puncto $a+u$; et construe puncto $(a+u)+v$. Vectore $(a+u+v)-a$, independente de puncto a, vale, per definitione, $u+v$.



Idea de « vectore » pertine ad Geometria; illo habe plure applicatione ad Mechanica et ad Physica, et exprime omni elemento, que habe magnitudine, directione et sensu.

Si vectore repraesenta vi (fortia), summa de duo vectore repraesenta « resultante » de duo vi.

$a,b,c\varepsilon\mathrm{p}\ .\ u,v,w\varepsilon\mathrm{v}\ .\supset.$

·1 $u+v\ \varepsilon\,\mathrm{v}$ ·2 $u+v=(a+u+v)-a$ $[\ \mathrm{P}4{\cdot}8\ .\supset.\ \mathrm{P}\ ]$

·3 $a+u+v=a+(u+v)$ $[\ \mathrm{P}{\cdot}2\,.\,\mathrm{Oper}+a\ .\supset.\ \mathrm{P}\ ]$

·4 $u+v=v+u$ $[\ \mathrm{P}4{\cdot}7\ .\supset.\ \mathrm{P}\ ]$ Comm+

·5 $u+(v+w)=(u+v)+w$ Assoc+
$[\ \mathrm{P}{\cdot}3\ .\supset.\ a+[(u+v)+w]=[a+(u+v)]+w=[(a+u)+v]+w=$
$(a+u)+(v+w)=a+[u+(v+w)]\ .\supset.\ \mathrm{P}\ ]$

·6 $(a-b)+(b-c)=a-c$
$[\ \mathrm{P}4{\cdot}7{\cdot}3{\cdot}4\ .\supset.\ c+(a-b)+(b-c)=c+(b-c)+(a-b)=b+(a-b)=a\ ]$

·7 $(a-b)+(b-a)=0$ $[\ (c|a)\mathrm{P}{\cdot}6\supset\mathrm{P}\ ]$

·8 $u+v=u+w\ .=.\ v=w$
$[\ u+v=u+w.=.a+(u+v)=a+(u+w).=.(a+u)+v=(a+u)+w.=.\ v=w\ ]$

·9 $u+0=u$ $[\ (b|c)\mathrm{P}{\cdot}6\supset\mathrm{P}\ ]$

✻ 6. $u,v\varepsilon\mathrm{v}\ .\ a,b\varepsilon\mathrm{p}\ .\supset.$

·0 $-u=\iota\,\mathrm{vct}\frown x\ni(u+x=0)$ Df

·01 $u-v=u+(-v)$ Df

Si u es vectore, $-u$ indica illo vectore que addito ad u da pro resultatu 0. Es repetitione de definitione super numero relativo (I § 9 P 1·3).

·1 $-u \varepsilon \mathrm{v}$ ·2 $u-u=0$

[$a,b\varepsilon\mathrm{p} \,.\, u=b-a \,.\, x=a-b$. P5·7 .⊃. $u+x=0$ (1)

(1) .⊃. $\exists\, \mathrm{v} \frown x\ni(u+x=0)$ (2)

$x,y\varepsilon\mathrm{v} \,.\, u+x=0 \,.\, u+y=0$. P5·8 .⊃. $x=y$ (3)

(1) . (2) . (3) . I §7 P2·2 .⊃ P]

·3 $-(-u)=u$ ·4 $-(a-b)=b-a$

* 7. $u,v\varepsilon\mathrm{v} \,.\, m,n\varepsilon N_0$.⊃:

·0 $0u=0 \quad . \quad (m+1)u=mu+u \quad . \quad (-m)u=-(mu)$ Df

Producto de vectore per numero m integro positivo (P 7), aut negativo (P 8), aut fracto (P 9) es definito ut in Arithmetica.

·1 $mu\,\varepsilon\,\mathrm{v}$ [$(m,n)|(b,a)$ §× 1·1 Dm .⊃. P]

·2 $(m+n)u=mu+nu$ Distrib(×,+)
[$(m,n,u)|(b,c,a)$ §× 1·2 Dm .⊃. P]

·3 $m(u+v)=mu+mv$ Distrib(×,+)
[$(m,u,v)|(c,a,b)$ §×1·3 Dm .⊃. P)

·4 $um=mu$ Comm × Df

·5 $(mn)u=m(nu)$ Assoc ×
[$(m,n,u)|(c,b,a)$ §×1·5 Dm .⊃. P]

·6 $m\varepsilon N_1 \,.\, mu=0$.⊃. $u=0$ Pp

Nos satisfac ad totos Pp praecedente, et non ad Pp·6, si p indica punctos de uno circumferentia, et nos dice que $a-b=c-d$ si nos pote fer punctos a et b ad coincide cum c et d per uno rotatione circa centro. 0 repraesenta identitate, et rotatione repetito pote produce identitate.

·7 $mu=mv$.⊃. $u=v$
[Hp .⊃. $mu-mv=0$. P·3 .⊃. $m(u-v)=0$. P·6 .⊃. Ths]

* 8. $u,v\varepsilon\mathrm{v} \,.\, m,n\varepsilon\mathrm{n}$.⊃. P7·1·2·3·4·5

* 9. $u,v\varepsilon\mathrm{v} \,.\, m,n\varepsilon N_1 \,.\, a,b\varepsilon\mathrm{r}$.⊃

·0 $u/m = \imath\,\mathrm{v} \frown v\ni(mv=u)$ Df

·1 $\exists\,\mathrm{v} \frown v\ni(mv=u)$ Pp

Si nos substitue « n » ad « pnt », omni Pp praecedente, es satisfacto, sed non P9·1.

·2 $u/m \,\varepsilon\, \mathrm{v} \,.\, m(u/m) = u$

·3 $p,q \varepsilon \mathrm{n} \,.\, p/m = q/n \,.\supset.\, (up)/m = (uq)/n$

[Hp $.\supset.$ $pn = qm$ $.\supset.$ $u(pn) = u(qm)$. P8·5 $.\supset.$ $(up)n = (uq)m$. P·2 $.\supset$ $(up/m)mn = (uq/n)mn$. P7·7 $.\supset.$ Ths]

·4 $au = \iota\, v \ni [\, m \varepsilon \mathrm{N}_1 \,.\, p \varepsilon \mathrm{n} \,.\, p/m = a \,.\supset_{m,p}.\, v = up/m]$ Df

·5 $au \,\varepsilon \mathrm{v} \,.\, (a+b)u = au + bu \,.\, a(u+v) = au + av \,.$
$(ab)u = a(bu) = abu$

·6 $au = 0 \,.=.\, a = 0 \,.\cup.\, u = 0$

❋ 10.

Si a es puncto et m es numero (rationale), tunc ma indica systema de numero m et de puncto a. Si m es mensura de « massa », tunc ma repraesenta puncto cum massa, vel « puncto materiale » de Physica.

Dato duo puncto a et b, et duo numero correspondente m et n, nos vol repraesenta per expressione $(ma+nb)/(m+n)$ puncto vocato in Mechanica « barycentro de puncto a et b, cum pondo (pondere, massa) m et n ».

Expressione considerato, si a et b es numero, es vocato « valore medio arithmetico inter a et b, cum pondo m et n ».

Si nos pone $(ma+nb)/(m+n) = x$, et nos opera super ce aequalitate, nunc sine sensu, ut in Arithmetica, nos habe:

$$ma + nb = (m+n)x,$$

et in fine: $$m(a-x) + n(b-x) = 0,$$

que habe sensu determinato. Ergo nos pone:

·0 $a \varepsilon \mathrm{p} \,.\, m \varepsilon \mathrm{r} \,.\supset.\, ma = (m,a)$ Df

·1 $a,b \varepsilon \mathrm{p} \,.\, m,n \varepsilon \mathrm{r} \,.\, m+n -= 0 \,.\supset.$
$(ma+nb)/(m+n) = \iota\, \mathrm{p} \frown x \ni [m(a-x) + n(b-x) = 0]$ Df

·2 $a,b \varepsilon \mathrm{p} \,.\, m \varepsilon \mathrm{r} \,.\supset.\, ma - mb = m(a-b)$ Df

·3 Hp·1 $.\supset.$ $ma + nb = (m+n)[(ma+nb)/(m+n)]$ Df

·4 $m \varepsilon \mathrm{r} - \iota 0 \,.\, a \varepsilon \mathrm{p} \,.\, u \varepsilon \mathrm{v} \,.\supset.\, u + ma = ma + u = m(a + u/m)$ Df

$m \varepsilon \mathrm{N}_1 \,.\, a \varepsilon \mathrm{p}\, \mathrm{f}\, 1 \cdots m \,.\, x \varepsilon \mathrm{r}\, \mathrm{f}\, 1 \cdots m \,.\supset:$

·5 $p \varepsilon \mathrm{p} \,.\supset.\, \Sigma(x_r a_r | r, 1 \cdots m) = \Sigma(x, 1 \cdots m)p + \Sigma[x_r(a_r - p) \,|\, r, 1 \cdots m]$

·6 $\Sigma(x,1\cdots m)=0 \,.\supset.\, \Sigma(x_r a_r|r,1\cdots m)\,\varepsilon\mathrm{v}$

·7 $\Sigma(x,$ » $)-=0 \,.\supset.\, \Sigma($ » $)/\Sigma(x,1\cdots m)\,\varepsilon\mathrm{p}$

Si nos considera m puncto $a_1\ a_2\ a_3 \dots a_m$, et m numero, aut pondo correspondente $x_1\ x_2\ x_3 \dots x_m$, tunc expressione:

$$x_1a_1+x_2a_2+x_3a_3+\dots+x_ma_m$$

si summa de pondo $x_1+x_2+\dots+x_m$ es nullo vale vectore. Si summa de pondo, vel pondo totale non es nullo, tunc summa de puncto cum pondo, diviso per pondo totale, repraesenta puncto, dicto « barycentro de puncto dato, cum pondo relativo ».

Archimede voca barycentro κέντρον τοῦ βάρεος. Carnot a.1801 defini illo per solo idea geometrico et voca « centre des moyennes distances » (p.154). Expressione $\Sigma xa/\Sigma x$ pro barycentro occurre in Möbius, *Barycentrische Calcul*, a.1827 t.1 p.37.

✻ 11.

Per duo idea primitivo praecedente nos pote defini plure objecto geometrico; sed nos non pote defini distantia de duo puncto. Occurre novo idea primitivo, que nos sume ab Mechanica.

Si vectore u repraesenta «fort-ia», et vectore v repraesenta spatio descripto per suo puncto de applicatione, tunc «labore de fortia u, respondente ad spatio v» vale « producto de u per projectione orthogonale de v super u ». Producto hic considerato habe signo. Nos indica ce producto per $u\times v$, que nos lege « producto interno, vel scalare » de vectore u per vectore v. Ergo, si v' es projectione de v super u:

$$u\times v = u\times v'$$

vel si per mod$u$ nos intellige suo valore numerico, et si nos adopta notatione de trigonometria:

$$u\times v = \mathrm{mod}u \times \mathrm{mod}v \times \cos(u,v).$$

Analysi de explicationes nunc exposito es longo; ergo nos sume $u\times v$ ut idea primitivo, determinato per Pp sequente:

$u,v,w\varepsilon\mathrm{v} \,.\supset.$

·1	$u\times v\ \varepsilon\mathrm{q}$	Pp	
·2	$u\times v = v\times u$	Pp	Comm ×
·3	$(u+v)\times w = u\times w + v\times w$	Pp	Distrib(×,+)
·4	$u^2 = u\times u$	Df	

·5 $u\varepsilon \text{v-}\iota 0 .\supset. u^2 \varepsilon \text{Q}$ Pp

P·3 exprime que projectione super w de summa $u+v$ vale summa de projectiones de u et de v.

Ergo subsiste calculo geometrico simile ad algebrico; omni identitate de secundo gradu super quantitate repraesenta aliquo theorema de Geometria. Plure exemplo in P 13, 14, 15, 16.

Coincidentia formale de novo calculo cum antiquo es multo commodo, et conveni elige notationes ita ut coincidentia fi magno. Novo calculo non pote coincide in omni suo parte cum antiquo. Ita calculo super numero relativo « n » es simile, quasi identico, ad calculo super numeros arithmetico N_0; calculo super rationales R, et calculo super irrationales Q es simile ad praecedentes; sed semper existe aliquo differentia.

$u\times v=0$, quando duo vectore es orthogonale. Non es necesse, ut in theoria de N_0, n, r, q, que uno factore es nullo. Producto de vectores non es associativo. Ergo identitates de Algebra de gradu tres non habe sensu in theoria actuale.

Euclide indica producto $u\times v$ per longo periphrasi; vide P14.

H. Grassmann a.1846 t.1 p.345, vide proprietates commutativo (P·2) et distributivo (P·3), et voca illo « innere Product ». Notatione $u\times v$ es adoptato per Resal a.1862, Somoff,...

Producto $u\times v$ resulta quoque, in modo indirecto, de quaterniones de Hamilton (vide « quaternione »), et de producto alternato de Grassmann a.1862 (vide operatione α). Plure auctore de libro de electricitate voca $a\times b$ « producto scalare », vocabulo derivato ab theoria de quaterniones, cum variatione de sensu.

✻ 12. $u,v\varepsilon \text{v} . m\varepsilon \text{N}_1 . p\varepsilon \text{n} . x\varepsilon \text{r} .\supset.$

·1 $0\times u=0$ [Distrib$(\times,+)$ $.\supset.$ $0\times u+0\times u=0\times u$ $.\supset.$ P]

·2 $(mu)\times v=m(u\times v)$

[$m=1$ $.\supset.$ P (1)

$m\varepsilon \text{N}_1 . (mu)\times v=m(u\times v) .\supset. [(m+1)u]\times v=(mu+u)\times v=$
$(mu)\times v+u\times v=m(u\times v)+u\times v=(m+1)(u\times v)$ (2)

(1) . (2) . Induct $.\supset.$ P]

·21 $(-u)\times v=-(u\times v)$ [$u\times v+(-u)\times v=(u-u)\times v=0$]

·22 $(-mu)\times v=-m(u\times v)$ [P·2 . P·21 $.\supset.$ P]

·3 $(pu)\times v=p(u\times v)$ [= P·1·2·22]

·31 $(u/m)\times v=(u\times v)/m$ [$m[(u/m)\times v]=(mu/m)\times v=u\times v$]

·32 $(up/m)\times v=(u\times v)p/m$ [P·3 . P·31 $.\supset.$ P]

·4 $(xu)\times v=x(u\times v)$ [P·32 $.\supset.$ P]

✻ 13. $u,v,w \varepsilon \mathrm{v}$.Ↄ.

·1 $(u+v)^2 = u^2+2u\times v+v^2$

·2 $(u+v+w)^2 = u^2+v^2+w^2+2u\times v+2u\times w+2v\times w$

·3 $(u+v)\times(u-v) = u^2-v^2$

·4 $u\times v = [(u+v)^2-(u-v)^2]/4$ Dfp

Ergo, nos pote substitue u^2 ad $u\times v$ ut idea primitivo. Resulta simplificatione in Pp 11·1·2, sed complicatione in ·3.

·5 $(u+v)^2+(u-v)^2 = 2(u^2+v^2)$ { LAGNY ParisM. a.1706 p.319:
« Dans tout parallelogramme $[a,a+u,a+v,a+u+v]$ la somme des quarrez des deux diagonales est égale à la somme des quarrez des quatre côtez ».

·6 $(u+v)^2-(u-v)^2 = 4u\times v$

·7 $(u+v+w)^2+(u+v-w)^2+(u+w-v)^2+(v+w-u)^2=4(u^2+v^2+w^2)$
{ LEGENDRE *Géom.* p.227:
« ... dans tout parallélépipède, la somme des carrés des quatre diagonales est égale à la somme des carrés des douze arêtes. »}

·8 $(u+v+w)^2+u^2+v^2+w^2 = (u+v)^2+(v+w)^2+(w+u)^2$

·9 $(u-v)^2+(v-w)^2+(w-u)^2+(u+v+w)^2 = 3(u^2+v^2+w^2)$

✻ 14. $a,b,c \varepsilon \mathrm{p}$.Ↄ.

·1 $(a-b)^2 = (a-c)^2+(b-c)^2-2(a-c)\times(b-c)$

·2 $(a-b)^2 = (a-c)^2+(b-c)^2+2(a-c)\times(c-b)$
{ EUCLIDE II P12 P13 }

In omni triangulo, quadrato de uno latere aequa summa de quadrato de alio duo latere, plus aut minus duplo producto (interno) de duo latere, considerato ut vectore; vel duplo producto de uno latere per projectione de altero super primo; vel duplo producto de valore absoluto de duo latere per cosinus de angulo comprehenso.

·3 $(a-b)\times(b-c)=0 .=. (a-c)^2 = (a-b)^2+(b-c)^2$
{ PYTHAGORA; vide Plutarcho *Symp.* VIII c.4 }

·4 $(a-c)\times(b-c)=0 .=. (a-c)^2 = (a-c)\times(a-b)$

·5 $2[a-(b+c)/2]^2 = (a-b)^2+(a-c)^2-(b-c)^2/2$
{ APOLLONIO: vide P15·7 }

Vectore $a-(b+c)/2$ vocare « mediana » de triangulo abc.

·6 $2(c-b)\times[a-(b+c)/2] = (a-b)^2-(a-c)^2$

✻ 15. $a,b,c,d,x,y \varepsilon \mathrm{p}$.Ↄ.

·0 $(a-b)\times(c-d)+(b-c)\times(a-d)+(c-a)\times(b-d)=0$

« Si in tetrahedro $abcd$ latere ab es orthogonale ad suo opposito cd, vel $(a-b)\times(c-d)=0$, et si latere bc es orthogonale ad suo opposito ad, vel

$(b-c)\times(a-d)=0$, tunc $(c-a)\times(b-d)=0$, et latere ca es orthogonale ad suo opposito bd ».

Vel, « si in triangulo abc, puncto d jace super perpendiculare de c ad ab, et super perpendiculare de a ad bc, tunc illo jace etiam super perpendiculare de b ad ac ». { Euler, pro punctos super recta. }

·1 $(a-b)^2+(b-c)^2+(c-d)^2+(d-a)^2=(a-c)^2+(b-d)^2+4[(a+c)/2-(b+d)/2]^2$ { EULER PetrNC. a.1748 t.1 p.66 }

·11 $2(a-b)\times(c-d)=(a-d)^2+(b-c)^2-(a-c)^2-(b-d)^2$
{ CARNOT a.1806 ; STAUDT JfM. a.1842 p.252 }

·2 $(x-a)\times(x-b)=0 .=. [x-(a+b)/2]^2=(b-a)^2/4$
{ THALETE; vide DIOGENE LAERTIO I 24 : « *φησὶ Παμφίλη πρῶτον* (THALETE) *καταγράψαι κύκλου τὸ τρίγωνον ὀρθογώνιον καὶ θῦσαι βοῦν.*»

« Si angulo axb es recto, tunc distantia de puncto x ad puncto $(a+b)/2$ es constante ».

·3 $(x-a)^2=(x-b)^2 .=. [x-(a+b)/2]\times(b-a)=0$

·4 $m\varepsilon \mathrm{R}-\iota 1 .\supset:$
$(x-a)^2=m(x-b)^2 .=. [x-(mb-a)/(m-1)]^2=(b-a)^2m/(m-1)^2$

{ ·3, ·4 APOLLONIO t.2 p.116 : « *ἐὰν ἀπὸ δύο δεδομένων σημείων εὐθεῖαι κλασθῶσιν, καὶ ᾖ τὰ ἀπ' αὐτῶν δοθέντι χωρίῳ διαφέροντα, τὸ σημεῖον ἅψεται θέσει δεδομένης εὐθείας·*
ἐὰν δὲ ὦσιν ἐν λόγῳ δοθέντι, ἤτοι εὐθείας ἢ περιφερείας· » }

·6 $n\varepsilon \mathrm{N}_1 . a\varepsilon \mathrm{p}\ \mathrm{F}\ 1\cdots n . g=(\Sigma a)/n . x\varepsilon \mathrm{p} .\supset.$
$\Sigma(x-a)^2=n(x-g)^2+\Sigma(g-a)^2$

{ APOLLONIO t.2 p.116: « *ἐὰν ἀπὸ ὁσωνοῦν δεδομένων σημείων κλασθῶσιν εὐθεῖαι πρὸς ἑνὶ σημείῳ, καὶ ᾖ τὰ ἀπὸ πασῶν εἴδη ἴσα δοθέντι χωρίῳ, τὸ σημεῖον ἅψεται θέσει δεδομένης περιφερείας·* » }

·7 $n\varepsilon \mathrm{N}_1 . a\varepsilon \mathrm{p\,f}\ 1\cdots n . a\varepsilon \mathrm{q\,f}\ 1\cdots n . \Sigma(x, 1\cdots n)-=0 . g=\Sigma(x_r a_r|r, 1\cdots n)/\Sigma(x, 1\cdots n) . p\varepsilon \mathrm{p} .\supset.$
$\Sigma[x_r(p-a_r)^2|r, 1\cdots n]=\Sigma[x_r(g-a_r)^2|r, 1\cdots n]+\Sigma(x, 1\cdots n)(p-g)^2$

Si n es numero naturale, et si a es successione de n puncto, vel si $a_1\ a_2 \ldots a_n$ es puncto, et si x es successione de n quantitate, vel si $x_1\ x_2 \ldots x_n$ es quantitate, et summa de illos non es nullo, et si nos voca g barycentro de punctos a cum pondo x, et si nos sume puncto arbitrario p, tunc:

Summa de quadratos de distantias de puncto p ad differente puncto a, multiplicato per correspondente pondo x, vale summa analogo calculato pro barycentro, plus pondo totale multiplicato per quadrato de distantia de puncto p ad barycentro g.

Summa considerato es dicto « momento de inertia de systema $x_1a_1 \ldots x_n a_n$ relativo ad puncto p ».

✻ 16·1 $a,b,c\varepsilon p\,.\,m,n\varepsilon q\,.\supset.$
$[(m+n)a-(mb+nc)]^2 = m(m+n)(a-b)^2+n(m+n)(a-c)^2-mn(b-c)^2$
} Stewart a.1746 {

·2 $a,b,c,d\varepsilon p\,.\,m,n,p\varepsilon q\,.\supset.$
$[(m+n+p)a-(mb+nc+pd)]^2 = (m+n+p)[m(a-b)^2+n(a-c)^2$
$+p(a-d)^2] -mn(b-c)^2-mp(b-d)^2-np(c-d)^2$

mod dist

✻ 21. $u,v\varepsilon v\,.\supset$ ·0 $\text{mod}u = \sqrt{(u^2)}$ Df

·1 $\text{mod}\,0 = 0$ ·11 $\text{mod}u = 0 \,.=.\, u=0$

·2 $\text{mod}(-u) = \text{mod}u$

·3 $x\varepsilon r\,.\supset.\ \text{mod}(xu) = \text{mod}x\ \text{mod}u$
[$\text{mod}(xu) = \sqrt{[(xu)^2]} = \sqrt{(x^2u^2)} = \sqrt{(x^2)}\sqrt{(u^2)} = \text{mod}x\ \text{mod}u$]

·4 $\text{mod}(u\times v) \leqq \text{mod}u\ \text{mod}v$
[$x\varepsilon r\,.\supset.\ (xu+v)^2 \gtreqless 0\,.\supset.\ x^2u^2+2xu\times v+v^2 \gtreqless 0 :\supset.$
$(\text{mod}u)^2(\text{mod}v)^2 \gtreqless (u\times v)^2\,.\supset.$ P]

·5 $\text{mod}(u+v) \leqq \text{mod}u+\text{mod}v$
[P·4 $.\supset.\ \text{mod}(u+v) = \sqrt{(u^2+2u\times v+v^2)} \leqq$
$\sqrt{[(\text{mod}u)^2+2\text{mod}u\ \text{mod}v+(\text{mod}v)^2]}\,.\supset.$ P]

✻ 22. $a,b\varepsilon p\,.\,u,v\varepsilon\,\text{Cls}'p\,.\supset:$
·1 $d(a,b) = \text{dist}(a,b) = \text{mod}(b-a)$ Df
·2 $d(a,v) = l_{\prime}\,d(a,y)\,|y`v$ . $d(u,b) = l_{\prime}\,d(x,b)\,|x`u$ Df
·3 $d(u,v) = l_{\prime}\,d(x,y)\,|(x,y)`(u{:}v)$ Df
$d(a,b)$, vel $\text{dist}(a,b)$ indica distantia de a ad b; a et b es puncto (P·1), aut classe de puncto vel figura (P·3).

✻ 23. $a,b,c,d\ \varepsilon p\,.\supset.$ ·1 $d(a,b) \leqq d(a,c)+d(c,b)$ [= P21·5]
} Euclide I P20:
« *Παντός τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονές εἰσι.* » {

·2 $d(a,b) = d(a,c) = d(b,c) = 1\,.\supset.$
$d[(a+b)/2, c] = \sqrt{3}/2$.
$d[(a+b+c)/3, a] = /\sqrt{3}$ } Euclide XIII P12:
« *Ἐὰν εἰς κύκλον τρίγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ τριγώνου πλευρὰ δυνάμει τριπλασίων ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου.* » {
Mediana et radio de circulo circumscripto ad triangulo regulare.

·3 $d(a,b) = d(a,c) = d(a,d) = d(b,c) = d(b,d) = d(c,d) = 1 .\supset.$
$d[(a+b)/2, (c+d)/2] = \surd(/2) \;.\; d[(a+b+c)/3, d] = \surd(2/3) \;.$
$d[(a+b+c+d)/4, a] = \surd(3/8)$ { EUCLIDE XIII P13:
« ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος. » }

Distantia de latere opposito, altitudo, radio de sphaera circumscripto ad tetrahedro regulare.

❋ 24.

Si nos sume ut idea primitivo « puncto », et relatione inter tres puncto a,b,c, que nos indica per $d(a,c) = d(b,c)$
lege « a et b es aequidistante de c », nos pote defini omni ente de Geometria.

Definitione de puncto medio de duo puncto a et b, indicato per $(a+b)/2$:

·1 $a,b\varepsilon p .\supset. (a+b)/2 = \iota\, p \frown x\ni[d(x,a) = d(x,b) :$
$y\varepsilon p \,.\, d(y,a) = d(x,a) \,.\, d(y,b) = d(x,b) .\supset_y . y=x]$ Dfp

Df de aequalitate de duo vectore:

·3 $a,b,c,d\varepsilon p .\supset: a-b = c-d .=. (a+d)/2 = (b+c)/2$ Dfp

Ergo habe sensu serie de Prop 1-10.

$u,v\varepsilon v .\supset.\cdot$

·4 $\mathrm{mod}\,u = \mathrm{mod}\,v .=: a\varepsilon p .\supset_a . d(a,a+u) = d(a,a+v)$ Dfp

Df de aequalitate de modulo de duo vectore.

Tunc P22·1 defini distantia de duo puncto.

·5 $u\times v = 0 .=: \mathrm{mod}(u+v) = \mathrm{mod}(u-v)$ Dfp

Df de relatione $u\times v = 0$, « vectores u et v es perpendiculare ».

·6 $u,v\varepsilon v .\supset:$
$\mathrm{mod}\,u \gtreqless \mathrm{mod}\,v .=. \exists\, v\frown z\ni[u\times z = 0 \,.\, \mathrm{mod}\,u = \mathrm{mod}(v+z)]$ Dfp

Df de relatione $\gtreqless$ inter duo long-ore; unde nos deduce relatione $>$.

vct q

❋ 25. (vct | cx) §cx P11
(pnt » »

·1 $u\varepsilon \mathrm{Cls'p} .\supset. \lambda u = p\frown a\ni[d(a,u) = 0]$ Dfp

❋ 26·1 $u\varepsilon v \,.\, x\varepsilon q .\supset. xu = \iota\, \{ \lambda[(r\frown\theta x)u] \frown \lambda[(r\frown x/\theta)u] \}$ Df

Df de producto de numero irrationale per vectore.

·2 $x\varepsilon q\text{-}r \,.\, u\varepsilon v .\supset. xu\, \varepsilon v$ Pp

·3 (q | r) P6·5·6, P7, P12·9

$*$ 27·1 $i\varepsilon \mathrm{v}{-}\iota 0 \,.\supset.\, \exists\, \mathrm{v}{-}(\mathrm{q}i)$ Pp

·2 $i\varepsilon \mathrm{v}{-}\iota 0 \,.\, j\varepsilon \mathrm{v}{-}(\mathrm{q}i) \,.\supset.\, \exists\, \mathrm{v}{-}(\mathrm{q}i+\mathrm{q}j)$ Pp

·3 $i\varepsilon \mathrm{v}{-}\iota 0 \,.\, j\varepsilon \mathrm{v}{-}(\mathrm{q}i) \,.\, k\varepsilon \mathrm{v}{-}(\mathrm{q}i+\mathrm{q}j) \,.\supset.\, \mathrm{v}=\mathrm{q}i+\mathrm{q}j+\mathrm{q}k$ Pp

Si i es vectore non nullo, qi significa « producto de aliquo numero reale per i » vel vectore parallelo ad i ».

Pp·1 dice que existe vectores non parallelo ad vectore dato; ille ne es satisfacto si nos considera solo vectores pertinente ad recta dato.

Si i es vectore non nullo, et j es vectore non parallelo ad i, tunc qi+qj significa « vectore summa de duo vectore, uno parellelo ad i, et altero ad j »; vel « vectore complanare cum i et j ».

Pp·2 dice que existe vectores non complanare cum duo vectore non parallelo dato. Ille ne es satisfacto si nos considera solo vectores in plano dato.

Pp·3 dice que spatio que nos considera habe tres dimensione. Ille ne es satisfacto si nos substitue « vct » per « Cx4 ».

Ce tres Pp, necessario in aliquo casu, es minus interessante.

·4 $\mathrm{Hp}{\cdot}3 \,.\, x,y,z\varepsilon\mathrm{q} \,.\, xi+yj+zk=0 \,.\supset.\, x=0 \,.\, y=0 \,.\, z=0$

[$i,j,k\varepsilon\mathrm{v} \,.\, x,y,z\varepsilon\mathrm{q} \,.\, xi+yj+zk=0 \,.\, x-=0 \,.\supset.$

$i=(-y/x)j+(-z/x)k \,.\supset.\, i\varepsilon\, \mathrm{q}j+\mathrm{q}k$ (1)

Hp. (1). Transp $.\supset.$ P]

·5 $\mathrm{Hp}{\cdot}3 \,.\, x,y,z,x',y',z'\varepsilon\mathrm{q} \,.\, xi+yj+zk=x'i+y'j+z'k \,.\supset.$
$x=x' \,.\, y=y' \,.\, z=z'$

·6 $\mathrm{Hp}{\cdot}3 \,.\, o\varepsilon\mathrm{p} \,.\supset.\, \mathrm{p}=o+\mathrm{q}i+\mathrm{q}j+\mathrm{q}k$

Numeros x,y,z que figura in P·4·5 vocare « coordinatas de vectore $xi+yj+zk$ relato ad vectores fundamentale i,j,k ». Illos et vocare « coordinatas de puncto $o+xi+yj+zk$ relato ad origine o et ad ipsos vectore ».

Quantitates x,y,z,t es « coordinatas barycentrico » de $xa+yb+zc+td$, si $a,b,c,d\varepsilon$ pnt; et « projectivo », si a,b,c,d es summa de punctos.

$*$ 28.

$o\varepsilon\mathrm{p} \,.\, i\varepsilon \mathrm{v}{-}\iota 0 \,.\, j\varepsilon \mathrm{v}{-}\mathrm{q}i \,.\, k\varepsilon \mathrm{v}{-}(\mathrm{q}i+\mathrm{q}j) \,.\, x,y,z,x',y',z',m\varepsilon\mathrm{q} \,.\supset.$

·1 $(xi+yj+zk)+(x'i+y'j+z'k)=(x+x')i+(y+y')j+(z+z')k$
[Assoc+ $\supset$ P]

·2 $m(xi+yj+zk)=mxi+myj+mzk$ [Distrib($m\times$,+) $\supset$ P]

·3 $(o+x'i+y'j+z'k)-(o+xi+yj+zk)=$
$(x'-x)i+(y'-y)j+(z'-z)k$ [Assoc+ $\supset$ P]

·4 $(xi+yj+zk)\times(x'i+y'i+z'k)=xx'i^2+yy'j^2+zz'k^2+(xy'+x'y)$
$i\times j+(xz'+x'z)i\times k+(yz'+y'z)j\times k$ [Distrib($\times$,+) $\supset$ P]

$i^2=j^2=k^2=1 \,.\, i\times j=j\times k=k\times i=0 \,.\supset.$

·5 $(xi+yj+zk)\times(x'i+y'j+z'k)=xx'+yy'+zz'$ [Distrib($\times$,+)$\supset$P]

·6 $(xi+yj+zk)^2 = x^2+y^2+z^2$ [Distrib(×,+) ⊃ P]

·7 $\mathrm{mod}(xi+yj+zk) = \surd(x^2+y^2+z^2)$ [Df mod . P·6 . ⊃ P]

·8 $\mathrm{dist}(o+xi+yj+zk, o+x'i+y'j+z'k) =$
$\surd[(x'-x)^2+(y'-y)^2+(z'-z)^2]$ [Df dist . P·7 . ⊃ P]

·1 Coordinatas de summa de duo vectore.
·2 » de producto de vectore per numero.
·3 » de vectore differentia de duo puncto.
·4 Producto de duo vectore de dato coordinatas.
·5-·8 Casu de coordinatas orthogonale.

pnt –

✻ 29. $a,b\varepsilon \mathrm{p}$.⊃.

·1 $a\,\bar{}\,b = a+\theta(b-a)$. $a \vdash b = a+\Theta(b-a)$ Df

Indica segmento de punctos comprehenso inter a et b. Existe Geometria de positione fundato super idea primitivo « pnt » et « segmento ». Vide « Peano *Principii di Geometria* a.1889 » et RdM. a.1894 p.51-90.

·2 $a\varepsilon \mathrm{p}$. $u\varepsilon$ Cls'p .⊃. $a \vdash u = \mathrm{p} \frown x\ni\, \exists\, u \frown y\ni(x\varepsilon a \vdash y)$ Df

·3 $a,b\varepsilon \mathrm{p}$. $c\varepsilon a \vdash b$.⊃. $\mathrm{d}(a,c)+\mathrm{d}(c,b) = \mathrm{d}(a,b)$

·4 $a,b,c\varepsilon \mathrm{p}$. $d\varepsilon a\,\bar{}\,b\,\bar{}\,c$.⊃. $\mathrm{d}(d,b)+\mathrm{d}(d,c) \leqq \mathrm{d}(a,b)+\mathrm{d}(a,c)$
} Euclide I P21 {

[Hp . $e\varepsilon a \vdash c$. $d\varepsilon b \vdash e$.⊃. $\mathrm{d}(d,c) \leqq \mathrm{d}(d,e)+\mathrm{d}(e,c)$. Oper $+\mathrm{d}(b,d)$.⊃. $\mathrm{d}(b,d)+\mathrm{d}(d,c) \leqq \mathrm{d}(b,d)+\mathrm{d}(d,e)+\mathrm{d}(e,c)$. P·3 .⊃. $\mathrm{d}(b,d)+\mathrm{d}(d,c) \leqq \mathrm{d}(b,e)+\mathrm{d}(e,c)$. $\mathrm{d}(b,e) \leqq \mathrm{d}(b,a)+\mathrm{d}(a,e)$.⊃. $\mathrm{d}(b,d)+\mathrm{d}(d,c) \leqq \mathrm{d}(b,a)+\mathrm{d}(a,e)+\mathrm{d}(e,c) = \mathrm{d}(b,a)+\mathrm{d}(a,c)$]

U

✻ 30. $u\varepsilon \mathrm{v}-\iota 0$.⊃. ·0 $\mathrm{U}u = u/\mathrm{mod}u$ Df

·1 $\mathrm{mod\,U}u = 1$. $\mathrm{U}(-u) = -\mathrm{U}u$. $a\varepsilon \mathrm{Q}$.⊃. $\mathrm{U}\,au = \mathrm{U}u$

·2 $i,j,k\varepsilon \mathrm{v}$. $i^2=j^2=k^2=1$. $i\times j=j\times k=k\times i=0$. $x,y,z\varepsilon \mathrm{q}$.
$x^2+y^2+z^2>0$.⊃. $\mathrm{U}(xi+yj+zk) = (xi+yj+zk)/\surd(x^2+y^2+z^2)$
[Df U ⊃ P]

Uu, lege « unitate de u », es introducto per Hamilton. Responde ad « sgn$u$ » de Arithmetica pag.94.

vct lin Subst

✻ 31. (v | Cx) §Subst

·1 $u\varepsilon \mathrm{v}$.⊃. $(u\times v|v, \mathrm{v})\ \varepsilon\ (\mathrm{qFv})\mathrm{lin}$

§2 recta p_2 plan p_3

Ce § contine propositiones relativo ad symbolos, frequente in lingua commune, et minus importante pro calculo geometrico.

✻ 1·1 $a\varepsilon p \,.\, u\varepsilon v\text{-}\iota 0 \,.\supset.\, \text{recta}(a,u) = a+\text{q}u$ Df

Recta que transi per a, et es parallelo ad vectore u.

·11 $\text{Hp}\cdot 1 \,.\, m\varepsilon \text{q-}\iota 0 \,.\supset.\, \text{recta}(a,u) = \text{recta}(a, mu)$

·2 $p_2 = x\ni\{\exists\,(a,u)\ni[a\varepsilon p \,.\, u\varepsilon v\text{-}\iota 0 \,.\, x = \text{recta}(a,u)]\}$ Df

= « recta ».

·3 $\text{Hp}\cdot 1 \,.\, b\varepsilon p \,.\supset.\, \text{d}[b, \text{recta}(a,u)] = \sqrt{\{(b-a)^2 - [(b-a)\times \text{U}u]^2\}}$

·4 $a\varepsilon p \,.\, b\varepsilon \text{p-}\iota a \,.\supset.\, \text{recta}(a,b) = \text{recta}(a, b-a) = a+\text{q}(b-a)$ Df

✻ 2·1 $a\varepsilon p \,.\, u\varepsilon v\text{-}\iota 0 \,.\, v\varepsilon \text{v-q}u \,.\supset.\, \text{plan}(a,u,v) = a+\text{q}u+\text{q}v$ Df

Plano determinato per uno puncto et duo vectore.

·2 $p_3 = x\ni\{\exists(a,u,v)\ni[a\varepsilon p \,.\, u\varepsilon v\text{-}\iota 0 \,.\, v\varepsilon \text{v-q}u \,.\, x = \text{plan}(a,u,v)]$ Df

= « plano ».

·3 $a\varepsilon p \,.\, b\varepsilon \text{p-}\iota a \,.\, c\varepsilon \text{p-recta}(a,b) \,.\supset.$
$\text{plan}(a,b,c) = \text{plan}(a,b-a,c-a)$ Df

cmp|| cmp⊥

✻ 3. $u\varepsilon v\text{-}\iota 0 \,.\, v,w\varepsilon v \,.\supset.$ ·0 $(\text{cmp}||u)v = (v\times \text{U}u)\text{U}u$ Df

« componente parallelo ad u de v ».

·01 $(\text{cmp}\perp u)v = v - (\text{cmp}||u)v$ Df

« componente normale ad u de v ».

·1 $(\text{cmp}||u)(v+w) = (\text{cmp}||u)v + (\text{cmp}||u)w$

·11 » ⊥ » ⊥ » ⊥ »

·2 $(\text{cmp}||u)v = 0 \,.=.\, u\times v = 0 \quad : \quad (\text{cmp}\perp u)v = 0 \,.=.\, v\varepsilon \text{q}u$

·3 $(\text{comp}||u, \text{v}), (\text{comp}\perp u, \text{v}) \,\varepsilon\, (\text{vFv})\text{lin}$

proj

✻ 4·1 $x\varepsilon p \,.\, a\varepsilon p_2 \,.\supset.$
$(\text{proj}a)x = \iota\, a \frown y\ni[z\varepsilon a \,.\supset_z.\, (y-x)\times(y-z) = 0]$ Df

= « projectione super recta a de x ».

·2 $x \varepsilon \mathrm{p} . a \varepsilon \mathrm{p}_3 .\supset.$
$(\mathrm{proj}a)x = \iota\, a \frown y\ni[z \varepsilon a .\supset_z. (y-x)\times(y-z) = 0]$ Df
= « projectione super plano a de x ».

·3 $x \varepsilon \mathrm{p} . a \varepsilon \mathrm{p}_2 .\supset. \mathrm{d}(x,a) = \mathrm{d}[x, (\mathrm{proj}a)x]$ Dfp

·4 » $. a \varepsilon \mathrm{p}_3 .\supset.$ » » Dfp

✻ 5. Transl

$u,v \varepsilon \mathrm{v} .\supset.$ ·0 $\mathrm{Transl}u = [(p+u)|p, \mathrm{p}]$ Df
= « translatione repraesentato per vectore u ».

·1 $(\mathrm{Transl}v)(\mathrm{Transl}u) = \mathrm{Transl}(u+v)$

·2 $m \varepsilon \mathrm{N}_1 .\supset. (\mathrm{Transl}u)^m = \mathrm{Transl}\, mu$

·3 $(\mathrm{Transl}u)^{-1} = \mathrm{Transl}(-u)$

Sym

✻ 6. $a,b \varepsilon \mathrm{p} . u \varepsilon \mathrm{v} .\supset:$

·0 $\mathrm{Sym}a = \{[a+(a-x)]|x, \mathrm{p}\}$ Df
= « symmetria relativo ad a ».

·1 $(\mathrm{Sym}a)^2 = (\mathrm{idem}, \mathrm{p})$ ·2 $(\mathrm{Sym}a)^{-1} = \mathrm{Sym}a$

·3 $(\mathrm{Sym}b)(\mathrm{Sym}a) = \mathrm{Transl}\, 2(b-a)$

·4 $\mathrm{Transl}u = [\mathrm{Sym}(a+u/2)]\, (\mathrm{Sym}a)$

·5 $(\mathrm{Transl}u)(\mathrm{Sym}a) = \mathrm{Sym}(a+u/2)$

·6 $(\mathrm{Sym}a)(\mathrm{Transl}u) = \mathrm{Sym}(a-u/2)$

✻ 7. $a \varepsilon \mathrm{p}_2 .\supset.$ ·0 $\mathrm{Sym}a = \{[2(\mathrm{proj}a)x - x]|x, \mathrm{p}\}$ Df

·1 $(\mathrm{Sym}a)^2 = (\mathrm{idem}, \mathrm{p})$ ·2 $(\mathrm{Sym}a)^{-1} = \mathrm{Sym}a$

·3 $a,b \varepsilon \mathrm{p}_2 .\supset. [(\mathrm{Sym}a)(\mathrm{Sym}b)]^{-1} = \mathrm{Sym}b\mathrm{Sym}a$

✻ 8. $a \varepsilon \mathrm{p}_3 .\supset.$ P7·0-·2

·1 $a \varepsilon \mathrm{p} . u,v \varepsilon \mathrm{v} . v -= 0 . u \times v = 0 .\supset.$
$\mathrm{Sym}(a+u/2+\mathrm{q}v)\mathrm{Sym}(a+\mathrm{q}v) = \mathrm{Transl}u$

$a \varepsilon \mathrm{p} . u,v,w \varepsilon \mathrm{v} - \iota 0 . u \times v = u \times w = w \times u = 0 .\supset.$

·2 $\mathrm{Sym}(a+\mathrm{q}u) = \mathrm{Sym}(a+\mathrm{q}v)\, \mathrm{Sym}(a+\mathrm{q}w)$

·3 $\mathrm{Sym}(a+\mathrm{q}u)\, \mathrm{Sym}(a+\mathrm{q}v+\mathrm{q}w) = \mathrm{Sym}a$

·4 $\mathrm{Sym}(a+\mathrm{q}v+\mathrm{q}w)\, \mathrm{Sym}(a+u+\mathrm{q}v+\mathrm{q}w) = \mathrm{Transl}\, 2u$

✻ 9. Motor

·0 Motor $= m\ni\{\exists(a,b)\ni[a,b\,\varepsilon\,\mathrm{p}_2\,.\,m=(\mathrm{Sym}a)(\mathrm{Sym}b)]\}$ Df

« Motor » indica motu de corpore rigido, rato ut transformatione de puncto in puncto. Nos pote defini illo ut producto de duo symmetria.

·1 $u\varepsilon\mathrm{v}\ .\supset.\ \mathrm{Transl}u\ \varepsilon$ Motor [P8·1 ⊃ P]

·2 $a\varepsilon\mathrm{p}_2\ .\supset.\ \mathrm{Sym}a\ \varepsilon$ Motor [P8·2 ⊃ P]

Translatione et symmetria circa axi es motor.

·3 $m,n\varepsilon$ Motor $.\supset.\ mn\ \varepsilon$ Motor

·4 $m\varepsilon$ Motor $.\supset.\ m^{-1}\ \varepsilon$ Motor [P7·3 ⊃ P]

·5 $m\varepsilon\,\mathrm{pFp} : x,y\varepsilon\mathrm{p}\,.\supset_{x,y}.\,\mathrm{d}(mx,my)=\mathrm{d}(x,y)] :\supset:$
$m\varepsilon$ Motor $:\cup:\ a\varepsilon\mathrm{p}_3\,.\supset_a.\ (\mathrm{Sym}a)m\ \varepsilon$ Motor

Si m es transformatione de puncto in puncto, et si distantia de duo puncto x,y semper aequa distantia de correspondentes mx,my, tunc aut m es Motor, aut, si a es plano, producto de m per symmetria circa a es Motor.

·6 $m,n,p\varepsilon$ Motor $.\supset.$
$\exists(a,b,c)\ni[a,b,c\varepsilon\mathrm{p}_2\,.\,(\mathrm{Sym}a)m=(\mathrm{Sym}b)n=(\mathrm{Sym}c)p]$

{ Halphen a.1882 AnnN. s.2 t.1 p.299:

« Trois positions quelconques d'une même figure dans l'espace sont les symétriques d'une seule et même figure prise respectivement par rapport à trois droites ». }

·7 $m\varepsilon$ Motor $.\supset.\ \exists\,\mathrm{p}_2\cap x\ni(m'x=x)$ { Mozzi a.1763 p.5:

« Il movimento si riduce a due altri, uno dei quali... di rotamento... e l'altro sarà rettilineo... e parallelo all'asse di rotazione ».

Chasles a.1830 *Bulletin de Férussac* t.14 p.324:

« Quand on a dans l'espace un corps solide libre, si on lui fait éprouver un déplacement fini quelconque, il existera toujours dans ce corps, une certaine droite indéfinie, qui après le déplacement, se retrouvera au même lieu qu'auparavant ». }

·8 $a\varepsilon\mathrm{p}\ .\ m\varepsilon$ Motor $.\ ma=a\ .\supset.\ \exists\mathrm{p}_2\cap y\ni(x\varepsilon y\ .\supset_x.\ mx=x)$

Omni motu que tene fixo puncto a es rotatione circa axi per a.

{ Euler, *Formulae generales pro translatione quacunque corporum rigidorum*, PetrNC. t.20 p.202 }

·9 $a,b,a',b'\varepsilon\mathrm{p}\ .\ \mathrm{d}(a,b)=\mathrm{d}(a',b')\ .\ b-a\ -=b'-a'\ .\supset.\ \exists(n,y)\ni$
$[n\varepsilon$ Motor $.\ y\varepsilon\mathrm{p}_2\ .\ na=a'\ .\ nb=b' : x\varepsilon y\ .\supset_x.\ nx=x]$

Si distantia de punctos a ad b aequa distantia de a' ad b', tum existe translatione aut rotatione circa axi, que fer a in a', et b in b'.

✻ 10. Homot

$a,b,c,p\varepsilon\mathrm{p}\,.\,h,k\varepsilon\mathrm{q}\,.\,u\varepsilon\mathrm{v}\;.\supset.$

·0 $\mathrm{Homot}(c,k) = \{[c+k(p-c)]|p,\mathrm{p}\}$ Df

$=$ « Homothetia de centro c et de ratione k ».

·1 $\mathrm{Homot}(c,1)p = p\;.\;\mathrm{Homot}(c,-1)p = (\mathrm{Sym}c)p$

·2 $\mathrm{Homot}(c,k)b - \mathrm{Homot}(c,k)a = k(b-a)$

·3 $k -= 1\;.\supset.\;\mathrm{Homot}(c,k)\mathrm{Transl}u = \mathrm{Homot}[c+uk/(1-k), k]$

·4 $hk -= 1\;.\supset.$

$[\mathrm{Homot}(b,k)]\,[\mathrm{Homot}(a,h)] = \mathrm{Homot}[a+(b-a)(1-k)/(1-hk), hk]$

·5 $k -= 0\;.\supset.\;[\mathrm{Homot}(b,/k)]\,[\mathrm{Homot}(a,k)] = \mathrm{Transl}\,(b-a)(1-/k)$

·6 $m\varepsilon\mathrm{N}_1\;.\supset.\;[\mathrm{Homot}(c,k)]^m = \mathrm{Homot}(c,k^m)$

cos sin

✻ 11. $u,v\varepsilon\,\mathrm{v}-\iota 0\;.\supset.$ ·0 $\cos(u,v) = (\mathrm{U}u)\times(\mathrm{U}v)$ Df cos

·1 $\mathrm{mod}\cos(u,v) = \mathrm{mod}\,(\mathrm{cmp}\,||\,u)\mathrm{U}v$

·2 $\cos(u,v) = \cos(v,u)$

$[\cos(u,v) = \mathrm{U}u\times\mathrm{U}v = \mathrm{U}v\times\mathrm{U}u = \cos(u,v)]$

·21 $\cos(-u,v) = -\cos(u,v)$

$[\cos(-u,v) = \mathrm{U}(-u)\times\mathrm{U}v = (-\mathrm{U}u)\times\mathrm{U}v = -(\mathrm{U}u\times\mathrm{U}v) = -\cos(u,u)]$

·3 $\cos(-u,-v) = \cos(u,v)$ [P·21 .⊃. P]

·31 $-1 \leqq \cos(u,v) \leqq 1$

·4 $\cos(u,u) = 1.\;\cos(u,-u) = -1$ [Df cos .⊃. P]

·5 $\cos(u,v) = (u\times v)/[\mathrm{mod}u\times\mathrm{mod}v]$

·6 $u\times v = \mathrm{mod}u\;\mathrm{mod}v\;\cos(u,v)$

[P·5 . Oper×(mod$u$ mod$v$) .⊃. P]

✻ 12. $u,v\varepsilon\,\mathrm{v}-\iota 0\;.\supset.$ ·1 $\sin(u,v) = \mathrm{mod}\,(\mathrm{cmp}\perp u)\mathrm{U}v$ Df sin

·2 $\sin(u,v) = \sin(v,u) = \sin(-u,v)$

·3 $\sin(u,u) = 0$ ·4 $0 \leqq \sin(u,v) \leqq 1$

·5 $[\sin(u,v)]^2 + [\cos(u,v)]^2 = 1$

$[u,v\varepsilon\mathrm{v}-\iota 0\;.\supset.\;v = v(\mathrm{cmp}\perp u)\mathrm{U}v + v(\mathrm{cmp}\,||\,u)\mathrm{U}v$ (1)

(1) $.\supset.\;1 = [(\mathrm{cmp}\perp u)\mathrm{U}v]^2 + [(\mathrm{cmp}\,||\,u)\mathrm{U}v]^2$ (2)

P11·1 , 12·1 . (2) .⊃. P]

·6 $\sin(u,v) = \sqrt{[1-\cos(u,v)^2]}$ Dfp

·7 $v -\varepsilon\mathrm{q}u\;.\supset.\;\sin(u,v) = \cos[v,(\mathrm{cmp}\perp u)v]$ Dfp

❋ 13. $i, j, k \varepsilon \mathrm{v} \,.\, i^2 = j^2 = k^2 = 1 \,.\, i \times j = j \times k = k \times i = 0 \,.\, x, y, z, x', y', z' \varepsilon \mathrm{q} \,.\, -(x = y = z = 0) \,.\, -(x' = y' = z' = 0) \,.\supset .$

·1 $\cos(xi + yj + zk, i) = x / \sqrt{}(x^2 + y^2 + z^2) \,.$

» » $, j) = y /$ » »

» » $, k) = z /$ » » [Df cos $\supset$ P]

·2 $\cos(xi + yj + zk, x'i + y'i + z'k) =$

$(xx' + yy' + zz') / \sqrt{}[(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2)]$ [Df cos $\supset$ P]

·3 $\sin(xi + yj + zk, x'i + y'j + z'k) =$

$\sqrt{}\{[(yz' - y'z)^2 + (zx' - x'z)^2 + (xy' - x'y)^2] / [(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2)]\}$

[$\ldots = \sqrt{}[1 - \cos(xi + yj + zk, x'i + y'j + z'k)^2] = \ldots$]

Vide continuatione in § π.

❋ 14. coord (coordinata)

$i \varepsilon \mathrm{v} - \iota 0 \,.\, j \varepsilon \mathrm{v} - \mathrm{q} i \,.\, k \varepsilon \mathrm{v} - (\mathrm{q} i + \mathrm{q} j) \,.\, u \varepsilon \mathrm{v} \,.\supset:$

·0 $\mathrm{coord}(u; i, j, k) = \iota\mathrm{q} \cap x \ni (u - xi \;\varepsilon\; \mathrm{q} j + \mathrm{q} k)$ Df

coord(u; i,j,k) significa « primo coordinata de vectore u, relato ad vectores i,j,k ».

·1 $\mathrm{coord}(u + v; i, j, k) = \mathrm{coord}(u; i, j, k) + \mathrm{coord}(v; i, j, k)$

·2 $\mathrm{coord}(mu; i, j, k) = m\,\mathrm{coord}(u; i, j, k)$

·3 $u = [\mathrm{coord}(u; i, j, k)]i + [\mathrm{coord}(u; j, k, i)]j + [\mathrm{coord}(u; k, i, j)]k$

·4 $i' \varepsilon \mathrm{v} - \iota 0 \,.\, j' \varepsilon \mathrm{v} - \mathrm{q} i' \,.\, k' \varepsilon \mathrm{v} - (\mathrm{q} i' + \mathrm{q} j') \,.\supset .$

$\mathrm{coord}(u; i', j', k') = \mathrm{coord}(u; i, j, k)\, \mathrm{coord}(i; i', j', k') +$

$\mathrm{coord}(u; j, k, i)\, \mathrm{coord}(j; i', j', k') + \mathrm{coord}(u; k, i, j)\, \mathrm{coord}(k; i', j', k')$

Transformatione de coordinatas, Euler t. 2, p. 17.

·5 $\mathrm{coord}(u; i, j, k) = (1, 0, 0) / (i, j, k)\, u$ Dfp

·6 $i, j, k \varepsilon \mathrm{v} \,.\, i^2 = j^2 = k^2 = 1 \,.\, i \times j = i \times k = j \times k = 0 \,.\supset:$

$u \varepsilon \mathrm{v} \,.\supset .\, \mathrm{coord}(u; i, j, k) = u \times i$

§3 A quaternio

❋ 1. $u,v \varepsilon \mathrm{v} - \iota 0 \,.\, u^2 = v^2 \,.\, u \times v = 0 \,.\supset.$

·0 $v/u = (v,-u)/(u,v)$ Df

Si *u* et *v* es vectore, non nullo, aequales in valore absoluto, et orthogonales, tunc v/u indica substitutione que ad vectores *u* et *v* fac corresponde vectores *v* et $-u$. Ce substitutione es definito super vectores $\mathrm{q}u+\mathrm{q}v$, id es complanare cum *u* et *v*; et habe proprietates de unitate imaginario de Algebra.

Wessel a.1797 et Bellavitis a.1832 opera super uno unitate imaginario, id es multiplica vectores in plano fixo per numeros imaginario.

$i = v/u \,.\, x,y,x',y' \varepsilon \mathrm{q} \,.\supset.$

·1 $iu = v \,.\, iv = -u \,.\, i(xu+yv) = xv - yu$

·2 $\mathrm{mod}\, i(xu+yv) = \mathrm{mod}(xu+yv)$ ·3 $\mathrm{mod}\, i = 1 \,.\, i^2 = -1$

·4 $(x'+iy')(xu+yv) = (x'x - y'y)u + (xy'+x'y)v$

·5 $\mathrm{mod}(x+iy) = \sqrt{(x^2+y^2)}$

❋ 2.

·0 $\mathrm{A} = i \ni \exists (u,v) \ni (u,v \varepsilon \mathrm{v} - \iota 0 \,.\, u^2 = v^2 \,.\, u \times v = 0 \,.\, i = v/u)$ Df

A = «absoluto», indica omni substitutione de forma indicato.

Cayley introduce vocabulo «absoluto». Hamilton ute expressione complexo «right radial».

·1 $i \varepsilon \mathrm{A} \,.\supset.\, i^2 = -1 \,.\, \mathrm{mod}\, i = 1$

·2 $x,y,x',y' \varepsilon \mathrm{q} \,.\, i,j \varepsilon \mathrm{A} \,.\, x+iy = x'+jy' \,.\supset.\, x = x' \,.\, y^2 = y'^2$

[$iy = (x'-x)+jy' \,.\supset.\, -y^2 = (x'-x)^2 + 2(x'-x)y'j - y'^2 \,.\supset.$

$y'^2 - y^2 - (x'-x)^2 = 2(x'-x)yj \,.\supset.\, [y'^2 - y^2 - (x'-x)^2]^2 = -4(x'-x)^2 y'^2$

$.\supset.\, (y'^2-y^2)^2 - 2(y'^2-y^2)(x'-x)^2 + (x'-x)^4 + 4(x'-x)^2 y'^2 = 0$

$.\supset.\, (y'^2-y^2)^2 + 2(y^2+y'^2)(x'-x)^2 + (x'-x)^4 = 0$

$.\supset.\, [(x'-x)^2 + y^2 + y'^2]^2 - 4y^2y'^2 = 0$

$.\supset.\, [(x'-x)^2 + (y'-y)^2][(x'-x)^2 + (y+y')^2] = 0$

$.\supset.\, (x'-x)^2 + (y'-y)^2 = 0 \,.\cup.\, (x'-x)^2 + (y+y')^2 = 0$

$.\supset:\, x'-x = 0 \,.\, y'-y = 0 :\cup:\, x'-x = 0 \,.\, y+y' = 0$

$.\supset:\, x'-x = 0 : y'-y = 0 \,.\cup.\, y'+y = 0 :\supset \mathrm{P}$]

·3 $\mathrm{Hp}\cdot 2 \,.\, y -\!= 0 \,.\supset.\, j = i\, \mathrm{sgn}(y'/y)$

✻ 3·0 quaternio $=$ qtr $=$ q $+$ qA Df

« Quaternio » es omni expressione de forma $x+iy$, ubi x et y es numero reale, et i es uno unitate imaginario.

Hamilton introduce vocabulo « quaternio », nam illo es determinato per 4 numero reale (P6·1).

$a,b\varepsilon$ qtr . $x,y\varepsilon$q . $i\varepsilon$A .⊃.

·1 real$a$ $=$ ɿ q ⌢ x϶(a—x ε qA) .

Imag$a$ $=$ a—real$a$. Ka $=$ real$a$— Imag$a$ Df

real$a$ $=$ Sa (lege: scalare de a) de Hamilton.

(« Scalare » deriva ab L « scala ». Vide editione de a.1899 p. 11, non deriva de Anglo, contra opinione de aliquo Physico).

$=$ Invariante de gradu 1 de substitutione a.

Imag$a$ lege « termine (parte) imaginario de a », responde ad i imag$a$ de §q'.

$=$ Va (lege: vectore de a) de Hamilton. Nota que illo non es vectore, sed substitutione repraesentabile per vectore. Vide § a.

·2 real($x+iy$) $=$ x . Imag($x+iy$) $=$ iy .

K($x+iy$) $=$ x—iy . mod($x+iy$) $=$ √(x^2+y^2)

·3 $a$$=$real$a$+Imag$a$.

real$a$ $=$ (a+Ka)/2 . Imag$a$ $=$ (a—Ka)/2 Dfp

·4 KKa $=$$a$

·5 mod$a$ $=$√[(real$a$)²—(Imag$a$)²] $=$ √(a×Ka) Dfp

·6 a-$=$0 .⊃. Ua $=$ a/mod$a$ Df

·7 » .⊃. Ka $=$ /(a mod$a$)² . K/a $=$ /Ka

✻ 4. $a,b\varepsilon$v . a -$=$ 0 .⊃: ·0 b/a $=$ ɿ qtr ⌢ x϶($b$$=$$xa$) Df

·1 $b\varepsilon$ qa .⊃. b/a εq

·3 (comp||a)b $=$ a real(b/a) . (comp⊥a)b $=$ aImag(b/a)

·4 real b/a $=$ (b×a)/a^2

·5 mod(b/a) $=$ (mod$b$)/(mod$a$)

·6 b-$=$0 .⊃. U(b/a) $=$ (Ub)/(Ua)

·7 » .⊃. /(b/a) $=$ a/b

·8 quaternio $=$ v/(v-ι0) Dfp

✻ 5. $x,y\varepsilon$ qtr .⊃: ·1 $x+y$ $=$

ɿ qtr ⌢ z϶($u\varepsilon$ Variab$x$⌢Variab$y$.⊃$_u$. zu $=$ xu+yu) Df

·2 yx $=$

ɿ qtr ⌢ z϶($u\varepsilon$ Variab$x$. xu ε Variab$y$.⊃$_u$. zu $=$ yxu) Df

Summa et producto de duo quaternio complanare, id es cum idem Variabilitate, es definito in §Subst. H a m i l t o n defini ce operationes super quaterniones non complanare.

Pro summa duo quaternio x et y, sume uno vectore u pertinente ad ambo plano de x et de y. Tunc habe sensu xu, yu, et $xu+yu$, que es vectore. Quaternio que, operato super u, da idem resultatu, es summa quaesito.

Analogo es Df de producto.

$a,b,c\varepsilon \mathrm{v} \,.\, a-=0 \,.\supset:$ ·3 $(b+c)/a = b/a+c/a$

·4 $b -= 0 \,.\supset.\, (c/b)(b/a) = c/a$

·2 $x,y,z\varepsilon \mathrm{qtr} \,.\supset.\, x+y=y+x \,.\, x+(y+z)=x+y+z \,.$
$\mathrm{real}(x+y) = \mathrm{real}x+\mathrm{real}y \,.\, \mathrm{Imag}(x+y) = \mathrm{Imag}x+\mathrm{Imag}y \,.$
$\mathrm{K}(x+y) = \mathrm{K}x+\mathrm{K}y$

·5 $x,y,z\varepsilon \mathrm{qtr} \,.\supset.\, (x+y)z = xz+yz \,.\, x(y+z) = xy+xz \,.$
$x(yz) = xyz$

✻ 6. $a,b,c\varepsilon \mathrm{v} \,.\, a^2 = b^2 = c^2 = 1 \,.\, a\times b = b\times c = c\times a = 0 \,.\, i =$
$c/b \,.\, j=a/c \,.\, k = b/a \,.\supset.$

·0 $i^2=j^2=k^2=-1 \,.\, ij=k \,.\, jk=i \,.\, ki=j \,.\, ji=-k \,.\, kj=-i \,.$
$ik=-j \,.\, i(jk)=(ij)k=ijk=-1$

·1 $\mathrm{qtr} = \mathrm{q}+\mathrm{q}i+\mathrm{q}j+\mathrm{q}k$

$m,x,y,z,m',x',y',z'\varepsilon \mathrm{q} \,.\, u=m+xi+yj+zk \,.\, u'=m'+x'i+y'j+z'k \,.\supset$

·2 $u=0 \,.=.\, m=x=y=z=0$

·3 $u=u' \,.=.\, m=m' \,.\, x=x' \,.\, y=y' \,.\, z=z'$

·4 $u+u' = (m+m')+(x+x')i+(y+y')j+(z+z')k$

·5 $u'u = (mm'-xx'-yy'-zz')+(m'x+mx'+y'z-z'y)i+$
$(m'y+my'+z'x-x'z)j+(m'z+mz'+x'y-y'x)k$

·6 $\mathrm{real}u = m \,.\, \mathrm{Imag}u = ix+jy+zk \,.\, \mathrm{K}u = m-ix-jy-zk \,.$
$\mathrm{mod}u = \sqrt{(m^2+x^2+y^2+z^2)} \,.$

·7 $u-=0 \,.\supset.$

$\mathrm{U}u = (m+ix+jy+kz)/\sqrt{(m^2+x^2+y^2+z^2)} \,.$

$\mathrm{real}\,\mathrm{U}u = m/\sqrt{(m^2+x^2+y^2+z^2)} \,.$

$\mathrm{mod}\,\mathrm{Imag}\mathrm{U}u = \sqrt{(x^2+y^2+z^2)}/\sqrt{(m^2+x^2+y^2+z^2)} \,.$

$/u=(m-ix-jy-kz)/(m^2+x^2+y^2+z^2) \,.$

$u'/u = [(mm'+xx'+yy'+zz')+(mx'-m'x+yz'-y'z)i+(my'-m'y$
$+zx'-z'x)j+(mz'-m'z+xy'-x'y)k]/(m^2+x^2+y^2+z^2) \,.$

$(xa+yb+zc)/(x'a+y'b+z'c) = [(xx'+yy'+zz)+(zy'-yz')i$
$+(xz'-zx')j+(yx'-ry')k]/(x'^2+y'^2+z'^2)$

§4 a (producto alterno)

* 1·0 $u,v,w \varepsilon \mathrm{v} \,.\, i \varepsilon \mathrm{v}\text{-}\iota 0 \,.\, j \varepsilon \mathrm{v}\text{-}\mathrm{q}i \,.\, k \varepsilon \mathrm{v}\text{-}(\mathrm{q}i+\mathrm{q}j) \;.\supset\therefore$

$(uavaw)/(iajak) =$

$\mathrm{Dtrm}\{[\,\mathrm{coord}(u;\, i,j,k),\; \mathrm{coord}(u;\, j,k,i),\; \mathrm{coord}(u;\, k,i,j)\,],$

$[\quad » \; v \quad » \quad v \quad » \quad v \quad],$

$[\quad » \; w \quad » \quad w \quad » \quad w \quad]\}$ Df

$u,v,w \varepsilon \mathrm{v} \;.\supset\therefore$

·1 $uavaw = 0 \;.=:\; i \varepsilon \mathrm{v}\text{-}\iota 0 \,.\, j \varepsilon \mathrm{v}\text{-}\mathrm{q}i \,.\, k \varepsilon \mathrm{v}\text{-}(\mathrm{q}i+\mathrm{q}j) \;.\supset_{i,j,k}.$
$(uavaw)/(iajak) = 0$ Df

·2 $uauav = 0$

·3 $uavaw = 0 \;.=.$
$\exists (x,y,z) \ni [x,y,z \varepsilon \mathrm{q} \,.\, \text{-}(x=y=z=0) \,.\, xu+yv+zw=0]$ Dfp

·4 $\mathrm{coord}(u;i,j,k) = [(uajak)/(iajak)]$

uavaw vocare « producto alterno » vel « producto exteriore » de vectores u,v,w. Illo vocare etiam « trivectore » (v^3, vide P2·0).

uavaw repraesenta parallelepipedo constructo super u,v,w, considerato in grand-ore et in sensu.

Grassmann, *Ausdehnungslehre* a.1844 indica illo per $[uvw]$.
Plure A. adopta notatione uvw. Pro majore claritate, me adde signo de multiplicatione sub forma a, initiale de « alterno ».

Idem operatione occurre in :
Hamilton a.1845, sub forma S(uvw), vide P26·4.
De Saint Venant (ParisCR. a.1845 d.258).
Cauchy a.1853 s.1 t.11 p.444.
Vide bibliographia in RdM. a.1895 p.179.

·0 Ratione de duo trivectore *uavaw* et *iajak* vale determinante de coordinatas de u,v,w relato ad i,j,k.

·1 Df de trivectore nullo.

* 2·0 $\mathrm{v}^3 = [(uavaw) \mid (u;v;w)]`(\mathrm{v}{:}\mathrm{v}{:}\mathrm{v}) = \mathrm{v}a\mathrm{v}a\mathrm{v}$ Df
$=$ « trivectore »

$a,b \varepsilon \mathrm{v}^3 \,.\, m \varepsilon \mathrm{q} \;.\supset\therefore$

·1 $x \varepsilon \mathrm{v}^3\text{-}\iota 0 \;.\supset.\; a/x \,\varepsilon \mathrm{q}$

·2 $a=b \;.=:\; x \varepsilon \mathrm{v}^3\text{-}\iota 0 \;.\supset_x.\; a/x = b/x$ Df

·3 $a+b = \imath \mathrm{v}^3 \frown c \ni [x \varepsilon \mathrm{v}^3\text{-}\iota 0 \;.\supset_x.\; c/x = a/x + b/x]$ Df

·4 $ma = \;»\; [\quad » \quad c/x = m(a/x)]$ Df

·5 $a+b, ma \,\varepsilon\, \text{v}^3$ ·6 $a -= 0 \,.\supset.\, \text{v}^3 = \text{q}a$

v^3 es systema lineare ad uno dimensione.

·7 $-a = (-1)a \;.\; a-b = a+(-b)$ Df

✻ 3. $u,u',v,w \varepsilon \text{v} \,.\, m \varepsilon \text{q} \,.\supset.$

·1 $uavaw = -vauaw = -uawav$

·2 $(u+u')avaw = uavaw + u'avaw$

·3 $(mu)avaw = m(uavaw)$

✻ 4. $u,v,w \varepsilon \text{v} \,.\supset.$ ·0 $(uav)aw = uavaw$ Df

·01 $\text{v}^2 = [(uav)\,|(u;v)]\,\text{‘}(\text{v}\text{:}\text{v}) = \text{v}a\text{v}$ Df

$=$ « bivectore »

$a,b \varepsilon \text{v}^2 \,.\, m \varepsilon \text{q} \,.\supset\therefore$ ·02 $aau \,\varepsilon\, \text{v}^3$

·1 $a=0 \,.=:\, x \varepsilon \text{v} \,.\supset_x.\, aax = 0$ Df

·11 $uau = 0$

·12 $uav = 0 \,.=.\, \exists(x,y) \ni [x,y \varepsilon \text{q} \,.\, -(x=y=0) \,.\, xu+yv=0]$ Dfp

·2 $a=b \,.=:\, x \varepsilon \text{v} \,.\supset_x.\, aax = bax$ Df

·3 $a+b = \iota\, \text{v}^2 \frown c \ni [x \varepsilon \text{v} \,.\supset_x.\, cax = aax+bax]$ Df

·4 $ma =$ » $= m(aax)]$ Df

·5 $a+b, ma \,\varepsilon \text{v}^2$

·6 $(a+b)au = aau+bau \,.\, (u+v)aw = uaw+vaw$

·61 $m(aau) = (ma)au = aa(mu)$

·7 $uav = -vau$

·8 $uavaw -= 0 \,.\supset.\, \text{v}^2 = \text{q}vaw+\text{q}wau+\text{q}uav$

·9 $uaa = aau$ Df

Nos decompone trivectore *uavaw* in producto *a* de novo objecto *uav* per *w*. *uav* es « producto alternato » de vectores *u*,*v*. Vocare « bivectore ».

Nos repraesenta illo per parallelogrammo constructo super *u* et *v*, considerato in magnitudo et orientatione.

·2 Duo bivectore es aequales, si producto de illos per idem vectore es aequales, id es, si parallelogrammos es in plano parallelo, et habe idem grand-ore et sensu.

·8 v^2 es systema lineare ad 3 dimensione.

Bivectore in Mechanica vocare « copula » F. « couple » (Poinsot a.1803).

Calculo super areas cum orientatione occurre in Chelini, *Saggio di Geometria analitica trattata con nuovo metodo*, Roma a.1838.

❋ 5. $x \varepsilon \mathrm{p} \,.\, y \varepsilon \mathrm{p} - \iota x \,.\, z \varepsilon \mathrm{p} - \mathrm{recta}(x,y) \,.\, t \varepsilon \mathrm{p} - \mathrm{plan}(x,y,z) \,.$
$a,b,c,d \varepsilon \mathrm{p} \,.\supset.$

·0 $(a\alpha b\alpha c\alpha d)/(x\alpha y\alpha z\alpha t) =$
$(b-a)\alpha(c-a)\alpha(d-a) \,/\, (y-x)\alpha(z-x)\alpha(t-x)$ Df

·01 $a\alpha b\alpha c\alpha d = 0 \,.=.\, (b-a)\alpha(c-a)\alpha(d-a) = 0$ Df

= « punctos a,b,c,d es complanare ».

Si $a,b,c,d \varepsilon \mathrm{p}$, $a\alpha b\alpha c\alpha d$, producto alternato de quatuor punctos, repraesenta tetraedro que habe pro vertices ce punctos, considerato in grand-ore et in sensu.

$a\alpha b\alpha c\alpha d = 0$, quando punctos es in uno idem plano.

Si tedraedro ne es nullo, dicere destrorso, vel positivo (Möbius), si homo cum caput in a, et pedes in b, vide c ad sinistra et d ad dextera.

Expressione $a\alpha b\alpha c\alpha d$ vocare etiam momento de puncto a ad triangulo aut plano $b\alpha c\alpha d$, momento de segmento (vi) $a\alpha b$ ad segmento (axi, recta) $c\alpha d$, momento de triangulo $a\alpha b\alpha c$ ad puncto d.

❋ 6·0 $\mathrm{p}^4 = \mathrm{p}\alpha\mathrm{p}\alpha\mathrm{p}\alpha\mathrm{p}$ Df

= « quadripuncto » id es tetraedro considerato in grand-ore et in sensu.

$u,v \varepsilon \mathrm{p}^4 \,.\, \Psi \varepsilon \mathrm{p}^4 - \iota 0 \,.\, m \varepsilon \mathrm{q} \,.\supset.$ ·1 $u/\Psi \,\varepsilon \mathrm{q}$

·2 $u = v \,.=.\, u/\Psi = v/\Psi$ ·3 $(u+v)/\Psi = u/\Psi + v/\Psi$

·4 $u+v \,\varepsilon \mathrm{p}^4$ ·5 $(mu)/\Psi = m(u/\Psi)$

·6 $mu \,\varepsilon \mathrm{p}^4$ ·7 $\mathrm{p}^4 = \mathrm{q}\Psi$

·8 $-u = (-1)u \,\,.\,\, u-v = u+(-v)$ Df

·1 Si Ψ es un quadripuncto non nullo, ratio de quadripuncto u ad Ψ, vel mensura de u, pro unitate de mensura Ψ, es quantitate.

·7 p^4 constitue systema homogeneo ad Ψ.

❋ 7. $a,b,c,d,e \varepsilon \mathrm{p} \,.\supset.$

·1 $a\alpha b\alpha c\alpha d = -b\alpha a\alpha c\alpha d = +b\alpha c\alpha a\alpha d = -b\alpha c\alpha d\alpha a$

·2 $a\alpha a\alpha b\alpha c = 0$

·3 $a\alpha b\alpha c\alpha d - a\alpha b\alpha c\alpha e + a\alpha b\alpha d\alpha e - a\alpha c\alpha d\alpha e + b\alpha c\alpha d\alpha e = 0$

{·1-·0 Möbius, *Der barycentrische Calcul*, a.1827 §20}

Lege de commutatione in producto alternato: « si nos permuta duo factore, producto muta signo ».

* 8. $a,b,c,d \varepsilon \mathrm{p} \ .\supset.$

·1 $(aabac)ad = (aab)a(cad) = aa(bacad) = aabacad$ Df

Assoc a

·2 $(aab)ac = aa(bac) = aabac$ Df

·3 $\mathrm{p}^3 = \mathrm{p}a\mathrm{p}a\mathrm{p} \ . \ \mathrm{p}^2 = \mathrm{p}a\mathrm{p} \ . \ \mathrm{p}^1 = \mathrm{p}$ Df

$\mathrm{p}^3 =$ « tripuncto ». $\mathrm{p}^2 =$ « bipuncto ».

·4 $r,s \varepsilon \mathrm{N}_1 \ . \ r+s \leqq 4 \ . \ x \varepsilon \mathrm{p}^r \ . \ y \varepsilon \mathrm{p}^s \ .\supset. \ xay \ \varepsilon \ \mathrm{p}^{r+s}$

·5 $r,s,t \varepsilon \mathrm{N}_1 \ . \ r+s+t \leqq 4 \ . \ x \varepsilon \mathrm{p}^r \ . \ y \varepsilon \mathrm{p}^s \ . \ z \varepsilon \mathrm{p}^t \ .\supset.$
$xa(yaz) = (xay)az$ Assoc a

·6 $u \varepsilon \mathrm{p}^3 \ .\supset\therefore \ u=0 \ .=: \ x \varepsilon \mathrm{p} \ .\supset_x. \ uax=0$ Df

$=$ « area de u es nullo ».

·7 $u,v \varepsilon \mathrm{p}^3 \ .\supset\therefore \ u=v \ .=: \ x \varepsilon \mathrm{p} \ .\supset_x. \ uax = vax$ Df

$=$ « triangulos u et v jace in idem plano, et habe idem area et sensu ».

·8 $u \varepsilon \mathrm{p}^2 \ .\supset\therefore \ u=0 \ .=: \ x \varepsilon \mathrm{p} \ .\supset_x. \ uax=0$ Df

$=$ « long-ore de u es nullo ».

·9 $u,v \varepsilon \mathrm{p}^2 \ .\supset\therefore \ u=v \ .=: \ x \varepsilon \mathrm{p} \ .\supset_x. \ uax = vax$ Df

$=$ « lineas u et v habe commune recta, longore et sensu ».

* 9·0 $r,s \varepsilon \mathrm{N}_1 \ . \ r+s \leqq 4 \ . \ n \varepsilon \mathrm{N}_1 \ . \ a \varepsilon \ \mathrm{p}^r \mathrm{f} 1 \cdots n \ . \ m \varepsilon \ \mathrm{qf} 1 \cdots n \ . \ b \varepsilon \mathrm{p}^s \ .\supset.$
$[\Sigma(m_z a_z | z, 1 \cdots n)]ab = \Sigma[m_z(a_z ab) | z, 1 \cdots n]$ Df

Distrib$(a,+)$

·1 $r \varepsilon \ 1 \cdots 4 \ .\supset. \ \varphi^r = x \ni \{ \exists(n,m,a) \ni [n \varepsilon \mathrm{N}_1 \ . \ m \varepsilon \ \mathrm{qf} 1 \cdots n \ .$
$a \varepsilon \ \mathrm{p}^r \mathrm{f} 1 \cdots n \ . \ x = \Sigma(m_z a_z | z, 1 \cdots n)] \}$ Df

$\varphi^r =$ « forma de gradu r ».

·2 $\varphi^4 = \mathrm{p}^4$

·3 $r \varepsilon \ 1 \cdots 3 \ . \ u \varepsilon \ \varphi^r \ . \ x \varepsilon \ \mathrm{p}^{4-r} \ .\supset. \ uax \ \varepsilon \ \mathrm{p}^4$

$r \varepsilon \ 1 \cdots 3 \ . \ u,v \varepsilon \ \varphi^r \ . \ m \varepsilon \mathrm{q} \ .\supset\therefore$

·4 $u=0 \ .=: \ x \varepsilon \ \mathrm{p}^{4-r} \ .\supset_x. \ uax=0$ Df

·5 $u=v \ .=: \ x \varepsilon \ \mathrm{p}^{4-r} \ .\supset_x. \ uax = vax$ Df

·51 $u=v \ .=: \ (uax|x, \mathrm{q}^{4-r}) = (vax|x, \mathrm{q}^{4-r})$

·6 $u+v = \iota \ \varphi^r \frown z \ni [x \varepsilon \ \mathrm{p}^{4-r} \ .\supset_x. \ zax = uax+vax]$ Df

·7 $mu = \iota \ \varphi^r \frown z \ni [x \varepsilon \ \mathrm{p}^{4-r} \ .\supset_x. \ zax = m(uax)]$ Df

·8 $r,s\varepsilon N_1 . r+s \leqq 4 . n\varepsilon N_1 . a\varepsilon \varphi^r . b\varepsilon p^s f1\cdots n . m\varepsilon qf1\cdots n .\supset.$
$aa[\Sigma(m_z b_z|z,1\cdots n)] = \Sigma[m_z(aab_z)|z,1\cdots n]$ Df
Distrib$(a,+)$

·9 $r,s,t\,\varepsilon N_1 . r+s+t \leqq 4 . a\varepsilon\varphi^r . b\varepsilon\varphi^s . c\varepsilon\varphi^t .\supset.$
$aabac = (aab)ac$ Df

Nos suppone p. ex. $r=1, s=3$. Tunc P·0 considera serie de n punctos a_1, $a_2,\ldots a_n$, et serie de numeros reale, positivo vel negativo, $m_1,m_2,\ldots m_n$; b es tripuncto; ce P pone per Df:

$$(m_1a_1+m_2a_2+\ldots+m_na_n)ab = m_1(a_1ab)+m_2(a_2ab)+\ldots+m_n(a_nab).$$

Secundo membro, summa de quadripunctos, es reductibile ad uno quadripuncto. Id es, nos pone per Df proprietate distributivo de signo a ad $+$. In Mechanica, systema de punctos cum numeros correspondente, dicto « massa », vocare « systema materiale ». Ergo per Df, momento de systema materiale $m_1a_1+m_2a_2+\ldots$ ad plano, vel tripuncto, b, vale summa de momentos de punctos, cum massa correspondente.

·1 Nos voca « forma de gradu 1 » omni expressione de forma $m_1a_1+m_2a_2+\ldots$, id es systema materiale.

·4 Nos dice que uno forma de gradu 1 es nullo, si es nullo suo producto per tripuncto arbitrario: id es, si es nullo suo momento ad omni plano.

·5 Duo forma es aequale, per Df, si es aequale suo producto per tripuncto arbitrario.

Nunc nos suppone $r=2$, $s=2$. Nos habe serie de bipunctos; in Mechanica systema de vi, que nos considera applicato ad corpore rigido.

P·0 dice que, per Df, momento de systema de vi ad axi (p^2) b, es summa de momento de vi dato.

P·4 dice que systema de p^2 es nullo vel que corpore es in equilibrio, si suo momento ad omni axi es nullo.

In modo analogo per forma de gradu 3. Forma de gradu 4 es reductibile ad uno p^4, secundo P·2.

P·8 da Df de producto alternato de duo forma. Exprime proprietate distributivo de operatione a ad secundo factore.

❋ 10. $a,b,c,d,e\varepsilon p .\supset:$

·1 $aaa=0$ [P7·2 . $b,c\varepsilon p .\supset_{b,c}. aaaabac=0$: Df9·4 :$\supset$. P]

·2 $aab=-baa$ [P7·1 . $c,d\varepsilon p .\supset_{c,d}. aabacad=-baaacad$: Df 9·7 :$\supset$ P

·3 $aabac=-baaac=+bacaa$

·4 $aabacad=(aabac-aabad+aacad-bacad)ae$
[P7·3 . Distrib$(a,+)$.$\supset$. P]

·5 $aabacad=0 .\supset. aabac-aabad+aacad-bacad=0$
[Hp . P·4 .$\supset$: $e\varepsilon p .\supset_e . (aabac-\ldots+\ldots-\ldots)ae=0 .\supset.$ Ths]

·6 $aabacad = 0 \;.\supset.\; aabac = (aab + bac + caa)ad$
[P·5·2 . Distrib$(a,+)$ $.\supset.$ P]

·7 $aabac = 0 \;.\supset.\; aab + bac + caa = 0 \;.\; aab = (a - b)ac$ [P·6⊃P]

·8 $aab = 0 \;.\supset.\; a = b$
[Hp . P·7 $.\supset:$ $c\varepsilon$p $.\supset.$ $(a-b)ac = 0 \;:\supset:\; a-b = 0 \;:\supset:\; a = b$]

❋ 11. $r\varepsilon 1\cdots 3 \;.\; u,u',u''\varepsilon\varphi^r \;.\; m,m'\varepsilon$q $.\supset.$ ·0 $u + u' \varepsilon \varphi^r$

·1 $u + u' = u' + u$
[$x\varepsilon$ p^{4-r} $.\supset.$ $(u+u')ax = uax + u'ax = u'ax + uax = (u'+u)ax$]

·2 $u + (u' + u'') = u + u' + u''$ ·3 $u + 0 = u$

·4 $mu \;\varepsilon\; \varphi^r$ ·5 $m(u + u') = mu + mu'$

·6 $(m + m')u = mu + m'u$ [$x\varepsilon$ p^{4-r} $.\supset.$ $[(m+m')u]ax = (m+m')(uax)$
$= m(uax) + m'(uax) = (mu)ax + (m'u)ax = (mu + m'u)ax$]

❋ 12. $r,s\varepsilon 1\cdots 3 \;.\; r+s \leqq 4 \;.\; u,u'\varepsilon\varphi^r \;.\; v,v'\varepsilon\varphi^s \;.\; m\varepsilon$q $.\supset:$

·0 $uav \;\varepsilon\; \varphi^{r+s}$ ·1 $uav = (-1)^{rs} vau$ Comma

·2 $u = u' \;.\; v = v' \;.\supset.\; uav = u'av'$
[P9·5 . $u = u'$. $x\varepsilon$p^{4-r} $.\supset.$ $uax = u'ax$ (1)
$u = u'$. $r+s<4$. $x\varepsilon$p^s . $y\varepsilon$ p^{4-r-s} $.\supset.$ $xay\;\varepsilon$ p^{4-r} . (1) $.\supset.$
$ua(xay) = u'a(xay)$. Assoc a $.\supset.$ $(uax)ay = (u'ax)ay$ (2)
$u = u'$. $r+s<4$. $x\varepsilon$p^s . (2) $.\supset:$ $y\varepsilon$p^{4-r-s} $.\supset_y.$ $(uax)ay = u'ax)ay$ (3)
$u = u'$. $r+s<4$. $x\varepsilon$p^s . (3) $.\supset.$ $uax = u'ax$ (4)
$u = u'$. $r+s\leqq 4$. $x\varepsilon$p^s . (1) . (4) $.\supset.$ $uax = u'ax$ (5)
$u = u'$. $n\varepsilon$N$_1$. $x\varepsilon$ p^s f $1\cdots n$. $m\varepsilon$ qf$1\cdots n$. $v = \Sigma(m_z x_z \,|\, z, 1\cdots n)$. (5) $.\supset.$
$uav = \Sigma(m_z\, uax_z \,|\, z, 1\cdots n) = \Sigma(m_z\, u'ax_z \,|\, z, 1\cdots n) = u'av$ (6)
$u = u'$. $v = v'$ $.\supset.$ $uav = u'av$. $vau' = v'au'$. $u'av = (-1)^{rs} v'au' =$
$(-1)^{rs} u'av'$ $.\supset.$ $uav = u'av'$]

P9·8 defini producto alterno de duo forma, per medio de repraesentatione de secundo; ce Df non es homogeneo; ergo nos debe demonstra P·2.

·3 $(u + u')av = uav + u'av \;.\; ua(v + v') = uav + uav'$

·4 $m(uav) = (mu)av = ua(mv)$ ·5 $ua0 = 0$

·6 $t\varepsilon$N$_1$. $r+s+t \leqq 4$. $w\varepsilon\varphi^t$ $.\supset.$ $(uav)aw = ua(vaw)$ Assoc$a$

❋ 13. $\mathrm{v} = \mathrm{p} - \mathrm{p} \;.\; \mathrm{v}^2 = \mathrm{v}a\mathrm{v} \;.\; \mathrm{v}^3 = \mathrm{v}a\mathrm{v}a\mathrm{v}$ Df

Relatione inter quatuor puncto $a - b = c - d$, que §vct 2·0 sume ut primitivo, resulta nunc definito ut casu particulare de aequalitate de duo formes de gradu 1. Vectore se praesenta quale summa de duo puncto cum coefficientes $+1$ et -1.

$i,j,k \varepsilon \mathrm{v} \,.\, iajak -= 0 \,.\supset.$ ·1 $\mathrm{v} = \mathrm{q}i + \mathrm{q}j + \mathrm{q}k$

·2 $x,y,z,x',y',z' \varepsilon \mathrm{q} \,.\, u = xi+yj+zk \,.\, u' = x'i+y'j+z'k \,.\supset.$
$uau' = (yz'-y'z)jak + (zx'-z'x)kai + (xy'-x'y)iaj$

·3 $\mathrm{v}^2 = \mathrm{q}(iaj) + \mathrm{q}(jak) + \mathrm{q}(kai)$

·4 $\mathrm{Hp}\cdot 2 \,.\, x'',y'',z'' \varepsilon \mathrm{q} \,.\, u'' = x''i + y'' + z''k \,.\supset.$
$uau'au'' = \mathrm{Dtrm}\begin{bmatrix}(x,y,z),\\(x',y',z'),\\(x'',y'',z'')\end{bmatrix} iajak$

·5 $\mathrm{v}^3 = \mathrm{q}\, iajak$

❋ 14·1 $a,b,c,d \varepsilon \mathrm{v} \,.\supset.\, aabacad = 0$

·2 $a \varepsilon \mathrm{v} - \iota 0 \,.\, b \varepsilon \mathrm{v}^2 \,.\, aab = 0 \,.\supset.\, \exists\, \mathrm{v} \frown c \ni (b = aac)$

·3 $a \varepsilon \mathrm{v}^2 - \iota 0 \,.\, b \varepsilon \mathrm{v}^3 \,.\supset.\, \exists\, \mathrm{v} \frown c \ni (b = aac)$

·4 $a,b \varepsilon \mathrm{v}^2 \,.\supset.\, \exists\, \mathrm{v} - \iota 0 \frown x \ni (aax = 0 \,.\, bax = 0)$

·5 $a,b \varepsilon \mathrm{v}^2 \,.\supset.\, a+b\, \varepsilon \mathrm{v}^2$

❋ 15. $a_1,a_2,a_3,a_4 \varepsilon \varphi^1 \,.\, \Psi = a_1aa_2aa_3aa_4 \,.\, \Psi -= 0 \,.\supset:$

·1 $\varphi^1 = \mathrm{q}a_1 + \mathrm{q}a_2 + \mathrm{q}a_3 + \mathrm{q}a_4$

·11 $x_1,x_2,x_3,x_4 \varepsilon \mathrm{q} \,.\, a = x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4 \,.\supset.$
$x_1 = (aa_2aa_3aa_4)/\Psi \,.\, x_2 = a_1aaaa_3aa_4/\Psi \,.\, \ldots$
[Hp . Oper $aa_2aa_3aa_4 = x_1\Psi \,.\supset.\, \mathrm{P}$]

·2 $\mathrm{Hp}\cdot 11 \,.\, x_1',x_2',x_3',x_4' \varepsilon \mathrm{q} \,.\, a' = x_1'a_1 + x_2'a_2 + x_3'a_3 + x_4'a_4 \,.\supset.$
$aa' = (x_1x'_2 - x_2x'_1)a_1aa_2 + (x_1x'_3 - x_3x'_1)a_1aa_3 + (x_1x'_4 - x_4x'_1)a_1aa_4 +$
$(x_2x'_3 - x_3x'_2)a_2aa_3 + (x_2x'_4 - x_4x'_2)a_2aa_4 + (x_3x'_4 - x_4x'_3)a_3aa_4$

·3 $\varphi_2 = \mathrm{q}a_1aa_2 + \mathrm{q}a_1aa_3 + \mathrm{q}a_1aa_4 + \mathrm{q}a_2aa_3 + \mathrm{q}a_2aa_4 + \mathrm{q}a_3aa_4$

·31 $y_{12},y_{13},y_{14},y_{23},y_{24},y_{34} \varepsilon \mathrm{q} \,.\, l = y_{12}a_1aa_2 + y_{13}a_1aa_3 + \ldots \,.\supset.\, y_{12} =$
$laa_3aa_4/\Psi \,.\, y_{13} = -laa_2aa_4/\Psi \,.\, \ldots$ [Hp . Oper $aa_3aa_4 \,.\supset.\, \mathrm{P}$]

·4 $\mathrm{Hp}\cdot 2 \,.\, x_1'',x_2'',x_3'',x_4'' \varepsilon \mathrm{q} \,.\, a' = x_1''a_1 + x_2''a_2 + x_3''a_3 + x_4''a_4 \,.\supset.$
$$aa'aa'' = \mathrm{Dtrm}[(a_2aa_3aa_4,\ -a_1aa_3aa_4,\ a_1aa_2aa_4,\ -a_1aa_2aa_3),\\ (x_1, x_2, x_3, x_4),\\ (x'_1, x'_2, x'_3, x'_4),\\ (x''_1, x''_2, x''_3, x''_4)]$$

·41 Hp·11 . Hp·31 $.\supset.\ aal = (x_2 y_{34} - x_3 y_{24} + x_4 y_{23}) a_2 a a_3 a a_4 + \ldots$

·5 $\varphi^3 = \mathrm{q} a_1 a a_2 a a_3 + \mathrm{q} a_1 a a_2 a a_4 + \mathrm{q} a_2 a a_3 a a_4$

·51 $z_1, z_2, z_3, z_4 \varepsilon \mathrm{q} \,.\, t = z_1 a_2 a a_3 a a_4 - z_2 a_1 a a_3 a a_4 + z_3 a_1 a a_2 a a_4 - z_4 a_1 a a_2 a a_3$
$.\supset.\ x_1 = a_1 a t / \Psi \,.\, x_2 = a_2 a t / \Psi \,.\, \ldots$ [Hp . Oper $a a_1$ $.\supset.$ P]

·6 Hp·4 . $x_1''', x_2''', x_3''', x_4''' \varepsilon \mathrm{q} \,.\, a''' = x_1''' a_1 + \ldots .\supset.$

$$a a' a a'' a a''' = \mathrm{Dtrm}\,[(x_1,\ x_2,\ x_3,\ x_4), (x'_1,\ x'_2,\ x'_3,\ x'_4), (x''_1, x''_2, x''_3, x''_4), (x'''_1, x'''_2, x'''_3, x'''_4)]\ \Psi$$

·61 Hp·31 . $y'_{12}, y'_{13}, y'_{14}, y'_{23}, y'_{24}, y'_{34} \varepsilon \mathrm{q} \,.\, l' = y'_{12} a_1 a a_2 + \ldots .\supset.$
$l a l' = (y_{12} y'_{34} - y_{13} y'_{24} + y_{14} y'_{23} + y_{23} y'_{14} - y_{24} y'_{13} + y_{34} y'_{12}) \Psi$

·62 Hp·31 $.\supset.\ lal = 2 (y_{12} y_{34} - y_{13} y_{24} + y_{14} y_{23}) \Psi$

·63 Hp·11 . Hp·51 $.\supset.\ aat = (x_1 z_1 + x_2 z_2 + x_3 z_3 + x_4 z_4) \Psi$

Es dato quatuor forma a_1, a_2, a_3, a_4 de gradu 1, independentes, id es, producto de que ne es nullo, tunc:

·1 Omni forma a de gradu 1 est reductibile ad forma $x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4$. Numeros $x_1 \ldots$ vocare « coordinatas » de a, ad formas a_1, a_2, a_3, a_4, que vocare « formas fundamentale ».

·11 Ce coordinatas exprimere ut ratione de p^4.

·2 Expressione de producto alternato de duo φ^1 de que nos nosce coordinatas.

·3 Omni forma de gradu 2 es reductibile ad summa de 6 producto ad 2 ad 2 de 4 forma fundamentale, multiplicato per coefficientes numerico. Ce 6 coefficiente vocare « coordinatas de forma de gradu 2 ».

P·31 exprime illos ut ratione de p^4.

·5 Omni forma de gradu 3 es summa de productos ad 3 ad 3 de 4 forma fundamentale. 4 coefficientes es « coordinatas de forma ».

Nos debe ce coordinatas generale (que plure A. tribue ad Cayley), ad Grassmann JfM. t.24 p.262-282, 372-380, et *Ausdehnungslehre* a.1844 §87: « Ich nenne die n Strecken, welche ein System n-ter Stufe bestimmen [nos habe $n=4$; Strecke $= \varphi^1$] ..., sofern jede Strecke des Systems durch sie ausgedrückt werden soll [exprime P·1], ... die Grundmasse dieses System,... die Produkte von m Grundmassen Richtmasse m-ter Stufe ».

Vide et Werke, t.1. p.195 §§117—124.

Si formas fundamentale es uno puncto et tres vectore de modulo 1, tunc de 4 coordinatas de puncto, primo es unitate, et cetero 3 es coordinatas que Descartes a.1637, defini in generale, et applica solo ad plano.

Si formas fundamentale es 4 puncto, coordinatas de puncto es barycentrico, considerato par Möbius a.1827, §26, que da expressione ·11.

6 coordinata de φ^2, considerato ut systema de fort-ia, si formas fundamentale es cartesiano, occurre in Poinsot a.1804; vocare «characteristicas de systema».

Cayley a.1860 (vide Papers t.7 p.66) reinveni 6 coordinata de φ^2.

Plücker a.1865, re-introduce coordinatas de φ^3.

✻ 16. ω

$o\varepsilon\mathrm{p} \,.\, i,j,k\varepsilon\mathrm{v} \,.\, \psi = iajak \,.\, \psi -= 0 \,.\, \Psi = oa\psi \,.\supset:$

·1 $a\varepsilon\varphi^1 \,.\supset.\, \omega a = (aa\psi)/\Psi$ Df ω

$=$ «massa de a».

·2 $a\varepsilon\varphi^2 \,.\supset.\, \omega a = i(aajak/\Psi) + j(aakai/\Psi) + k(aaiaj/\Psi)$ »

$=$ «vectore de a».

·3 $a\varepsilon\varphi^3 \,.\supset.\, \omega a = jak(aai/\Psi) + kai(aaj/\Psi) + iaj(aak/\Psi)$ »

$=$ «bivectore de a».

·4 $a\varepsilon\varphi^4 \,.\supset.\, \omega a = \psi(a/\Psi)$ »

$=$ «trivectore de a».

Operatione ω es casu particulare de «producto regressivo», theoria que non continere in Formulario. Vide meo *Calcolo geometrico*, a.1888, p.106.

Æqualitates ·1-·4 ne es homogeneo; secundo membro contine literas o,i,j,k, que ne figura in primo. Pro brevitate nos sume P·1-·4 ut Df.

$x,y,z,t,u,v\varepsilon\mathrm{q} \,.\supset.$ ·5 $\omega(uo+xi+yj+zt) = u$

·6 $\omega[x(oai)+y(oaj)+z(oak)+t(iaj)+u(jak)+v(kai)] = xi+yj+zk$

·7 $\omega[x(oaiaj)+y(oajak)+z(oakai)+t(iajak)] = x(iaj)+y(jak)+z(kai)$

·8 $r\varepsilon 1\cdots 3 \,.\, a,b\varepsilon\varphi^r \,.\supset.\, \omega(a+b) = \omega a + \omega b$ Distrib$(\omega,+)$

·9 » » $.\, m\varepsilon\mathrm{q} \,.\supset.\, \omega(ma) = m(\omega a)$ Comm$(\omega,\times m)$

✻ 17·0 $a\varepsilon\varphi^1 \,.\supset.\, \omega a\, \varepsilon\mathrm{q}$ ·1 $a\varepsilon\mathrm{p} \,.\supset.\, \omega a = 1$

·2 $a\varepsilon\mathrm{v} \,.\supset.\, \omega a = 0$ ·3 $a\varepsilon\varphi^1 \,.\, \omega a = 0 \,.\supset.\, a\varepsilon\mathrm{v}$

·4 $a\varepsilon\varphi^1 \,.\, \omega a -= 0 \,.\supset.\, a/(\omega a)\, \varepsilon\mathrm{p}$

·5 » » $.\supset.\, a\varepsilon\, \mathrm{qp}$

·6 $\varphi^1 = \mathrm{qp} \cup \mathrm{v}$

·1 Massa de puncto simplice vale 1.

·2 Massa de vectore vale 0.

·3 Omni forma de massa nullo es reductibile ad vectore.

·4 Si massa de forma a non es nullo, tunc $a/(\omega a)$ es puncto, dicto «barycentro» de a.

·5 Et a es puncto cum massa, in symbolo «qp».

·6 Omni forma de gradu 1 es puncto cum massa, aut vectore.

❋ 18·0 $a\varepsilon\varphi^2 .\supset. \omega a\, \varepsilon \mathrm{v}$

·1 $a,b\varepsilon \mathrm{p} .\supset. \omega(aab) = b - a$

·2 $a\varepsilon \mathrm{p} . b\varepsilon \mathrm{v} .\supset. \omega(aab) = b$

·3 $a,b\varepsilon\varphi^1 .\supset. \omega(aab) = (\omega a)b - (\omega b)a$

·4 $a\varepsilon\varphi^2 . \omega a = 0 .\supset. a\varepsilon \mathrm{v}^2$

·5 $a\varepsilon\varphi^2 . \omega a -= 0 . aaa = 0 .\supset. a\varepsilon\, \mathrm{p}^2$

·6 $a\varepsilon\varphi^2 .\supset: aaa = 0 .=. a\varepsilon\, \mathrm{p}^2 \cup \mathrm{v}^2 .=. a\varepsilon\varphi^1 a\varphi^1$

·7 $\varphi^2 = \mathrm{p}^2 + \mathrm{v}^2$ { POINSOT n.68 :

« Tant de forces que l'on voudra appliquées d'une manière quelconque à un corps, peuvent toujours se réduire à une seule force et à un couple unique ».

·4 Si a es forma de gradu 2, et suo vectore es nullo, tunc a es reductibile ad bivectore.

·5 Si vectore de forma a non es nullo, et producto aaa es nullo, tunc a es reductibile ad bipuncto.

·6 Si producto aaa es nullo, tunc a es reductibile ad bipuncto aut ad bivectore, id es ad producto de duo forma de gradu 1; et vice-versa.

·7 Omni forma de gradu 2 es reductibile ad summa: bipuncto plus bivectore».

❋ 19·0 $a\varepsilon\varphi^3 .\supset. \omega a\, \varepsilon \mathrm{v}^2$

·1 $a,b,c\varepsilon \mathrm{p} .\supset. \omega(aabac) = aab + bac + caa$

·2 $a,b,c\varepsilon\varphi^1 .\supset. \omega(aabac) = (\omega a)bac + (\omega b)caa + (\omega c)aab$

·3 $a\varepsilon\varphi^1 . b\varepsilon\varphi^2 .\supset. \omega(aab) = \omega(baa) = (\omega a)b + (\omega b)aa$

·4 $a\varepsilon\varphi^3 . \omega a = 0 .\supset. a\varepsilon \mathrm{v}^3$

·5 $a\varepsilon\varphi^3 . \omega a -= 0 .\supset. a\varepsilon \mathrm{p}^3$

·6 $\varphi^3 = \mathrm{p}^3 \cup \mathrm{v}^3$

·4 Si a es forma de gradu 3, et suo bivectore es nullo, tunc a es reductibile ad trivectore.

·5 Si suo bivectore non es nullo, a es reductibile ad tripuncto.

·6 Ergo omni forma de gradu 3 es tripuncto vel trivectore.

❋ 20·0 $a\varepsilon\varphi^4 .\supset. \omega a\varepsilon \mathrm{v}^3 . a/\Psi = \omega a/\psi$

·1 $a\varepsilon\varphi^1 . b\varepsilon\varphi^3 .\supset. \omega(aab) = (\omega a)b - aa(\omega b)$

·2 $a,b\varepsilon\varphi^2 .\supset. \omega(aab) = (\omega a)ab + aa(\omega b)$

·3 $a,b,c,d\varepsilon\varphi^1 .\supset. \omega(aabacad) = (\omega a)bacad - (\omega b)aacad$
$+ (\omega c)aabad - (\omega d)aabac$

✻ 21. I (indice)

$i,j,k \varepsilon \mathrm{v}$. mod$i$ = mod$j$ = mod$k$ = 1 . $i \times j = j \times k = k \times i = 0$. $\psi =$ $iajak$.⊃: ·0 I = $(i,j,k)/(jak, kai, iaj)$ Df

I, lege « indice » es transformatione lineare que ad bivectores jak, kai, iaj fac corresponde vectores i,j,k.

Poinsot, que considera v^2 ut « copula de for-tia », appella indice de bivectore « le représentant de l'axe et de la grandeur du couple ».

Grassmann a.1862 N.89 inveni ce operatione pro complexo de omni ordine, voca illo « Ergänzung = complemento », et indica per |, signo que nos deforma in I, secundo Hamilton.

Vectores unitario et orthogonale i,j,k es supposito tale que tetraedro $oa\psi$, ubi o es puncto, es positivo; p. ex., secundo digitos pollice indice et medio, id es primo secundo et tertio, de dextera (Gibbs).

·1 $a,b \varepsilon \mathrm{v}^2$. $m \varepsilon \mathrm{q}$.⊃. Ia ε v . I$(a+b)$ = Ia+Ib . I(ma) = mIa
Distrib(I,+) [DfI .⊃. I ε (vFv²)lin .⊃. P]

{ Poinsot a.1803 p.48 (n.61): « Si deux couples L et M sont représentés, pour leurs axes et pour leurs grandeurs, par les deux côtés AL et AM d'un parallélogramme ALGM, ces deux couples se composent en un seul G représenté, pour son axe et pour sa grandeur, par la diagonale AG de ce parallélogramme ». }

·2 $x,y,z \varepsilon \mathrm{q}$.⊃. I$(xjak+ykai+ziaj) = xi+yj+zk$

·3 $u,v \varepsilon \mathrm{v}$.⊃. $u \times v = (ua\mathrm{I}v)/\psi$ Dfp

·4 $a,b,c \varepsilon \mathrm{v}$.⊃. I$[aa\mathrm{I}(bac)] = (a \times c)b - (a \times b)c$

·5 $u,v \varepsilon \mathrm{v}^2$.⊃. $u \times v = (ua\mathrm{I}v)/\psi$ Df (v²|v) §vct P11-13

·51 $u,v \varepsilon \mathrm{v}^2$.⊃. $u \times v = (\mathrm{I}u) \times (\mathrm{I}v)$ Dfp

Df de producto interno de duo bivectore.

·6 mod$a$ = mod(Ia) Dfp

·7 $u \varepsilon \mathrm{v}$.⊃. Iu = ɿ v² ∩ xɜ[I$x = u$] Df
[(v²,v)|(v,v²)]P·1-·7

Si a es v², mod$a$ es suo « momento ». (Poinsot n.47).

Si a es vi (p²), et si b es p², de longore 1 (modωb = 1), tunc aab es « momento de vi a ad axi b » (Poinsot n.103).

$a,b \varepsilon \mathrm{p}^2$. modωa = modωb = 1 .⊃. aab = « momento de duo recta a et b ». (Cayley a.1865, Papers t.5 p.540).

Producto alternato et operatione I se praesenta in multitudo de quaestione de Mechanica, et es introducto per plure A., sub nomen differente.

Appell, *Traité de Mécanique rationnelle*, Paris a.1893, voca « vecteur » p² (quod produce ambiguitate cum nomenclatura plus antiquo et plus diffuso de Hamilton).

« Momento de vectore-p² a ad puncto b » $=$ I$\omega(aab)$.
« Momento relativo de duo vectore-p² a et b » $= aab$.
Distributivitate de a ad $+$ es dicto « theorema de Varignon ».
Königs, *Leçons de Cinématique*, Paris, a.1897 voca:
« Segment » $=$ p².
« Système de segments » $= \varphi^2$.
« Automoment du système a » $= aaa$.
« Vis » $=$ « screw de Ball » $= \varphi^2 \frown x \ni (\text{mod}\omega x = 1)$.
« Torseur (Ball) » $=$ « Dyname (Plücker) $= \varphi^2$.
« Complexe linéaire déterminé par a, forme de degré 2 » $= \varphi^2 \frown x \ni (aax = 0)$; illo es « système focal » de Chasles.

E. Carvallo, *Conférence sur les notions de calcul géométrique utilisées en Mécanique et en Physique*. AnnN. a.1902 p.333:
« Cycle » $=$ v².
« Flux » $=$ v³.
« Produit superficiel de deux vecteurs a et b » $= aab$.
« Produit vectoriel » $=$ I(aab).
« Produit algébrique » $= a \times b$.
« Produit de trois vecteurs a,b,c » $= aabac$.

✻ 22. $u,v,t \varepsilon$ v$-\iota 0$.⊃.

·1 $\sin(u,v) = \text{mod}(\text{U}u\, a\, \text{U}v)$ Dfp

·2 $\sin(u,v,t) = (\text{U}u\, a\, \text{U}v\, a\, \text{U}t)/\psi$ Df

« Sinu de trihedro ». Staudt, JfM. a.1842 t.24 p.255.

·3 $-1 \leqq \sin(u,v,t) \leqq +1$

·4 $(uavat)/\psi = \text{mod}u \times \text{mod}v \times \text{mod}t \times \sin(u,v,t)$

·5 $[\sin(u,v,t)]^2 =$
$1 - \cos(u,v)^2 - \cos(t,u)^2 - \cos(v,t)^2 + 2\cos(u,v)\cos(v,t)\cos(t,u)$
} Lagrange a.1799, Oeuvres t.7 p.334,341 {

✻ 23·1 $a,b,c,d \varepsilon$ v .⊃. $(aab) \times (cad) = (a \times c)(b \times d) - (a \times d)(b \times c)$

·2 $a,b \varepsilon$ vf1···3 .⊃. $[(a_1 a a_2 a a_3)/\psi] \times [(b_1 a b_2 a b_3)/\psi] =$
Dtrm$[(a_r \times b_s) | (r,s), 1 \cdots 3 : 1 \cdots 3]$

·3 $a,b \varepsilon$ vf1···4 .⊃. Dtrm$[(a_r \times b_s) | (r,s), 1 \cdots 4 : 1 \cdots 4] = 0$
} ·1-·3 Staudt a.1842 JfM. t.24 p.255 {

✻ 24·0 $r \varepsilon$ p²$-\iota 0$.⊃. rect$ar = \text{p} \frown x \ni (xar = 0)$ Df

·1 $a \varepsilon$ p . $r \varepsilon$ p²$-\iota 0$.⊃. d$(a, \text{rect}ar) = \text{mod}(\omega aar)/\text{mod}\omega r$

·2 $p \varepsilon$ p³$-\iota 0$.⊃. plan$p = \text{p} \frown x \ni (xap = 0)$ Df

·3 $a \varepsilon$ p . $p \varepsilon$ p³$-\iota 0$.⊃. d$(a, \text{plan}p) = \text{mod}\omega(aap)/\text{mod}\omega p$

✻ 25. $a\varepsilon \mathrm{v} \;.\supset.$

·0 $\text{ɪ}a = [\mathrm{I}(aax) \,|x,\ \mathrm{v} \frown x\ni(x\times a = 0)]$ Df

·1 $\mathrm{Variab}\ \text{ɪ}a = \mathrm{v}\frown x\ni(x\times a = 0)$ ·2 $(\text{ɪ}a)^2 = -a^2$

[$x\varepsilon\mathrm{v} \,.\, x\times a = 0 \;.\supset.\; (\text{ɪ}a)^2 x = (\text{ɪ}a)[\mathrm{I}(aax)] = \mathrm{I}aa[\mathrm{I}(aax)] =$
$(a\times x)a - (a\times a)x = -a^2 x$]

·0 Si a es vectore, $\text{ɪ}a$, lege « involutione de a », indica transformatione lineare scripto. Es operatione super vectores perpendiculare ad a. Suo quadrato vale numero $-a^2$. Nota que id es vero pro vectores perpendiculare ad a; nam si $\mathrm{mod}a = 1$, $[\mathrm{I}(aax)|x, \mathrm{v}]^2 = -(\mathrm{comp} \perp a)$.

·3 $\mathrm{qA} = \text{ɪ}\mathrm{v}$ ·4 $\mathrm{quaternio} = \mathrm{q} + \text{ɪ}\mathrm{v}$ Dfp

Omni involutione de forma $\text{ɪ}u$, ubi u es vectore, es producto de numero reale per substitutione indicato per A (absoluto) in §3, id es vale termine imaginario de quaternione. Relatione inter producto alternato de Grassman, et quaternione de Hamilton.

·5 $u\varepsilon\,\text{ɪ}\mathrm{v} \;.\supset.\; \mathrm{I}u = \imath\, \mathrm{v}\frown x\ni(u = \text{ɪ}x)$ Df

Si u es involutione, vel imaginario de quaternione, $\mathrm{I}u$, lege « Indice de u », es vectore que repraesenta involutione.

Hamilton, N.133 et 286, introduce l'operatione I, et suo inverso $\mathrm{I}^{-1} = \text{ɪ}$. In libro III, Hamilton tace signos I et ɪ, et identifica quaternione recto cum suo vectore. Nos ne pote seque ce conventione, contra notatione de Grassmann;, quare P·2 da ad a^2 valore opposito ad illo de §1.

✻ 26. $a,b,c\varepsilon\mathrm{v} \,.\, m,n\varepsilon\mathrm{q} \;.\supset.$

·1 $(m+\text{ɪ}a)+(n+\text{ɪ}b) = (m+n)+\text{ɪ}(a+b)$

[Hp . $w\varepsilon\mathrm{Variab}(m+\text{ɪ}a)\frown\mathrm{Variab}(n+\text{ɪ}b) \;.\supset.\; w\varepsilon\mathrm{v} \,.\, w\times a = 0 \,.\, w\times b = 0$.
$(m+\text{ɪ}a)w + (n+\text{ɪ}b)w = mw + \mathrm{I}(aaw) + nw + \mathrm{I}(baw) =$
$(m+n)w + \mathrm{I}[(a+b)aw] = [(m+n)+\text{ɪ}(a+b)]w$]

·2 $\mathrm{real}(m+\text{ɪ}a) = m \,.\, \mathrm{Imag}(m+\text{ɪ}a) = \text{ɪ}a$.
$\mathrm{K}(m+\text{ɪ}a) = m - \text{ɪ}a \,.\, \mathrm{mod}(m+\text{ɪ}a) = \sqrt{}(m^2+a^2)$

·3 $(\text{ɪ}b)(\text{ɪ}a) = -b\times a + \text{ɪ}\mathrm{I}(baa)$

·4 $(\text{ɪ}c)(\text{ɪ}b)(\text{ɪ}a) = -(b\times a)\text{ɪ}c + (c\times a)\text{ɪ}b - (c\times b)\text{ɪ}a + (aabac)/\psi$

·5 $a -= 0 \;.\supset.\; b/a = [a\times b + \text{ɪ}\mathrm{I}(aab)]/a^2$

✻ 28. posit

·1 $a\varepsilon\,\varphi^1 - \mathrm{v} \;.\supset.\; \mathrm{posit}a = a/(\omega a)$ Df

·11 $a,b\,\varepsilon\,\mathrm{v} - \iota 0 \;.\supset:\; \mathrm{posit}a = \mathrm{posit}b \;.=.\; a\varepsilon\,\mathrm{q}b$ Df

·12 $a,b\,\varepsilon\,\varphi^1 - \iota 0 \;.\supset:\;$ ThsP·11

·13 $\text{posit}`(\varphi^1\text{-v}) = \text{p}$

·2 $a \varepsilon \varphi^1 a \varphi^1\text{-}\iota 0 \;.\supset.\; \text{posit}a = \text{posit}`[\varphi^1\text{-}\iota 0 \frown x\ni(xaa = 0)]$ Df

·21 $a,b \varepsilon \varphi^1 a \varphi^1\text{-}\iota 0 \;.\supset:\; \text{posit}a = \text{posit}b \;.=.\; a \varepsilon \text{q}b$

·3 $a \varepsilon \varphi^3\text{-}\iota 0 \;.\supset.\; \text{posit}a = \text{posit}`[\varphi^1\text{-}\iota 0 \frown x\ni(xaa = 0)]$ Df

·31 $a,b \varepsilon \varphi^3\text{-}\iota 0 \;.\supset:\; \text{posit}a = \text{posit}b \;.=.\; a \varepsilon \text{q}b$

·4 $a,b \varepsilon \text{p} \,.\, a \text{-}= b \;.\supset.\; \text{posit}(aab) = \text{recta}(a,b) \cup \text{posit}(b - a)$

·41 $a,b \varepsilon \text{p} \,.\, a \text{-}= b \,.\, c \varepsilon \text{p-recta}(a,b) \;.\supset.\; \text{posit}(aabac) =$
$\text{plan}(a,b,c) \cup \text{posit}(b - a)a(c - a)$

« Posit », lege « positione », es simbolo introducto per Prof. Burali-Forti, (*Il metodo di Grassmann nella Geometria proiettiva*, Palermo R. a.1896,1897,1901,) et *Lezioni di Geometria metrico proiettiva*, Torino a.1904, p. 95) pro defini elementos de Geometria de positione ope Calculo geometrico de Grassmann.

·1 Si *a* es forma de gradu 1, non reductibile ad vectore, suo « positione » vale forma diviso per massa, id es suo barycentro.

·11 Si *a* et *b* es vectore non nullo, nos dice que illos habe idem positione, si es parallelo.

« Positione » de vectore vocare et « directione », vel « puncto ad infinito ».

·2 Si *a* es producto de duo forma de gradu 1, non nullo, suo « positione » vale positione de omni forma de gradu 1, non nullo, que habe cum *a* producto non nullo.

·21 Si *a* et *b* es producto de duo forma de gradu 1, nos dice que illos habe idem positione si *a* es producto de *b* per numero.

« Positione » de bivectore vocare et « recta ad infinito ».

·31 « Positione » de trivectore vocare et « plano ad infinito ».

VOCABULARIO IV.

§1.

332. ***Geometria,*** G γεωμετρία, A geometry, D geometrie, F géométrie, HI geometria, R geometrïja. ⊂ ge + -o- + -metria.

333. ge G γῆ, γέα = terra. ⊃ ge-o-daesia, ge-o-logia, ...
|| ? D gau, S gau-s.

334. -o- ⊃ anth-o-logia, phil-o-sophia, ellips-o-ide ...
Litera de unione in G, ut L -i- (9).

335. -metria ⊃ (332), stereo-metria, gonio-metria ...
⊂ metro — -o + -ia (143).

336. ***metro*** HI μέτρο-ν, A meter, metre, D metrum, F mètre, R metr'.
⊃ dia-metro, peri-metro, sym-metr-ico.
|| S matra. ⊂ me- + -tro (91).

337. ma-, me- (thema E), S mâ = mensura (verbo).
⊃ (secundo Van.) L ma-nu.
L ma-ter, matre, G μήτηρ, μητρί, G.dorico μάτηρ, ματρί, A mother, D mutter, R matı, mater-, S matar, matre.
L me-ti-re, A mete, D messe, R mjeritj.
L mensura (nomen), D mass, G metro, R mjera.
L mense, G μήν, D monat, S mâsa.
A moon, D mond, R mjesjas = L luna.

338. ***puncto,*** AF point, D punct, HI punto, R (non Math.) punct'.
⊂ punge — -e + -to (135). Vide nota 189.

339. ***punge*** HI. ⊃ A punge-nt. ⊂ pug- + -n- + -e.

340. pug- ⊃ L punge, pu-pug-i, pug-ile, pug-no.
|| G pyx, pyg-macho-s. || ? A fist, D faust.

341. -n-, sub forma -ne es suffixo:
li-ne — li-(to) = sine — si-(to) = cer-ne — cre-(vi).
Es «infixo» in thema de verbo, sub forma -n-:
lin-n-que — (re)-lic-(to), ju-n-ge — juga, fra-n-ge — frac-(to), pi-n-ge — pic-(to), stri-n-ge — stric-(to), sci-n-de — scis-(so), ta-n-ge — tac-to, ru-m-pe — rup-(to), la-m-be — lab-(io), te-m-pore — tepore.
|| G -ne, cri-ne = L cer-ne.
|| S -na-, ri-na-cti =|| L li-n-quit, ju-na-cti =|| L ju-n-git.

342. ***vectore,*** ADL vector, F vecteur, I vettore.
⊂ vehe — -e + -tore (138). Vide nota 164.

343. ***vehe*** ⊃ veh-iculo, vec-tore, via.
|| A wag, D wage, wege, be-wege, R ves-ti, S vah- ⊂ E veghe.
G och-o = || A wag-on = || L veh-iculo.
Nota : E gh ⊃ L h, G ch, S h, gh, AD g, R s, z.

344. ***barycentro*** (P10) (vocabulo scientifico; non existe in vocabularios.)
⊂ bary + centro. = centro de gravitate.

345. bary- G βαρύ-ς ⊃ bary-te (Chimica), bary-tono, bar-o-metro, ...
|| L grave.

346. ***grave*** AFHI. ⊃ grav-itate. || G bary- (345), S guru.

347. ***gravitate***, A gravity, D gravität, F gravité, H gravedad, I gravità
⊂ grave (316) — -e + -itate (8).

348. ***centro*** HI, G κέντρο-ν, AF centre, DL centrum, R tsentr'.
⊂ cen- (= L punge, || S çnath = neca) + -tro (91).

349. ***pondo***, ***pondere*** ⊃ ponder-a A, ponder-oso A, D pfund.
|| L pende, penso, pensa. pondere ⊂ pondo — -o + -ere (56).

350. ***distantia*** (P 21) AF distance, H distancia, I distanza, R distantsia.
⊂ distante — -e + -ia (143).

351. ***distante*** HI, AF distant. ⊂ dista + -nte (142).

352. ***dista*** HI. ⊂ di- (51) + sta (77).

§2.

353. ***recta*** (P1) H, F d-roite ⊂ di-recta, I retta. || A right.
⊂ recto — -o + -a (126, linea).

354. ***recto*** H, F d-roit, I retto. ⊃ rect-i-fica Dr.
|| A right, D recht. ⊂ rege — -e + -to (135; vide nota 189).

355. ***rege*** H, F régir, I regge. ⊃ rege-nte DR, reg-ula D.
|| G orege, D recke, S r̥g'u,

356. ***plano*** (P2) H, AF plan, plane, I piano, R plan'.
⊂ pla- + -no (160).

357. pla- ⊃ pla-no, pla-nta.
|| G pla-ty-s ⊃ F plat, I piatto, D platt.
G pla-tea FI ⊃ I piazza, AF place, D platz.
|| D fla-ch, A flat, S pra-thas, R plo-scij

358. ***componente*** (P3) DHI, A component, F composante.
⊂ compone + -nte (142).

359. ***compone*** HI, D componire. ⊂ com- (47) + pone (148).

360. ***parallelo*** I, AD parallel, F parallèle, H paralelo, R parallel̦.
⊂ G παράλληλο-ς ⊂ par- + allelo.

361. para, par- (ante vocale), παρά = contra, juxta, trans.
⊃ G para-bola ⌒ para-metro ⌒ para-grapho ⌒... || L prae (57).

362. allelo G ⊂ allo + allo (per dissimilatione de secundo elemento).
|| D ein-ander.

363. allo G, ἄλλος = || L alio. ⊃ G all-egoria, allo-tropia, par-all-axi.
Nota: G allo — L alio = G phyllo — L folio.

364. ***normale*** FI, ADFH normal, R normalı. ⊂ norma + -le (6).

365. ***norma*** HIR, D norm.
⊂ gno- (250; vide nota 249) + -ro (107) — -o + -ma (secundo V a n.).

366. -ma ⊃ nor-ma ⌒ fa-ma ⌒ ani-ma ⌒ for-ma ⌒ spu-ma ⌒ lacru-ma ⌒....
|| G -μή, gram-ma. ⊂ -mo (125) — -o + -a (37).

367. ***projectione*** (P4), AF projection, H proyeccion, I projezione, R projectsija. ⊂ projecto — -o + -ione (118).

368. ***projecto***, A project, D projekt, F projet, H proyecto, I progetto, projetto. ⊃ project-ile, project-ivo.
⊂ pro- (134) + jac- (vide nota 240) + -to (135).

369. jac-, jace, (infinito: jàcere).
⊃ ob-jec-to, ad-jec-tivo, con-jec-tura, -jec-ta = F jette, I getta
⊃ jacère (passivo de jàcere) ⊃ F gésir, H yacer, I giacère.
⊃ L jac-tura, jac-ulo, jac-ta,...

370. ***translatione*** (P5) ADFI.
⊂ trans (64) + lato (211) — -o + -ione (118).

371. ***symmetria*** (P6) G συμμετρία ADFHIR.
⊂ sym- (32) + metro (306) — -o + -ia (143).

372. ***motore*** (P9) I, LAH motor, F moteur. ⊂ mo- + -tore (138).

373. ***motu*** HI, F mouvement. ⊂ mo- + -tu (100).

374. ***move***, mo-, AH move, F mouv-oir, I muove.
⊃ mo-bile, mo-mento, mo-tu, mo-tore, mu-ta.
|| G a-mey-o-, S me = R mje-natı = L muta.
(Grassmann puta: move — mea = mone — mene).

375. ***homothetia*** (P10) (vocabulo scientifico).
⊂ homo + the- (23) + -to (135) — o- + -ia (143).

376. homo-, G ὁμό-ς. ⊃ homo-logo, homo-nymo, homo-gen-eo,..
= || L sim-ile (198).

377. ***sinu***, LADFR sinus, HI seno.
traductione in L (anno 1500) de vocabulo Arabo *g'ib* = L sinu = F pli, introducto per astronomo Al Battani a. 800, pro indica chorda plicato in duo parte.

378. ***chorda,*** G χορδή, A cord, chord, D chorde, F corde, H cuerda, I corda, R chorda.
$u, v \varepsilon$ vct .⊃. chorda $(u,v) = \mathrm{mod}(Uu - Uv)$.
Ptolemaeo a. 150 calcula chorda de arcus de 0^0 ad 180^0.

379. ***cosinu*** = complementi sinu.

380. ***coordinata*** (P14) L scientifico, ADHIR, F coordonnée.
⊂ co- (47) + ordina + -to (135) — -o + -a (126, linea).

381. ***ordina*** HI, A order, F ordonne, D ordne.
⊂ ordine (297) — -e + -a (92).

***

Ideographia repraesenta, ope pauco symbolo, innumero vocabulo commune. Me exprime aliquo vocabulo de Geometria elementare:

$u \varepsilon$ v-ι0 .⊃. qu = vectore parallelo ad u.
Qu = vectore parallelo in idem sensu de u.

$a \varepsilon$ p . $b \varepsilon$ p-ιa .⊃.
$a + Q(b-a)$ = radio de origine a et que transi per b.
$a + \theta(b-a)$ = segmento inter a et b, sine extremos.
$a + \Theta(b-a)$ = idem segmento cum extremos.

384. ***radio***, A ray, F rayon, H rayo, I raggio, LADR radius.
⊂ rad- + -io (43).

385. rad- = se dilata, extende, irradia.
⊃ rad-io, rad-ice, ramo (⊂ *rad-mo). ⊂ E vrad, S vardh, vradh.

386. ***origine***, FI, A origin, H origen. ⊂ ori- (298) + -gine.

387. -gine ⊃ ori-gine, vora-gine, lanu-gine, verti-gine.
⊂ (secundo Henry) -co (201) — -o + -ine (302).

388. ***segmento*** HI, ADFR segment. ⊂ seca — -a + -mento.
Nota: -c + m- ⊃ -gm-.

389. ***seca*** = divide. F scie, H sega, I sega, seca.
⊃ seca-nte, sec-tore, sec-tione.
|| A saw, D säge, D sehe = L vide, R sjec'ı.

390. -mento ⊃ aug-mento, mo-mento.
¦| -mate G -ματι, theore-mate, lem-mate, ... (60).
⊂ -men (251) + -to (135)
= -men; frag-mento = frag-men, medica-mento = medica-men,...

$u \varepsilon$ v . $v \varepsilon$ v-qu .⊃. $qu + qv$ = vectore coplanare cum u et v)

391. ***coplanare,*** com-planare. Vocabulo introdotto per Hamilton; A complanar, F coplanaire. ⊂ co- (47) + plano (356) — -o + -are.

392. -are = -ale (239). Suffixo -ale, post thema que contine litera *l*, sume forma -are: mi*l*it-a*r*e, popu*l*-a*r*e, *l*ine-a*r*e,..., per lege dicto « de dissimilatione ». In tardo latino existe forma: fi*l*i-a*l*e.

$a \varepsilon \mathrm{p} \,.\, u \varepsilon \mathrm{v} \,.\, v \varepsilon \mathrm{v}{-}\mathrm{q}u \,.\supset.$

$a+\mathrm{Q}u+\mathrm{Q}v =$ (angulo de vertice a et de lateres $a+\mathrm{Q}u$ et $a+\mathrm{Q}v$).
suo supplemento $= a+\mathrm{Q}u-\mathrm{Q}v$
suo opposito $= a-\mathrm{Q}u-\mathrm{Q}v$

393. ***angulo*** H, AF angle, I angolo. ⊃ tri-angul-atione DR.
|| R ugol'. ⊂ ange — -e + -ulo (224).

394. ***ange*** ⊃ (393), ang-ore, ang-usto, ang-ustia, anxio,... || D angst, enge, et G anc-ylosi, anc-yra ⊃ L ànc-ora, D angel = L hamo,... L unc-o, unc-ino = G oncino,... (V a n.).

395. ***vertice*** HI, LA vertex; || R verch', vers'ina.
⊂ verte (204) — -e + -ice.

396. -ice = vèrt-ice ∩ àp-ice ∩ ... rad-ice ∩ bisectr-ice ∩ ... ⊂ -i- (9) + -ce.

397. -ce ⊃ fero-ce, mer-ce, fa-ce, -a-ce, velo-ce.
(Secundo Grassmann, || -co N. 201).

398. ***latere,*** latus, H lado, I lato. ⊃ later-ale AD.

399. ***supplemento*** ADFHI. ⊂ sub (95) + ple (111) + -mento (390).

400. ***opposito,*** A opposite, F opposé, H oposito, I opposto.
⊃ opposit-ione DR. ⊂ ob- + posito (216).

401. ob = pro causa de, contra, ante. ⊃ ob-jecto, oc-curre.
|| G epi = epi-tome ∩ epi-graphe ∩ ..., S api, abhi, R o, ob',
A be- = be-get ∩ be-fore ..., D be-.

$a \varepsilon \mathrm{p} \,.\, u \varepsilon \mathrm{v}{-}\mathrm{q}u \,.\, w \varepsilon \mathrm{v}{-}(\mathrm{q}u+\mathrm{q}v) \,.\supset.$

$a+\mathrm{q}u+\mathrm{Q}v+\mathrm{Q}w =$ angulo dihedro.
$a+\mathrm{q}u$ es latere, A edge || L acie, F arête ⊂ L arista.
$a+\mathrm{q}u+\mathrm{Q}v$, $a+\mathrm{q}u+\mathrm{Q}w$ es facie-s.
$a+\mathrm{Q}u+\mathrm{Q}v+\mathrm{Q}w =$ angulo trihedro; a es vertice, $a+\mathrm{Q}u$, $a+\mathrm{Q}v$, ... es lateres, et $a+\mathrm{Q}u+\mathrm{Q}v$, $a+\mathrm{Q}u+\mathrm{Q}w$, ... es facies.

402. ***dihedro,*** diedro (non L, non G).
A dihedron, F dièdre. ⊂ di- (= duo, N. 114) + -hedro.

Si nos suppone ce vocabulo derivato ab vocabulo Graeco (quod non es), tunc nos debe scribe « diedro », sine *h*. Nam in G, in compositione de duo elemento, si secundo habe *h* ut initiale, ce *h* evanesce, si non seque *t*,*p*,*c*. Exemplo: an- + hydro ⊃ GL anydro. F anhydre, A anhydrous non deriva de vocabulo G, sed es composito moderno de duo elemento G.

403. -hedro ⊃ exa-hedro, tetra-hedro ... ⊂ hedra — -a + -o (182).

404. hedra G ἕδρα = L sede, facie. ⊃ ex-hedra, cat-hedra.
⊂ hed- + -ra. || L sella, D sessel, A settle, R sjedlo.

405. hed- G ἑδ-, || L sede.

406. ***sede*** (verbo), || S sad-, G hed- (365), D sitze, setze, R saditı̣. E sede.

407. -ra G -ρα = -ro (107) — -o + -a (37).

408. ***acie*** ‖ A edge, D ecke. ⊂ ace — -e + -ie.

409. ***ace*** = es acido; (in compositione) = es acuto.
⊃ ac-ido, ac-u, ac-uto, ac-re.
‖ G ac-ro-poli, ac-ro-bata, S aç, D ähre = arista, R ostro. ⊂ E ace.

410. -ie ⊃ spec-ie, ser-ie, fac-ie, progen-ie.

411. ***facie,*** AF face, H faz, I faccia. ⊃ super-ficie.
⊂ (secundo Van. et Grassmann) face — -e + -ie (371), = apparentia vel ⊂ (secundo Bréal) fac (137) + -ie, = factura.

412. ***face*** I. ⊃ fac-ula, I fiaccola, D fackel. ⊂ fa- + -ce.

413. fa- = vide-re, splende; clara, explica, loque.
⊃ fa-cie, fa-villa, fe-nestra, fo-co; fa-nte, fa-to, af-fa-bile, pro-fe-ssa.
‖ G pha-, pha-si, dia-pha-no, epi-pha-neia = L super-fi-cie, pha-e-nomeno, pho-s-phoro, pho-to: pro-phe-ta, eu-phe-mismo, pho-no.
‖ R bjelyj = candido, S bhâ = splende. ⊂ E bhâ.

414. ***parallelogrammo***, G *παραλληλόγραμμο-ν*, ADFHIR.
⊂ parallelo (330) + gramma — -a + -o (182).
Es figura $a+\theta u+\theta v$.

415. ***gramma*** G *γραμμή* = L linea, G *γράμμα* = L litera.
⊃ grammat-ico, F.intn. gramme. ⊂ graphe — -e + -ma (366).

416. graphe G ⊃ graph-ico, phono-graph-o,...
= L scribe (cum quo es ligato, secundo Van.).
‖ D kerbe = L sculpe.

417. ***parallelepipedo*** G *παραλληλεπίπεδο-ν* (Euclide l. 11 Prop. 25).
ADFHIR. ⊂ parallelo (330) — -o + epipede.
Es figura $a+\theta u+\theta v+\theta w$.

418. epipedo G = L plano. ⊂ epi + pedo.

419. epi G = L supra, = ‖ L ob. ⊃ epi-cyclo DR, epi-gramma. ‖ S api.

420. pedo G = L agro campo, solo. ‖ G podi = L pede.

421. ***orthohedro.*** Ita Mansion voca «parallelepipedo rectangulo».
⊂ ortho + hedro (363).

422. ortho G = L recto. ⊃ ortho-gon-ale, ortho-graphia, ...

$k \,\varepsilon\, \mathrm{Cls}'q \;.\supset.$

$k+\mathrm{q}u$ = cylindro, prisma, que projecta figura k secundo directione u.
$a+\mathrm{q}(k-a)$ = cono, pyramide, que projecta k de puncto a.

423. ***cylindro*** G *κύλινδρο-ς*, AD cylinder, F cylindre, HI cilindro, R tsilindr'. ⊂ cylinde — -e + -ro (107).

424. cylinde = L rota, rotula. ⊂ cyl- + -inde (= 0).

425. cyl- = rota. ⊃ G polo, L colu, R cole-so ⊃ I calesse.

426. ***prisma*** G *πρίσμα*, A prism, DHIR prisma, F prisme.
⊂ prize (= L seca, trunca) + -ma (60).

427. ***cono,*** *κῶνο-ς* A cone, F cône, HI cono, LDR conus.
⊂ co- (thema G = L cu-) + -no (160).

428. ***cuneo*** I, H cuneo, cuño, AF coin. ⊂ cu- + no (160) — -o + -eo.

429. cu- = acue. ⊃ cuneo. ‖ G co-no, A ho-ne, S ça, ça-na.

430. -eo ⊃ aur-eo ∩ ros-eo. ‖ G -eo: chrys-eo.

431. ***pyramide*** DF, A pyramid, HI piramide, R piramida.
⊂ G *πυραμίδι* ⊂ Aegyptio. Simile ad G pyr = foco, G pyro = grano.

Angulo (u,v) es recto = $(u \times v = 0)$
» » acuto = (» >0)
» » obtuso = (» <0)

432. ***acuto*** I, A acute, F aigu, H agudo. ⊂ acu + -to (135).

433. ***acu,*** F aigu-ille, H agu-ja, I ago
⊃ acu-e, acu-men, acu-leo. ⊂ ace (370) — -e + -u.

434. -u ⊃ ac-u, -tu (100), sens-u, flux-u, man-u, corn-u, sin-u, gen-u, gel-u, ... ‖ G -y = pol-y ∩ bar-y ∩ ...

Desinentia L -u in FHI coincide cum -o: L manu = I mano,...; sed mane in compositione, in vocabulos internationale:
man-u-ale, sens-u-ale, vis-u-ale, man-u-brio, flex-u-oso, luct-u-oso, corn-u-to, sin-u-oso, Gen-u-a = I Genova, ...

Vocabulos que termina in -o: campo, naso, oculo, vento, ...
produce: camp-ale, nas-ale, ocul-are, vent-oso, ...

435. ***obtuso*** H, A obtuse, F obtus, I ottuso.
⊂ ob (401) + tud- + -to (135); vide nota 174.

436. tud- ⊃ tu-n-de, con-tuso. ‖ S tud, G typ-te, D stosse ?.

437. ***sphaera*** de centro o et de radio r, rato ut superficie =
$$\mathrm{p} \frown x \ni [\mathrm{mod}(x - o) = r].$$
G *σφαῖρα*, AF sphere, D sphäre, IR sfera, H esfera.

§4.

438. ***alterno,*** altern-ato ADFHI. ⊂ altero — -o + -no (160).

439. ***altero,*** F autre, H otro, I altro. ‖ G allotero. ⊂ al- + -tero (269).

440. ***alio*** ‖ G allo (363), Gotico alja. ‖ ? S arja = socio.
⊂ al- + -io = ali + -o.

441. al-, ali (L. antiquo) ⊃ al-io, al-tero. =‖ A el-se. ‖ D elend ⊂ D antiquo ali-lanti, D El-sass.

442. ***indice*** FHI, LAD index. ⊂ in (113) + -dice.

443. -dice ⊃ in-dice, ju-dice, vin-dice. ⊂ dic.

444. ***dic, dice*** (L. antiquo) I, F di-re, H dec-ir.
⊃ dicta ADR, dictione ADR. ‖ G deic- = indica, dicê = justitia, dic-asterio, syn-dic-o, para-dig-ma. ‖ D zeihe, zeige, S diç.

445. ***positione*** ADFHIR. ⊂ posito (216) — -o + -ione (118).

IV

LIMITES

V. LIMITES.

§1 Lm lim

Vocabulo « limite » habe in Mathematica plure sensu. Idea plus simplice es indicato per l' = « limite supero » et $l_{,}$ « limite infero ». l' fac corresponde ad omni classe, uno numero determinato, finito aut infinito.

In secundo loco se praesenta classe indicato per λ = « classe limite », Λ = « limite generale, δ = « classe derivata » (pag. 139-142), que ad classe fac corresponde classe.

Nunc nos stude limite de successione et de functione.

Nos considera successione de quantitate, in numero infinito:

$$x_0\ x_1\ x_2\ x_3\ \dots\ x_n\ \dots$$

Id es, nos suppone que litera x cum uno ex indice 0, 1, 2,... n... repraesenta quantitate. Indice, scripto ad dextera et sub litera x, differ de variabile que comita x, solo per forma typographico. Ergo nos suppone que

$$x0,\ x1,\ x2,\ \dots\ xn,\ \dots$$

es quantitate; quod nos exprime per signo f de functione:

$$x \varepsilon \mathrm{qfN}_0$$

lege: « x es successione, vel serie, de quantitate ».

Nos considera valores de x respondente ad indice de m in post: $x`(m+\mathrm{N}_0)$. Suo classe limite generale $\Lambda x`(m+\mathrm{N}_0)$ varia cum m; et sume valores $\Lambda x`(m+\mathrm{N}_0)\,|m`\mathrm{N}_0$. Me voca « limes de x », et indica per « Lmx », classe parte commune ad omni classe $\Lambda x`(m+\mathrm{N}_0)$, pro $m = 0,1,2,$ ... Collecto diverso suo elemento, definitione de Lm sume forma:

❋ 1. $x \varepsilon \mathrm{qfN}_0 . \supset .$

·0 $\mathrm{Lm}x = \bigcap [\Lambda x`(m+\mathrm{N}_0)|m`\mathrm{N}_0]$ Df Lm

Si nos elimina signo $\bigcap$, vel substitue ad illo suo valore dato per definitione, P·0 fi:

·1 $\mathrm{Lm}x = a\ni[\, m\varepsilon \mathrm{N}_0 \,.\supset_m .\, a\varepsilon\, \Lambda\, x`(m+\mathrm{N}_0)\,]$ Dfp

[III. §5 P1·0 Df$\cap$.$\supset$. P]

« Classe limes de x, es classe composito ex omni objecto a tale que, si nos sume numero arbitrario (indice) m, semper a pertine ad classe limite generale de valores sumpto per x, de m in post ».

Nos pote elimina symbolo Λ, cum introductione de signo λ, l′, $\mathrm{l}_{\prime}$, que occurre in Df de Λ. Tunc definitione se decompone in tres propositione ·2·4·5 :

·2 $\mathrm{q}\frown \mathrm{Lm}x = a\ni[m\varepsilon \mathrm{N}_0 \,.\supset_m .\, a\varepsilon\, \lambda x`(m+\mathrm{N}_0)]$ Dfp

$$
\begin{array}{rlll}
[\ \text{Distrib}(\ni,\frown)\ .\supset. & \mathrm{q}\frown \mathrm{Lm}x & = a\ni[a\varepsilon \mathrm{q}\ .\ a\varepsilon\ \mathrm{Lm}x] & \\
\text{Df Lm}\ .\supset. & \text{»} & = a\ni[a\varepsilon \mathrm{q} : m\varepsilon \mathrm{N}_0 \,.\supset_m .\ a\varepsilon\ \Lambda x`(m+\mathrm{N}_0)] & \\
§\exists\ 2{\cdot}6\ .\supset. & \text{»} & = a\ni[m\varepsilon \mathrm{N}_0 \,.\supset_m .\ a\varepsilon \mathrm{q}\ .\ a\varepsilon\ \Lambda x`(m+\mathrm{N}_0)] & \\
\text{Distrib}(\varepsilon,\frown)\ .\supset. & \text{»} & \text{»} & .\ a\varepsilon\ \mathrm{q}\frown \Lambda x`(m+\mathrm{N}_0)] \\
§\Lambda\ {\cdot}2\ .\supset. & \text{»} & \text{»} & .\ a\varepsilon\ \lambda\, x`(m+\mathrm{N}_0)]\ ]
\end{array}
$$

« Limes finito de x es omni objecto (numero) a tale que, si nos sume numero m, seque, pro omni valore de m, que a es uno ex valores limite de classe de valores de x, de m in post ».

Et si nos vol elimina signo λ, nos habe definitione sequente, expresso per solo signo elementare :

·3 $a\varepsilon \mathrm{q} \,.\supset.\!\cdot\!.\ a\varepsilon\, \mathrm{Lm}x \ .=:$

$m\varepsilon \mathrm{N}_0 \,.\, h\varepsilon \mathrm{Q} \,.\supset_{m,h} .\ \exists\, (m+\mathrm{N}_0)\frown n\ni[\mathrm{mod}(x_n - a) < h]$ Dfp

$$
\begin{array}{rl}
[\ \mathrm{P}{\cdot}1\ .\supset.\!\cdot\!. & a\varepsilon\ \mathrm{Lm}x\ .=:\ m\varepsilon \mathrm{N}_0 \,.\supset_m .\ a\varepsilon\ \lambda\, x`(m+\mathrm{N}_0) \\
\text{Df}\ \lambda\ .\supset:: & \text{»}\ \ .=.\!\cdot\!.\, m\varepsilon \mathrm{N}_0 .\supset_m : h\varepsilon \mathrm{Q} .\supset_h .\exists\, x`(m+\mathrm{N}_0)\frown y\ni[\mathrm{mod}(y-a)<h] \\
\text{Import}\ .\supset.\!\cdot\!. & \text{»}\ \ .=:\ m\varepsilon \mathrm{N}_0\ .\ h\varepsilon \mathrm{Q}\ .\supset_{m,h} .\ \ \text{»}\ \ \text{»}\ \ \text{»}\ \ \text{»} \\
§`\ 1{\cdot}6\ .\supset.\!\cdot\!. & \text{»}\ \ \text{»}\ \ \text{»}\ \ \text{»}\ \ \text{»}\ \ \exists\, (m+\mathrm{N}_0)\frown n\ni[\mathrm{mod}(x_n - a)<h]\ ]
\end{array}
$$

« Si x es successione de quantitate, et si a es quantitate (finito), tunc nos dice que a es uno ex limes de successione x, quando, si nos sume ad arbitrio numero integro (indice) m, et ad arbitrio quantitate positivo h, seque pro omni valore de x ed de h, que nos pote determina aliquo indice, sequente m, et tale que differentia inter elemento correspondente in successione et quantitate a es, in valore absoluto, minore de h ».

Casu de limes $\pm\infty$ es plus simplice:

·4 $+\infty\ \varepsilon\ \mathrm{Lm}x\ .=.\ \mathrm{l}'x`\mathrm{N}_0 = \infty$ Dfp

[Df Lm $.\supset\therefore +\infty\ \varepsilon\ \mathrm{Lm}x\ .=:\ m\varepsilon\mathrm{N}_0\ .\supset_m .\ +\infty\ \varepsilon\ \Lambda x`(m+\mathrm{N}_0)$
Df $\Lambda\ .\supset\therefore$ » $.=:\ m\varepsilon\mathrm{N}_0\ .\supset_m .\ \mathrm{l}'x`(m+\mathrm{N}_0) = \infty$ (1)
$m\varepsilon\mathrm{N}_0\ .\supset:\ \mathrm{l}'x`(m+\mathrm{N}_0) = \infty\ .=.\ \mathrm{l}'x`\mathrm{N}_0 = \infty$ (2)
(1) . (2) $.\supset.$ P]

·5 $-\infty\ \varepsilon\ \mathrm{Lm}x\ .=.\ \mathrm{l}_{\prime}x`\mathrm{N}_0 = -\infty$ Dfp

« $+\infty$ es limes de successione x, quando limite supero de valores de x vale $+\infty$. Et in modo analogo pro $-\infty$ ».

❋ 2. $x\varepsilon\ \mathrm{qf}\,\mathrm{N}_0\ .\supset:$

·1 $\mathrm{l}_{\prime}\{[\mathrm{l}'x`(m+\mathrm{N}_0)]\,|m`\,\mathrm{N}_0\}\ \varepsilon\ \mathrm{Lm}x$

[Hp . $y = \mathrm{l}'x`(m+\mathrm{N}_0)|m\ .\ a = \mathrm{l}_{\prime}y`\mathrm{N}_0\ .\supset:$
$m,n\varepsilon\mathrm{N}_0\ .\supset.\ m+n+\mathrm{N}_0 \supset m+\mathrm{N}_0$
» . Oper $x`\ .\supset.\ x`(m+n+\mathrm{N}_0) \supset x`(m+\mathrm{N}_0)$
» . Oper $\mathrm{l}'\ .\supset.\ y(m+n) \leqq ym$ (1)
$m\varepsilon\mathrm{N}_0$. (1) $.\supset.\ a = \mathrm{l}_{\prime}y`(m+\mathrm{N}_0)\ .\supset.\ a\varepsilon\ \Lambda\, y`(m+\mathrm{N}_0)$ (2)
$m,n\varepsilon\mathrm{N}_0\ .\supset.\ y(m+n)\ \varepsilon\ \Lambda\, x`(m+n+\mathrm{N}_0)$ . (1) $.\supset.\ y(m+n)\ \varepsilon\ \Lambda x`(m+\mathrm{N}_0)$
$.\supset.\ y`(m+\mathrm{N}_0) \supset \Lambda\, x`(m+\mathrm{N}_0)$ (3)
$m\varepsilon\mathrm{N}_0$. (2) . (3) $.\supset.\ a\varepsilon\ \Lambda\, x`(m+\mathrm{N}_0)$ (4)
(4) . Df Lm $.\supset.\ a\varepsilon\ \mathrm{Lm}x$]

« Si in successione x de quantitates, nos considera limite supero de valores de x de m in post, illo depende de m. Limite infero de valores de limite supero præcedente es limes de x ».

·2 $\mathrm{l}_{\prime}\{[\mathrm{l}'x`(m+\mathrm{N}_0)]\,|m`\mathrm{N}_0\} = \max \mathrm{Lm}\,x$

·3 $\mathrm{l}'\{[\mathrm{l}_{\prime}$ » » » min »

·4 $\exists\ \mathrm{Lm}x$ [P2·1 $.\supset.$ P]

« Omni successione de quantitates habe limes ».

·5 $\lambda(\mathrm{q}\frown\mathrm{Lm}x) = \mathrm{q}\frown\mathrm{Lm}x$

[§λ 1·1 $.\supset.\ \mathrm{q}\frown\mathrm{Lm}x \supset \lambda(\mathrm{q}\frown\mathrm{Lm}x)$ (1)
$m\varepsilon\mathrm{N}_0\ .\ y\varepsilon\ \lambda(\mathrm{q}\frown\mathrm{Lm}x)$. P1·1 . P1·1 $.\supset.\ y\varepsilon\ \lambda\lambda\ x`(m+\mathrm{N}_0)$ (2)
» » . (2) . §λ 1·2 $.\supset.\ y\varepsilon\ \lambda\, x`(m+\mathrm{N}_0)$ (3)
(3) . Export $.\supset:\ m\varepsilon\mathrm{N}_0\ .\supset_m .\ y\varepsilon\ \lambda\, x`(m+\mathrm{N}_0)$ (4)
$y\varepsilon\ \lambda(\mathrm{q}\frown\mathrm{Lm}\,x)$. (4) . P1·0 $.\supset.\ y\varepsilon\,\mathrm{q}\frown\mathrm{Lm}\,x$ (5) (1) . (5) $.\supset.$ P]

✻ 3. $x \varepsilon \mathrm{qfN}_0 . \supset .$ ·0 $\lim x = \iota\, \mathrm{Lm} x$ Df lim

Si classe limes de successione x, que semper existe, consta ex uno solo individuo, casu multo interessante, tunc nos indica per $\lim x$, lege: limite de x, ce limes unico.

Per eliminatione de signo « Lm », Df gene propositiones ·1·2:

·1 $a \varepsilon \mathrm{q} . \supset \therefore a = \lim x .=:$

$h \varepsilon \mathrm{Q} . \supset_h . \exists\, \mathrm{N}_0 \frown m \ni [n \varepsilon m + \mathrm{N}_0 . \supset_n . \mathrm{mod}(x_n - a) < h]$ Dfp

$[\; a = \lim x :=:$

$\infty, -\infty -\varepsilon \mathrm{Lm} x . a = \mathrm{l}'[\mathrm{l}_{,} x`(m+\mathrm{N}_0) | m`\mathrm{N}_0] = \mathrm{l}_{,}[\mathrm{l}' x`(m+\mathrm{N}_0) | m`\mathrm{N}_0] :=:$

$h \varepsilon \mathrm{Q} . \supset_h . \exists\, \mathrm{N}_0 \frown m \ni [\mathrm{l}' x`(m+\mathrm{N}_0) < a+h] . \exists\, \mathrm{N}_0 \frown m \ni [\mathrm{l}_{,} x`(m+\mathrm{N}_0) > a-h] :=$

» $\exists\, \mathrm{N}_0 \frown m \ni [a+h > \mathrm{l}' x`(m+\mathrm{N}_0) \geqq \mathrm{l}_{,} x`(m+\mathrm{N}_0) > a-h] :=:$

» » $[n \varepsilon m+\mathrm{N}_0 . \supset_n . a+h > x_n > a-h] :=:$

» » » $\mathrm{mod}(x_n - a) < h] \;]$

« Si a es quantitate finito, tunc affirmatione de propositione « a vale limite de successione x », significa que, si nos sume ad arbitrio quantitate positivo h, semper nos pote determina indice m, tale que, pro omni indice n, superiore ad m, differentia inter x_n et a semper fi in valore absoluto minore de h ».

Membro definiente, vel secundo membro, contine literas h,m,n, que non occurre in membro definito, vel primo membro. Ce literas debe es apparente in secundo membro; in vero h es apparente quare figura ut indice ad primo signo $\supset$; m es apparente quare figura cum signo $\ni$; et n es apparente quare figura ut indice ad secundo signo $\supset$.

·2 $+\infty = \lim x .=: h \varepsilon \mathrm{Q} . \supset_h . \exists\, \mathrm{N}_0 \frown m \ni [n \varepsilon m + \mathrm{N}_0 . \supset_n . x_n > h]$

$-\infty$ ——————————————————— $. x_n < -h]$

« $+\infty$ es limite de successione x, quando ad omni quantitate positivo h responde aliquo indice m, de que in post, semper termine de serie supera h ». In modo analogo pro limite $-\infty$.

·3 $\lim x \;\varepsilon\; \mathrm{q} \cup \iota\infty \cup \iota{-\infty} .=. \max \mathrm{Lm} x = \min \mathrm{Lm} x$

« Conditione necessario et sufficiente, ut limite de x es quantitate finito aut infinito, es que maximo suo limes coincide cum minimo limes ».

·4 $\lim x \ \varepsilon \mathrm{q} \ .=:$

$h \varepsilon \mathrm{Q} \ . \supset_h . \ \exists \ \mathrm{N}_0 \frown m \ni [n \varepsilon \mathrm{N}_0 \ . \supset_n . \ \mathrm{mod}(x_m - x_{m+n}) < h]$

{ BOLZANO a.1817 p.35: « Wenn eine Reihe von Grössen

$F^1x, \ F^2x, \ldots \ F^nx, \ldots \ F^{n+r}x \ldots$

von der Beschaffenheit ist, dass der Unterschied zwischen ihren $n$ten Gliede F^nx und jedem späteren $F^{n+r}x$, sey dieses von jenem auch noch so weit entfernt, kleiner als jede gegebene Grösse verbleibt, wenn man n gross genug angenommen hat: so giebt es jedesmahl eine gewisse *beständige* Grösse und zwar nur *eine*, der sich die Glieder dieser Reihe immer mehr nähern ... » }

Versione: Si uno serie de gross-ia habe proprietate, que differentia inter suo membro de loco n et omni ultra,... minore que omni dato grossia mane, quando n grosso satis sumpto es; tunc existe semper uno determinato grossia, et uno solo, ad que membro de serie semper plus es prope.

·5 $\lim x \ \varepsilon \mathrm{q} \ .=:$

$h \varepsilon \mathrm{Q} \ . \supset_h . \ \exists \ \mathrm{N}_0 \frown m \ni [p,q \varepsilon m + \mathrm{N}_0 \ . \supset_{p,q} . \ \mathrm{mod}(x_p - x_q) < h]$

·4·5 Alio forma de conditione ut limite existe.

·6 $\lim x = \iota\, a \ni [u \varepsilon \mathrm{Cls}'\mathrm{N}_0 \ . \ \mathrm{l}'u = \infty \ . \supset_u . \ a \varepsilon \Lambda x`u]$ Dfp

NOTA

Idea de limite (non vocabulo), occurre in Euclide in mensura de pyramide (libro 12, P5), et in Archimede; ce limite es exprimibile per symbolo l′.

Pro functione crescente idea « lim » es reductibile ad « l′ » per P4·1.

Definitione de Wallis, a.1655 t.1 p.383, que quantitate variabile ad suo limite « continue propius accedere ita ut differentia tandem evadat quavis assignata minor; adeoque in infinitum continuata evanescet » conveni ad casu particulare de functione crescente, vel decrescente.

Definitio completo de « lim » es recente, a. 1860 circa. Vide citatione ad P42·1, et Formul. t. 4, p. 148.

Classe « Lm » occurre in Cauchy a.1821 p.30:

« ..... si l'on suppose que la variable x converge vers zéro, on aura

$$\lim\left(\left(\sin\frac{1}{x}\right)\right) = \mathrm{M}((-1, +1)),$$

attendu que l'expression $\lim\left(\left(\sin\frac{1}{x}\right)\right)$ admettra une infinité de valeurs comprises entre les valeurs extrêmes -1 et $+1$. »

Versione: $\mathrm{Lm}[\sin(1/x) \,|x, \ \mathrm{q} - \iota 0, \ 0] = (-1) \frown 1$

Vide et P22·5, P24·5.

Me defini « Lm » in RdM. a.1892 p.77, et *Sur la définition de la limite d'une fonction*, AJ. a.1894, t.17 p.38-68.

446. ***limite*** (Vide N. 253) AFHI. ⊂ lime — -e + -ite (255).

447. ***lime*** (L antiquo), ***limo*** = que es limite, es obliquo, transverso.
⊂ lic- (284) + -mo (125).

448. ***limes*** = Lm, es plurale de « lime ». ⊂ lime + -s.
Et es nominativo de « limite ».
Ita me indica idea « Lm ».

Formul. t.4, lege symbolos « Lm » et « lim » per « classe limite », et « valore limite ».

Scriptura commune $\lim_{y=x} fy$, in loco de lim(f,u,x) (P40), contine litera apparente y, et non contine litera reale u. Es symbolo incompleto.

✻ 4. const cres decr

·0 $u \, \varepsilon \, \mathrm{Cls'q} \;.\supset\therefore$

$f \varepsilon (\mathrm{qf}u)\mathrm{const} \;.=:\; f \varepsilon \mathrm{qf}u : x,y \varepsilon u \;.\supset_{x,y}.\; fx = fy$ Df

$f \varepsilon (\mathrm{qf}u)\,\mathrm{cres} \;.=:\; » \;: x,y \varepsilon u \,.\, x < y \,.\supset_{x,y}.\; fx < fy$ Df

» $\mathrm{cres}_0 \;.=:\;$ » $\leqq$ » Df

» $\mathrm{decr} \;.=:\;$ » $>$ » Df

» $\mathrm{decr}_0 \;.=:\;$ » $\geqq$ » Df

Ces P defini expressiones : « functione constante », « functione crescente », « functione crescente, quando varia », « functione decrescente », et « functione decrescente, quando varia ».

449. ***constante*** DFH, A constant, I costante. ⊂ consta + -nte (142).

450. ***consta*** HI = sta, es composito, habe per pretio.
⊃ HI costa = A cost = F coûte = D koste. ⊂ con- (47) + sta (77).

451. ***crescente*** I, A crescent, F croissante, H creciente.
⊂ cresce + -nte (142).

452. ***cresce*** I, H crece, F croî-t. ⊂ cre- + -sce.

453. ***cre-*** ⊃ cre-sce, in-cre-mento, cre-a = fac cre.
‖ G cra-, demo-cra-tia, S car.

454. -sce = cre-sce ⌢ no-sce ⌢ pa-sce ⌢ na-sce- ⌢ adole-sce-nte AFHI.
albe-sce = fi albo, albe = es albo, I fini-sce = L fini.
‖ G -sce, S -cc'a, A -sh, wa-sh, wi-sh, D -sch, for-sche, mi-sche.

455. ***decrescente*** I, A decrescent, F décroissante. ⊂ decresce + -nte.

456. ***decresce*** I, A decrease, F décroi-t, H descrece. ⊂ de + cresce.

·1 $f \varepsilon (\mathrm{qf}\,\mathrm{N}_0)\mathrm{cres}_0 \;.\supset.\; \lim f = \mathrm{l}'f`\mathrm{N}_0 \,.\, \lim f \,\varepsilon\, \mathrm{q} \cup \iota(+\infty)$

·2 » decr_0 » $\mathrm{l}_{\prime}f`\mathrm{N}_0$. » $\iota(-\infty)$

« Si f es successione crescente de quantitates, suo limite vale limite supero de suo valores ; ergo es semper determinato, finito aut infinito ».

❋ 5·1 $u\varepsilon\,(N_0 f N_0)\mathrm{sim}\ .\supset.\ \lim u = \infty$

[$a\varepsilon N_0 .\supset.\ \mathrm{Num}\ N_0 \frown n\ni(u_n\ \varepsilon\ 0\cdots a) < \mathrm{Num}\ 0\cdots a = a+1$ (1)

(1) . §max ·8 .⊃. $\max N_0 \frown n\ni(u_n \leqq a)\ \varepsilon N_0$ (2)

$a\varepsilon N_0 .\ m = \max N_0 \frown n\ni(u_n \leqq a)+1 .\ p\varepsilon\, m+N_0 .\supset.\ u_p > a .\supset.$ P]

·2 $x\varepsilon\ \mathrm{q}fN_0 .\ u\varepsilon\,(N_0 f N_0)\mathrm{rcp}\ .\supset.\ \mathrm{Lm}\ xu = \mathrm{Lm}x$

·3 » » sim .⊃. » ⊃ »

·4 Hp·3 . $\lim x\ \varepsilon\ \mathrm{q} \cup \iota\infty \cup \iota{-\infty}\ .\supset.\ \lim xu = \lim x$

❋ 6. $x,y\varepsilon\ \mathrm{q}fN_0 .\ m\varepsilon N_1 .\ a\varepsilon\mathrm{q}\ .\supset.$

·1 $\mathrm{Lm}(a+x_s)|s = a+\mathrm{Lm}x$

·2 $+\infty, -\infty\ -\varepsilon\ \mathrm{Lm}x \cup \mathrm{Lm}y\ .\supset.\ \mathrm{Lm}(x_s+y_s)|s \supset \mathrm{Lm}x + \mathrm{Lm}y$

·3 $\lim x\ \varepsilon\mathrm{q}\ .\supset.\ \mathrm{Lm}(x_s+y_s)|s = \lim x + \mathrm{Lm}y$

·4 $\lim x, \lim y\ \varepsilon\mathrm{q}\ .\supset.\ \lim(x_s+y_s)|s = \lim x + \lim y$

·5 $\lim x = \infty .\ -\infty\ -\varepsilon\ \mathrm{Lm}y\ .\supset.\ \lim(x_s+y_s)|s = \infty$

❋ 7. $x\varepsilon\ \mathrm{q}fN_0\ .\supset\therefore$

·1 $\mathrm{Lm}(-x) = -\mathrm{Lm}x$ Comm(Lm,−)

[Df Lm .⊃∴ $a\varepsilon\ \mathrm{Lm}-x .=: m\varepsilon N_0 .\supset_m.\ a\varepsilon\ \Lambda[(-x)‘(m+N_0)]$

§‘2·4 . §n2·2 .⊃∴ » , $a\varepsilon\ \Lambda-[x‘(m+N_0)]$ ⁝

Comm(Λ,−) .⊃∴ » $a\varepsilon\ -\Lambda x‘(m+N_0)]$ ⁝

Oper−‘ .⊃∴ » $-a\varepsilon\ \Lambda x‘(m+N_0)]$ ⁝

Df Lm .⊃. » .=. $-a\varepsilon\ \mathrm{Lm}x$

Oper− .⊃: » .=. $a\varepsilon\ -\mathrm{Lm}x$:⊃. Oper $a\ni$.⊃. P]

·2 $\lim x\ \varepsilon\ \mathrm{q} \cup \iota\infty \cup \iota{-\infty}\ .\supset.\ \lim -x = -\lim x$ Comm(lim,−)

❋ 8. $x,y\varepsilon\ \mathrm{q}fN_0\ .\supset.$

·1 $a\varepsilon\ \mathrm{q}-\iota 0\ .\supset.\ \mathrm{Lm}(a\times x) = a\times\mathrm{Lm}x$

·2 $\infty\ -\varepsilon\ \mathrm{Lm\ mod}x \cup \mathrm{Lm\ mod}y\ .\supset.\ \mathrm{Lm}(x_s\times y_s)|s \supset \mathrm{Lm}x\times\mathrm{Lm}y$

·3 $\lim x, \lim y\ \varepsilon\mathrm{q}\ .\supset.\ \lim(x_s\times y_s)|s = \lim x\times\lim y$

·4 $\lim x = \infty .\ 0\ -\varepsilon\ \mathrm{Lm}y\ .\supset.\ \lim(x_s\times y_s)|s = \infty$

❋ 9·0 $\lim /n\ |\ n = 0$

·1 $x\varepsilon\ \mathrm{Q}fN_0 .\ 0, \infty\ -\varepsilon\ \mathrm{Lm}x\ .\supset.\ \mathrm{Lm}/x = /\mathrm{Lm}x$

·2 $x\varepsilon\ (\mathrm{q}-\iota 0)fN_0 .\ \lim x\ \varepsilon\ \mathrm{q}-\iota 0\ .\supset.\ \lim/x = /\lim x$

·3 » » =0 » » ∞

·4 » » ∞ » » 0 Comm(lim, /)

·5 $x\varepsilon\ \mathrm{q}fN_0 .\ \lim(x_{n+1}-x_n)\ |n\ \varepsilon\ \mathrm{q} \cup \iota\infty \cup \iota(-\infty)\ .\supset.$

$\lim(x_n/n)|n = \lim(x_{n+1}-x_n)\ |\ n$ ⦃ Cauchy a.1821 p.63 ⦄

[= P20·2]

✻ 10·1 $a\varepsilon 1+\mathrm{Q} \;.\supset.\; \lim a^n \mid n = \infty$

[$n\varepsilon \mathrm{N}_1 \;.\supset.\; a^n > 1+n(a-1) : \lim\,[1+n(a-1)] \mid n = \infty \;:\supset \mathrm{P}$]

·2 $a\varepsilon\theta \;.\supset.\; \lim a^n \mid n = 0$

·3 $\mathrm{Lm}\,(-1)^n \mid n = \iota 1 \cup \iota(-1)$

Exemplo de functione cum duo limite.

·4 $x\varepsilon\, \mathrm{q\,f\,N}_0 \,.\, m\varepsilon \mathrm{N}_1 \;.\supset.\; \mathrm{Lm}\; x_s^{\,m} \mid s = (\mathrm{Lm}x)^m$

·5 » $.\; \lim x\; \varepsilon \mathrm{q} \;.\supset.\; \lim x_s^{\,m} \mid s = (\lim x)^m$

·6 $a\varepsilon 1+\mathrm{Q} \,.\, m\varepsilon \mathrm{N}_1 \;.\supset.\; \lim\,(a^n/n^m) \mid n = \infty$

[$n > m+1 \;.\; \S\mathrm{Q}\; 30{\cdot}5 \;.\supset.\; a{\upharpoonright}[n/(m+1)] > 1+n(a-1)/(m+1) >$
$n(a-1)/(m+1) \;.\; \mathrm{Oper}{\upharpoonright}(m+1) \;.\supset.\; a^n > [n(a-1)/(m+1)]^{m+1} \;.$
$\mathrm{Oper}\; /n^m \;.\supset.\; a^n/n^m > n[(a-1)/(m+1)]^{m+1}$]

✻ 11·1 $a\varepsilon\mathrm{Q} \;.\supset.\; \lim n^a \mid n = \infty \;.\; \lim n^{-a} \mid n = 0$

·2 $a\varepsilon\mathrm{Q} \;.\supset.\; \lim \sqrt[n]{a} \mid n = 1$

·3 $x\varepsilon\, \mathrm{Qf\,N}_0 \,.\, \lim x\; \varepsilon\mathrm{Q} \,.\, m\varepsilon\mathrm{q} \;.\supset.\; \lim x_s^{\,m} \mid s = (\lim x)^m$

·4 $\mathrm{Hp}{\cdot}3 \,.\, y\varepsilon\, \mathrm{q\,f\,N}_0 \,.\, \lim y\; \varepsilon\mathrm{q} \;.\supset.\; \lim\,(x_s {\upharpoonright} y_s) \mid s = (\lim x){\upharpoonright}(\lim y)$
Distrib(lim, $\upharpoonright$)

·5 $x\varepsilon\, \mathrm{QfN}_0 \,.\, \lim\,(x_{n+1}/x_n) \mid n\; \varepsilon\, \mathrm{Q}_0 \cup \iota\infty \;.\supset.$
$\lim(\sqrt[n]{x_n}) \mid n = \lim\,(x_{n+1}/x_n) \mid n$ \} Cauchy a.1821 p.63 \{

·6 $a,b\varepsilon\mathrm{Q} \;.\supset.\; \lim\,[(\sqrt[n]{a}+\sqrt[n]{b})/2]^n \mid n = \sqrt{(ab)}$

✻ 12·1 $a\varepsilon\theta \;.\supset.\; \lim\,(a{\upharpoonright})^n 1 \mid n\; \varepsilon\theta$
$—\,—\,—\,—\,—\,— = ɿ\,\mathrm{Q} \cap x \ni (a^x = x)$
\} Eisenstein JfM. a.1844 t.28 p.49 \{

·2 $a,b\varepsilon \mathrm{N}_1 \;.\supset.\; \lim[\}[b/(2a+x)] \mid x\{^r 0] \mid r = \sqrt{(a^2+b)} - a$

·3 $a\varepsilon\mathrm{Q} \,.\, n\varepsilon\, \mathrm{N}_1+1 \;.\supset.$
$\lim\}[\sqrt[n]{(a+x)} \mid x]^r\, 0\{ \mid r = ɿ\,\mathrm{Q} \cap x \ni (x^n - x - a = 0) \quad .$
$\lim\}[\sqrt[n]{(a-x)} \mid x]^r\, 0\{ \mid r = ɿ\,\mathrm{Q} \cap x \ni (x^n + x - a = 0)$
\} Joh. Bernoulli t.4 p.13:
« universaliter $\sqrt[n]{(a+\sqrt[n]{(a+\sqrt[n]{(a+\sqrt[n]{(a\; \&c.)})})})}$ pro aequatione habebitur $x^n - x - a = 0$ » \{

✻ 13·1 $a\varepsilon\mathrm{q} \;.\supset.\; \lim(a^n/n!) \mid n = 0$

·2 $\lim \sqrt[n]{(n!)} \mid n = \infty$ \} Cauchy a.1821 p.64 \{

Np ❋ 14.

·0 $\lim [\mathrm{Num}(\mathrm{Np} \cap 1 \cdots n)/n] | n = 0$ { LEGENDRE a.1797 p.464 }

·1 $\lim \{ [\max[\mathrm{Np} \cap (1+4(1\cdots n)^2)/\mathrm{N}_1]]/n \} | n = \infty$

{ TCHEBYCHEF; Vide MARKOFF a.1895 ParisCR. t.120 p.1032 }

·2 $\lim \{ \mathrm{Num}[\mathrm{Np} \cap (4\mathrm{N}_0+3) \cap (1\cdots n)] - \mathrm{Num}[\mathrm{Np} \cap (4\mathrm{N}_0+1) \cap (1\cdots n)] \} | n = \infty$

·3 lim{———————————— / ——————————————{|$n=1$

·4 lim{{———————————— — ——————————————{

$/\mathrm{Num}[\mathrm{Np} \cap 1 \cdots \mathrm{E}\sqrt{n}] \} | n = /2$

{·2-·3 TCHEBYCHEF a.1853 t.1 p.697; ·4 CESÀRO a.1896, NapoliR.}

❋ 15.

·1 $m \varepsilon \mathrm{N}_1 . \supset . \mathrm{Lm} \beta(n/m) | n = [0 \cdots (m-1)]/m$

Exemplo de functione cum m limes.

·2 $a \varepsilon \mathrm{R} . \supset . \mathrm{Lm}\, \beta(an) | n = [0 \cdots (\mathrm{dt}a - 1)]/\mathrm{dt}a$

·3 $a \varepsilon \mathrm{Q}-\mathrm{R} . \supset . \mathrm{Lm}\, \beta(an) | n = \Theta$

·4 $\mathrm{Lm}\, \beta \sqrt{} = \Theta$ $[\, a \varepsilon \theta \,.\, n \varepsilon \mathrm{N}_1 . \supset . \beta\sqrt{[(5 \times \mathrm{X}^{n-1})^2 + \mathrm{X}^n a + 1]} - a \,\varepsilon\, \theta \mathrm{X}^{-n} \,]$

Si de numero n nos extrahe radice, $\sqrt{n}$ et considera solo parte decimale $\beta\sqrt{n}$, tunc, si numero n varia, $\beta\sqrt{n}$ habe ut limes omni quantitate inter 0 et 1.

In vero, dato numero a inter 0 et 1, radice de numero scripto in demonstratione, habe n primos cifra decimale commune cum a.

·5 $f \varepsilon \mathrm{q\,f\,N}_0 . \lim f = \infty . \lim(f(n+1) - fn) | n = 0 . \supset . \mathrm{Lm}\, \beta f = \Theta$

·6 $\mathrm{Lm}\, [n - (\mathrm{E}\sqrt{n})^2] | n = \mathrm{N}_0 \cup \iota\infty$

·7 $\mathrm{Lm}[n - (\mathrm{E}\sqrt{n})^2 / [(\mathrm{E}\sqrt{n} + 1)^2 - n] \,|\, n = \mathrm{Q} \cup \iota 0 \cup \iota\infty$

Si nos considera numero n, suo differentia ab maximo quadrato in illo, et ad numero quadrato proximo superiore ad illo, tunc ratione de duo differentia habe pro limes omni quantitate positivo, et 0 et $+\infty$.

Pro valores de n: 0, 1, 2, 3 ..., functione sume valores:
0, 0, 1/2, 2/1, 0, 1/4, 2/3, 3/2, 4/1, 0, 1/6, 2/5, 3/4, 4/3, 5/2, 6/1, 0, 1/8, 2/7, 3/6, 4/5, 5/4, 6/3, 7/2, 8/1, 0, ...

❋ 16.

·1 $k \varepsilon 1+\mathrm{Q} . x \varepsilon \mathrm{q} . \supset . \mathrm{sgn} x = \lim (k^{xn} - k^{-xn})/(k^{xn} + k^{-xn}) | n$

·2 $x \varepsilon \mathrm{r} . \supset . \lim [\beta(n!x) + \beta(-n!x)] | n = \lim (\mathrm{sgn}\, \beta\, n!x) | n = 0$

·21 $x \varepsilon \mathrm{q}-\mathrm{r} . \supset .$ » » » » » $=1$

Expressione analytico de functione, que pro x rationale habe valore 0, et pro x irrationale habe valore 1. Peano a.1884, pag. XII.

Σ lim

❋ 20·1 $m \varepsilon \mathrm{Q} \,.\supset.\, \lim \Sigma(1\cdots n)^m / n^{m+1} \,|\, n = /(m+1)$

[§Σ 7·1·2 $.\supset.$ P]

·2 $u \varepsilon \mathrm{q\,f\,N_1} \,.\supset.\, \max \mathrm{Lm}[\Sigma(u, 1\cdots n)/n] \,|\, n \leqq \max \mathrm{Lm}\, u \,.$
$\quad \min \quad » \quad \geqq \min \quad »$

[$m \varepsilon \mathrm{N_1} \,.\supset.\, \mathrm{Lm}\Sigma(u, 1\cdots n)/n \mid n = \mathrm{Lm}\Sigma[u, 1\cdots(m+n)]/(m+n) \mid n$

$= \mathrm{Lm}\{\Sigma(u, 1\cdots m)/(m+n) + \Sigma[u, (m+1)\cdots(m+n)]/(m+n)\} \mid n$ (1)

$\mathrm{Lm}\Sigma(u, 1\cdots m)/(m+n) \mid n = 0$ (2)

$m \varepsilon \mathrm{N_1} \,.\, (1) \,.\, (2) \,.\supset.\, \mathrm{Lm}\Sigma(u, 1\cdots n)/n$

$= \mathrm{Lm}\Sigma[u, (m+1)\cdots(m+n)]/(m+n) \mid n$ (v)

$m, n \varepsilon \mathrm{N_1} \,.\supset.\, \Sigma[u, (m+1)\cdots(m+n)] \leqq n\,\mathrm{l}'u`(m+\mathrm{N_0})$

$\geqq n\,\mathrm{l}{,}u`(m+\mathrm{N_0})$ (4)

$(4) \,.\, m \varepsilon \mathrm{N_1} \,.\supset.$

$\max \mathrm{Lm}\Sigma[u, (m+1)\cdots(m+n)]/(m+n) \leqq \mathrm{l}'u`(m+\mathrm{N_0}) \times \lim n/(m+n) \mid n$
$\min \quad » \quad » \quad » \quad \geqq \mathrm{l}{,} \quad » \quad » \quad »$

$\mathrm{Lm}\, n/(m+n) \mid n = 1$ (5)

$(3) \,.\, (5) \,.\, m \varepsilon \mathrm{N_1} \,.\supset.\, \max \mathrm{Lm}\Sigma[u, (1\cdots n)]/n \mid n \leqq \mathrm{l}'u`(m+\mathrm{N_0})$ (6)

$(6) \,.\supset.\, \max \mathrm{Lm}\Sigma[\,\ldots\,]/n \mid n \leqq \mathrm{l}{,}u`(m+\mathrm{N_0}) \mid m`\mathrm{N_0} = \max\mathrm{Lm}\,u$

·4 $u \varepsilon \mathrm{Cls'Q} \,.\, \mathrm{Num}\, u \,\varepsilon \mathrm{N_1} \,.\supset.\, \lim [(\Sigma u^{n+1})/(\Sigma u^n)] \,|\, n = \max u$
{ D. Bernoulli PetrC. t.3 a.1728 }

❋ 21. Serie.

·0 $u \varepsilon \mathrm{q\,f\,N_0} \,.\supset.\, \Sigma(u, \mathrm{N_0}) = \lim [\Sigma(u, 0\cdots n)] \,|\, n$ Df

« Si u es serie de quantitate, tunc $\Sigma(u, \mathrm{N_0})$, que nos lege « summa de u, extenso ad omni numero $\mathrm{N_0}$ », es, per definitione, limite de summa de u, de 0 ad n, quando varia n (et tende ad infinito) ».

Summa de serie et indicare per $u_0+u_1+\ldots$, quando non es periculo de ambiguitate.

Serie es successione summando.

Serie u es dicto « convergente », si $\Sigma(u, \mathrm{N_0})\, \varepsilon \mathrm{q}$.

Si serie u es convergente, « resto » de serie u, post n termine, es differentia inter summa de serie $\Sigma(u, \mathrm{N_0})$, et summa de primos n termine: $\Sigma[u, 0\cdots(n-1)]$. Vale summa de serie $u_n+u_{n+1}+\ldots$

Functio u vocare et « termine generale » de serie.

457. ***serie*** AL series, F série, HI serie, R serija. ⊂ sere — -e + -ie (411).

458. sere = pone in serie. ⊃ ser-ie, ser-to, as-sere, as-ser-to, de-ser-to, dis-ser-tatione, in-sere, in-ser-tione D, ser-mone.
|| G eire = L necte, G hormo = L serto, monile de collo.

459. ***convergente*** FHI, AD convergent. $\subset$ converge + -nte (142).

460. ***converge*** (L scientifico) AFHI, D convergire. $\subset$ con- (47) + verge.

461. verge I. $\supset$ con-verge, di-verge || L urge, S varg'.

462. ad-verge (Leibniz) = converge. $\subset$ ad (41) + verge.

463. ***convergentia,*** AF convergence, D convergenz, I convergenza. $\subset$ converge + -ntia (144).

464. di-verge AFHI, D divergire. $\subset$ di- (51) + verge.
Habe plure sensu in differente auctore. Serie es divergente:
1⁰. Si non es convergente.
2⁰. Si $\infty = \lim \Sigma(u, 0 \cdots n)|n$.
3⁰. Si $\infty \,\varepsilon\, \mathrm{Lm}\, \Sigma(u, 0 \cdots n)|n$.

·01 $k \varepsilon \mathrm{Cls}'\mathrm{N}_0 \,.\, u \varepsilon \mathrm{qf}k \,.\supset.\, \Sigma(u,k) = \Sigma(u \min_{1+r} k|r, \mathrm{N}_0)$ Df

Si functio u es considerato in aliquo classe k de numeros, nos reduce $\Sigma(u,k)$ ad casu praecedente.

·1 $u \varepsilon \mathrm{Qf}\mathrm{N}_0 \,.\supset.\, \Sigma(u, \mathrm{N}_0) = \mathrm{l}'\Sigma(u, 0 \cdots n) \mid n`\mathrm{N}_0 \,.\, \Sigma(u, \mathrm{N}_0) \,\varepsilon\, \mathrm{Q} \cup \iota\infty$

[Hp $.\supset.$ $\Sigma(u, 0 \cdots n)|n \,\varepsilon\, (\mathrm{Qf}\mathrm{N}_0)\mathrm{cres}$. §lim 4·2 $.\supset.$ Ths]

« Si u es serie de quantitate positivo, suo summa jam considerato in Arithmetica III §15 Σ P12, es semper numero determinato finito aut infinito ».

·2 $u \varepsilon \mathrm{q} \,\mathrm{f}\, \mathrm{N}_0 \,.\, \Sigma(u, \mathrm{N}_0) \varepsilon \mathrm{q} \,.\supset.\, \lim u = 0$

[Hp . $n \varepsilon \mathrm{N}_1 \,.\supset.\, u_n = \Sigma(u, 0 \cdots n) - \Sigma(u, 0 \cdots (n-1))$ (1)

Hp . (1) $.\supset.$ $\lim u = \Sigma(u, \mathrm{N}_0) - \Sigma(u, \mathrm{N}_0) = 0$]

« In serie convergente, limite de termine generale vale 0 ».

$\lim u = 0$ es conditio necessario de convergentia, sed non sufficiente. Per ex. serie de reciprocos de numeros naturale, considerato in P23·1, habe suo termine generale que tende ad 0, et non es convergente.

·3 $u \varepsilon \mathrm{qf}\mathrm{N}_0 \,.\, \Sigma(u, \mathrm{N}_0) \,\varepsilon \mathrm{q} \,.\, h \varepsilon \mathrm{Q} \,.\, p \varepsilon \mathrm{N}_1 \,.\supset.$
$\exists\, \mathrm{N}_1 \cap m \ni \{ n \varepsilon\, m + \mathrm{N}_0 \,.\supset_n.\, \mathrm{mod}\Sigma[u, (n+1) \cdots (n+p)] < h \}$

[Hp . P·1 $.\supset.$ $\lim u = 0$ $.\supset.$ $\lim \Sigma[u, (n+1) \cdots (n+p)] = 0$ $.\supset.$ Ths]

Si u es serie convergente, tunc pro omni valore de quantitate positivo h (parvo ad arbitrio), et pro omni valore de numero p (magno ad arbitrio), semper existe indice m tale que pro omni indice $n \geqq m$, semper summa de p termine sequente termine de loco n, es in valore absoluto minore de h.

Aequivale ad P·2; exprime conditio necessario pro convergentia, sed non sufficiente. Per exemplo, in serie de reciprocos de numeros naturale, $\Sigma[/, (n+1) \cdots (n+p)] < p/n$, que fi $< h$, si $n > h/p$. Et serie non converge.

·4 $u \varepsilon \mathrm{qf}\mathrm{N}_0 \,.\supset\therefore\, \Sigma(u, \mathrm{N}_0) \,\varepsilon \mathrm{q} \,.=:\, h \varepsilon \mathrm{Q} \,.\supset_h.\, \exists\, \mathrm{N}_1 \cap m \ni \{ n \varepsilon\, m + \mathrm{N}_0 \,.$
$p \varepsilon \mathrm{N}_1 \,.\supset_{n,p}.\, \mathrm{mod}\Sigma[u, (n+1) \cdots (n+p)] < h \}$ { P3·4 $.\supset.$ P }

Conditio necessario et sufficiente pro convergentia de serie.

Differentia de P·3 et P·4 es notato in tractatos de Catalan, Mansion, Hagen; sed confusione mane hodie in aliquo tractato.

·5 $u, v \varepsilon \mathrm{q\,f\,N_0} \,.\, \Sigma(u,\mathrm{N_0}),\, \Sigma(v,\mathrm{N_0})\, \varepsilon \mathrm{q} \,.\supset.$

$\Sigma[(u_s+v_s)|s,\mathrm{N_0}] = \Sigma(u,\mathrm{N_0}) + \Sigma(v,\mathrm{N_0})$ Distrib(Σ,+)

[P·0 $.\supset.\ \Sigma[(us+vs)\,|s,\,\mathrm{N_0}] = \lim \Sigma[(us+vs)\,|s,\,0\cdots n]\,|n$

Distrib(Σ,+) $= \lim [\Sigma(u,\,0\cdots n) + \Sigma(v,\,0\cdots n)]\,|n$

Distrib(lim,+) $= \lim \Sigma(u,\,0\cdots n)\,|n + \lim \Sigma(v,\,0\cdots n)\,|n$

P·0 $= \Sigma(u,\mathrm{N_0}) + \Sigma(v,\mathrm{N_0})$]

·6 $u, a \varepsilon \mathrm{QfN_0} \,.\, \Sigma(u,\,\mathrm{N_0})\varepsilon\mathrm{Q} \,.\, \mathrm{l'}(a\text{‘}\mathrm{N_0})\varepsilon\mathrm{Q} \,.\supset.\ \Sigma(a_r u_r \mid r,\,\mathrm{N_0})\varepsilon\mathrm{Q}$

[Hp $.\supset.\ \Sigma(a_r\,u_r \mid r,\,\mathrm{N_0}) \leqq \Sigma[\mathrm{l'}(a\text{‘}\mathrm{N_0})\times u_r \mid r,\,\mathrm{N_0}] = \mathrm{l'}(a\text{‘}\mathrm{N_0})\times\Sigma(u,\,\mathrm{N_0})$]

* 22·1 $u \varepsilon \mathrm{qfN_0} \,.\, \Sigma(u,\mathrm{N_0})\,\varepsilon\mathrm{q} \,.\, a\varepsilon\mathrm{q} \,.\supset.\ \Sigma(au,\,\mathrm{N_0}) = a\Sigma(u,\mathrm{N_0})$

[Hp $.\supset.\ \Sigma(au,\mathrm{N_0}) = \lim \Sigma(au,\,0\cdots n)\,|n = \lim a\Sigma(u,\,0\cdots n)\,|n =$
$a \lim \Sigma(u,\,0\cdots n)\,|n = a\Sigma(u,\mathrm{N_0})$] Comm(Σ, $a\times$)

·2 $u, v \varepsilon \mathrm{QfN_0} \,.\, \Sigma(u,\mathrm{N_0}),\, \Sigma(v,\mathrm{N_0})\,\varepsilon\mathrm{Q} \,.\supset.$

$\Sigma\{\Sigma(u_m v_{n-m}|m,\,0\cdots n)\,|n,\,\mathrm{N_0}\} = \Sigma(u,\mathrm{N_0}) \times \Sigma(v,\mathrm{N_0})$

{ Cauchy a.1821 p.127 }

[$p\varepsilon\mathrm{N_1} \,.\supset.\ \Sigma[u,0\cdots\mathrm{E}(p/2)] \times \Sigma[v,0\cdots\mathrm{E}(p/2)] < \Sigma[\Sigma(u_m v_{n-m} \mid m,\,0\cdots n)|n,$
$0\cdots p] < \Sigma(u,\,0\cdots p) \times \Sigma(v,\,0\cdots p) \,:\supset.\ \mathrm{P}$]

Si u et v es serie de quantitate positivo, ambo convergente, tunc serie $u_0v_0+(u_0v_1+u_1v_0)+(u_0v_2+u_1v_1+u_2v_0)+\ldots$ ubi nos multiplica omni termine de primo serie pro omni termine de secundo, et collige productos que habe identico summa de indice, es convergente ad producto de summas de duo serie dato.

In vero, summa de termines de serie que resulta, de 0 ad p, es compraehenso inter productos de summas de termines de serie dato, de 0 ad p, et de 0 ad maximo integro in $p/2$. Ambo ce quantitates tende ad producto de summas de duo serie dato, quando p verge ad ∞. Ergo, quantitate compraehenso verge ad idem limite.

·3 $u, v \varepsilon \mathrm{q\,f\,N_0} \,.\, \Sigma(u,\mathrm{N_0}),\, \Sigma(v,\mathrm{N_0}),\, \Sigma[\Sigma(u_m v_{n-m}\,|m,\,0\cdots n)\,|n,\,\mathrm{N_0}]\,\varepsilon\mathrm{q}$
$.\supset.$ Ths P·2 { Abel a.1826 t.1 p.226 }

·4 $a\varepsilon\,(\mathrm{QfN_0})\,\mathrm{decr_0} \,.\, \Sigma(a,\mathrm{N_0}) = \infty \,.\, n\varepsilon\mathrm{N_1} \,.\, h\varepsilon\, 0\cdots(n-1) \,.\supset.$
$\Sigma(a,\ n\times\mathrm{N_0}+h) = \infty$

·5 $u\varepsilon\,\mathrm{QfN_0} \,.\, \Sigma(u,\mathrm{N_0})\,\varepsilon\mathrm{Q} \,.\supset.\ 0 = \min \mathrm{Lm}\ nu_n\,|n$

{ Abel t. 2, p. 199:

« pour qu'une série Σu_n soit convergente, il faut que la plus petite des imites de nu_n soit zéro ». }

·6 $u\varepsilon(\mathrm{Qf}N_0)\mathrm{decr} . \Sigma(u,N_0)\,\varepsilon Q .\supset. 0 = \lim nu_n|n$ [P·3 ⊃P]
{ Catalan, ParisCR. a.1886 }

·7 $u\varepsilon\,\mathrm{qf}N_0 . \mathrm{Lm}\,\Sigma(u,0\cdots n)|n \supset \mathrm{q} . a\varepsilon(\mathrm{Qf}N_0)\mathrm{decr} . \lim a = 0 .\supset.$
$\Sigma(au,N_0)\,\varepsilon\mathrm{q}$ { Abel t.1 p.222 }

✻ 23·0 $u\varepsilon\,\mathrm{Cls'Q} .\supset. \Sigma u =$
$= \mathrm{l'}\, x\ni\exists\,\mathrm{Cls'}u \cap v\ni(\mathrm{Num}v\ \varepsilon N_1 . x = \Sigma v)$ Df

Summa de quantitate positivo (in numero infinito) es limite supero de summas de numero finito de quantitates.

·01 $u\varepsilon\,\mathrm{Cls'Q} . \exists u .\supset. \Sigma u\ \varepsilon\ Q\cup\iota\infty$

·02 $x\varepsilon(\mathrm{Qf}N_0)\mathrm{sim} .\supset. \Sigma(x`N_0) = \Sigma(x,N_0)$

·1 $\Sigma/N_1 = \infty$ { Leibniz a.1673 MathS. t.1 p.49 }
[$n\varepsilon N_1+1 .\supset. \Sigma/[1\cdots(2^n)] > 1+n/2 :\supset.$ P]

·2 $a,b\varepsilon Q .\supset. \Sigma/(a+N_0 b) = \infty$

·3 $m\varepsilon 1+Q .\supset. \Sigma N_1^{-m}\,\varepsilon Q$

·4 » $/(m-1) < \Sigma N_1^{-m} < 1 + /(m-1)$

Stieltjes AM. a.1887 t.10 p.299, da valores de ΣN_1^{-m}, si $m\varepsilon 2\cdots70$, cum 32 cifra decimale. Vide V. §4 π P11·1.

·5 $m\varepsilon 1+Q .\supset. \Sigma(2N_0+1)^{-m} = (1-2^{-m})\Sigma N_1^{-m}$
{ Joh. Bernoulli t.4 p.11 }
[$\Sigma N_1^{-m} = \Sigma(2N_1)^{-m} + \Sigma(2N_0+1)^{-m} . \Sigma(2N_1)^{-m} = 2^{-m}\Sigma N_1^{-m} .\supset.$ P]

✻ 24. $u\varepsilon\,\mathrm{Qf}N_0 .\supset:$

·0 $h\varepsilon\theta : n\varepsilon N_0 .\supset_n. u_{n+1}/u_n < h :\supset. \Sigma(u, N_0) < u_0/(1-h)$
[Hp $.\supset: n\varepsilon N_0 .\supset. u_n < u_0 h^n :\supset. \Sigma(u,N_0) < u_0\Sigma(h^r|r, N_0) . \S\Sigma$ P12·1 $.\supset.$ P]

Si in uno serie de termine positivo, ratione de uno termine ad praecedente es semper minore de uno quantitate h minore de uno, tunc summa de serie habe valore (determinato et finito) minore de numero scripto.

Resto in serie considerato es summa de serie de idem natura; ergo es minore di $u_n/(1-h)$, si u_n es primo termine relicto.

Seque criterio de convergentia P·1. Hp de P·1 contine literas h et m que non figura in Ths. Si nos elimina h, criterio sume forma P·2. Si nos elimina m, nos habe P·3, de que P·4 es casu particulare.

·1 $h\varepsilon\theta . m\varepsilon N_1 : n\varepsilon m+N_0 .\supset_n. u_{n+1}/u_n < h :\supset. \Sigma(u,N_0)\,\varepsilon Q$
Hp . P·0 $.\supset. \Sigma(u,N_0) = \Sigma(u, 0\cdots(m-1)) + \Sigma(u,m+N_0) . \Sigma(u,m+N_0)\varepsilon Q .\supset$P

·2 $m\varepsilon N_1 . \mathrm{l'}[u(s+1)/us\,|s`(m+N_0)]\,\varepsilon\theta .\supset. \Sigma(u,N_0)\,\varepsilon Q$
[P·1 . Elim $h .\supset$P]

·3 $\max \mathrm{Lm}\, (u_{n+1}/u_n) \,|n < 1 \;.\supset.\; \Sigma(u,N_0)\, \varepsilon Q$
[P·2 . Elim m .⊃P]

·4 $\lim\, (u_{n+1}/u_n)|n \;\varepsilon\; \theta \cup \iota 0 \;.\supset.\; \Sigma(u,N_0)\, \varepsilon Q$ [P·3 ⊃ P]

·41 $\lim\, (u_{n+1}/u_n)\,|n \;\varepsilon\; 1+Q \;.\supset.\; \Sigma(u,N_0) = \infty$
[Hp .⊃. $\lim u = \infty$.⊃. Ths]
{ Cauchy a.1821 p.123:

« Si pour des valeurs croissantes de n, le rapport $\frac{u_{n+1}}{u_n}$ converge vers une limite fixe k, la série sera convergente toutes les fois que l'on aura $k<1$, et divergente toutes les fois que l'on aura $k>1$. »}

·5 $u \varepsilon\, \mathrm{QfN_0} \,.\, \max \mathrm{Lm}\, {}^n\!\sqrt{u_n}\, |n < 1 \;.\supset.\; \Sigma(u,N_0)\, \varepsilon Q$

·51 —————— ———— >1 .⊃. ——— $= \infty$
{ Cauchy a.1821 p.121:

« Cherchez la limite ou les limites vers lesquelles converge, tandis que n croît indéfiniment, l'expression $(u_n)^{\frac{1}{n}}$, et désignez par k la plus grande de ces limites, ou, en d'autres termes, la limite des plus grandes valeurs de l'expression dont il s'agit. La série sera convergente si l'on a $k<1$, et divergente si l'on a $k>1$. »}

Versione: quaere limite aut limes verso que converge, dum n cre ad infinito, expressione $u_n \upharpoonright\!/n$, et indica per k plus grande limes. Serie converge, si $k<1$, et diverge si $k>1$ ».

[Hp·5 .⊃. $\exists\, (h;m)\ni(h\varepsilon\theta \,.\, m\varepsilon N_1 : n\varepsilon\, m+N_1 \;.\supset_n.\; {}^n\!\sqrt{u_n} < h\,)$ (1)
Hp·5 . $h\varepsilon\theta \,.\, m\varepsilon N_1 : n\varepsilon\, m+N_1 \;.\supset_n.\; {}^n\!\sqrt{u_n} < h$:⊃.
$\Sigma(u, m+N_0) < \Sigma(h^n\,|n, m+N_0) = h^m/(1-h)$ (2)
(1) . (2) .⊃. P·5]
[Hp·51 . $m\varepsilon N_1$.⊃. $\exists\, (m+N_0) \cap n\ni(u_n>1)$.⊃. $\lim u -= 0$.⊃. P·51]

✻ 25·1 $u\varepsilon\, (\mathrm{QfN_0})\mathrm{decr} \,.\, \lim u = 0 \;.\supset.\; \Sigma[(-1)^n u_n\, |n, N_0]\; \varepsilon\; \theta u_0$

Si u es successione de quantitate positivo, decrescente, ad limite 0, tunc serie $u_0 - u_1 + u_2 - \ldots$ converge ad fractione de primo termine u_0. Id es:

Serie de quantitate alterno positivo et negativo, que decresce, et verge ad 0, es convergente.

{ Leibniz a.1713 MathS. t.3 p.987:

« quandocunque series constat ex membris alternatim positivis et privativis et membra ipsa decrescunt in infinitum, series est advergens »}

Resto de serie considerato, post plure termine, es serie de idem natura; suo primo termine es primo termine relicto. Ergo resto de serie considerato es fractione de primo termine relicto.

✻ 26·1 $u \varepsilon \mathrm{qf}N_0 . \Sigma(\mathrm{mod}u, N_0) \varepsilon Q .\supset. \Sigma(u,N_0) \varepsilon \mathrm{q}$

[Hp $.\supset. \Sigma[(\mathrm{mod}u+u)+(\mathrm{mod}u-u), N_0]/2 = \Sigma(\mathrm{mod}u, N_0) \varepsilon Q .\supset.$
$\Sigma(\mathrm{mod}u+u, N_0), \Sigma(\mathrm{mod}u-u, N_0) \varepsilon Q_0 .\supset.$
$\Sigma[(\mathrm{mod}u+u)-(\mathrm{mod}u-u), N_0]/2 = \Sigma(u, N_0) \varepsilon \mathrm{q}$]

{ CAUCHY a.1821 p.129 }

Si u es serie de quantitate relativo, et si serie formato per valores absoluto de u converge, tunc et serie dato converge.

In vero, serie u es differentia inter summa de suo termines positivo $(\mathrm{mod}u + u)/2$, et de summa de suo termines negativo, considerato in valore absoluto $(\mathrm{mod}u - u)/2$. Summa de ce duo serie es finito, tunc etc.

·2 $u \varepsilon \mathrm{qf}N_0 . \Sigma(\mathrm{mod}u, N_0) \varepsilon Q . v \varepsilon (N_0 \mathrm{f} N_0)\mathrm{rcp} .\supset. \Sigma(uv, N_0) = \Sigma(u,N_0)$

{ DIRICHLET JfM. a.1829 t.4 p.157 }

·3 $u \varepsilon \mathrm{qf}N_0 . \Sigma(u+\mathrm{mod}u, N_0) = \infty . \Sigma(u-\mathrm{mod}u, N_0) = -\infty .$
$\lim u = 0 . h \varepsilon \mathrm{q} \cup \iota \pm\infty .\supset. \exists (N_0 \mathrm{f} N_0)\mathrm{rcp} \cap v \ni [\Sigma(uv, N_0) = h]$

{ RIEMANN a.1854 p.221 }

✻ 27·1 $u,v \varepsilon \mathrm{qf}N_0 . \Sigma(\mathrm{mod}u, N_0), \Sigma(\mathrm{mod}v, N_0) \varepsilon Q .\supset.$
$\Sigma(u,N_0) \times \Sigma(v,N_0) = \Sigma[\, \Sigma(u_m v_{n-m} \mid m, 0\cdots n) \mid n, N_0]$

{ CAUCHY a.1821 p.132 }

[$p \varepsilon N_1 .\supset. \mathrm{mod}\{\Sigma(u, 0\cdots p) \times \Sigma(v, 0\cdots p) - \Sigma[\Sigma(u_m v_{n-m} \mid m, 0\cdots n) \mid n, 0\cdots p]\}$
$< \Sigma(\mathrm{mod}u, 0\cdots p) \times \Sigma(\mathrm{mod}v, 0\cdots p) -$
$\Sigma[\Sigma(\mathrm{mod}u_m \mathrm{mod}v_{n-m} \mid m, 0\cdots m) \mid n, 0\cdots p]$ (1)

(1) . P22·2 $.\supset. \lim\{\Sigma(n, 0\cdots p) \times \Sigma(v, 0\cdots p) -$
$\Sigma[\Sigma(n_m v_{n-m} \mid m, 0\cdots n) \mid n, 0\cdots p]\} \mid p = 0 .\supset.$ P]

Si serie u et v converge in valore absoluto, tunc lice multiplica illos cum regula P22·2.

·2 $u,v \varepsilon \mathrm{qf}N_0 . \Sigma(\mathrm{mod}u, N_0) \varepsilon Q . \Sigma(v,N_0) \varepsilon \mathrm{q} .\supset.$ ThsP·1

{ MERTENS a.1875 JfM. t.79 p.182 }

·3 $a,b \varepsilon (Q\mathrm{f}N_0)\mathrm{decr}_0 . \lim a = \lim b = 0 .\supset:$
$\Sigma[(-1)^r a_r \mid r, N_0] \times \Sigma[(-1)^r b_r \mid r, N_0] = \Sigma\{\Sigma[(-1)^n a_r b_{n-r} \mid r, 0\cdots n] \mid n, N_0\}$
$.=. \lim b_n \Sigma(a, 0\cdots n) \mid n = \lim a_n \Sigma(b, 0\cdots n) \mid n = 0$

{ PRINGSHEIM AmericanT. a.1901 p.411 }

Ths P·1 subsiste et in alio Hp. Vide Pringsheim, Encykl. IA3 p.96.

✻ 28.

·1 $u \varepsilon Q\mathrm{f}N_0 . h \varepsilon Q . \infty \mathrm{-}\varepsilon \mathrm{Lm}\, n^{1+h} u_n \mid n .\supset. \Sigma(u,N_0) \varepsilon Q$

{ CAUCHY id. }

·2 $u \varepsilon \mathrm{Qf}N_0 . \Sigma(u,N_0) = \infty . a \varepsilon Q .\supset.$
$\Sigma\{u_n/[\Sigma(u,0\cdots n)]^{1+a}|n, N_0\} \varepsilon Q$ { ABEL t.2 p.198 }

·3 $u \varepsilon \mathrm{Qf}N_0 \Sigma(u,N_0) = \infty .\supset. \Sigma\{[u_{n+1}/\Sigma(u,0\cdots n)]|n, N_0\} = \infty$
{ ABEL a.1828 t.1 p.400 }

·4 $u \varepsilon \mathrm{Qf}N_0 . \Sigma(u,N_0) = \infty .\supset. \Sigma[u_n/\Sigma(u,0\cdots n)|n, N_0] = \infty$
{ DINI a.1867 p.43 }

·5 $u \varepsilon \mathrm{qf}N_0 . \Sigma(u,N_0) = \infty .\supset.$
$\Sigma[u_{r+1}/\{\Sigma(u,0\cdots r)\times\Sigma[u,0\cdots(r+1)]\}|r, N_0] = /u_0$

·6 $u \varepsilon \mathrm{qf}N_0 . \Sigma(u,N_0) \varepsilon q .\supset.$
$\Sigma[u_{r+1}/\{\Sigma(u,0\cdots r)\times\Sigma[u,0\cdots(r+1)]\}|r, N_0] = /u_0 - /\Sigma(u,N_0)$

$[n\varepsilon N_1 .\supset. \Sigma\{u_{r+1}/[\Sigma(u,0\cdots r)\times\Sigma(u,0\cdots(r+1)]\}r, 0\cdots n\} = /u_0 - /\Sigma(u,0\cdots(n+1))]$
{ F. D'ARCAIS RdM. a.1895 t.5 p.186 }

·7 $x\varepsilon q . \mathrm{mod}x<1 .\supset. \Sigma\{x^r/[(1-x^{r+1})(1-x^{r+2})]|r, N_0\} = (1-x)^{-2}$

·71 » >1 » » $= x(1-x)^{-2}$
{ D'ARCAIS RdM. t.5, p.187 }

Exemplo de functio analytico, que sub conditiones differente, repraesenta functiones algebrico differente.

·8 $x\varepsilon q . \mathrm{mod}x<1 .\supset. \Sigma\{x\uparrow(2\uparrow r+1)/[1-x\uparrow(2r)]|r, N_1\} = /(1-x)$

·81 ——— >1 ——— $= x/(1-x)$
{ TANNERY J., DarbouxB. s.2 t.5 p.182 }

❋ 29.

·1 $x\varepsilon\theta .\supset. \Sigma[x^n/(1-x^n)|n, N_1] = \Sigma\{\mathrm{Num}(N_1 \cap n/N_1)\times x^n|n, N_1\}$
{ LAMBERT *Architechtonik* a.1771 t.2 p.507 }

·2 $x\varepsilon\theta .\supset. \Sigma[nx^n/(1-x^n)|n, N_1] = \Sigma\{\mathrm{Num}[N_1\cap n/(N_1+1)]x^n|n, N_1\}$
{ EULER PetrNC. t.5 a.1760 p.70 }

·3 $1 = /2!+2/3!+3/4!+\ldots = \Sigma[n/(n+1)!|n, N_1]$
{ Joh. BERNOULLI a.1692 t.1 p.525 }

·4 $a,b\varepsilon N_1 . D(a,b) = 1 .\supset. \Sigma /\{Np \cap (aN_0+b)\} = \infty$
{ DIRICHLET a.1837 t.1 p.313 }

❋ 31. BINOMIO DE NEWTON.

·1 $m\varepsilon q . x\varepsilon q . \mathrm{mod}x<1 .\supset. (1+x)^m = \Sigma[C(m,n)x^n|n, N_0]$

Si m es quantitate, et si x es quantitate, minore, in valore absoluto, de uno, tunc binomio $(1+x)^m$ vale summa de serie:

$1+mx+m(m-1)/2!\,x^2+\ldots+m(m-1)\ldots(m-n+1)/n!\,x^n+\ldots$

Hoc es importante theorema, ita exposito per suo inventore:
{ Newton 13 Junii a.1676:

«Sed Extractiones Radicum multum abbreviantur per hoc *Theorema*.

$$\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

ubi P+PQ significat Quantitatem cujus Radix, vel etiam Dimensio quævis, vel Radix Dimensionis, investiganda est, P primum terminum quantitatis ejus; Q, reliquos terminos divisos per primum. Et $\frac{m}{n}$ numeralem Indicem dimensionis ipsius P+PQ: Sive dimensio illa integra sit; sive (ut ita loquar) fracta; sive affirmativa, sive negativa.

Nam, sicut analystae, pro aa, aaa, &c. scribere solent a^2, a^3, &c. sic ego, pro $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{C\,a^5}$ &c. scribo $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, ...

Et sic pro $\frac{aa}{\sqrt{C : a^3+bbx}}$, scribo $aa\times\overline{a^3+bbx}|^{-\frac{1}{3}}$

... Denique, pro terminis inter operandum inventis in quoto, usurpo A, B, C, D, &c. Nempe A pro primo termino $P^{\frac{m}{n}}$; B pro secundo $\frac{m}{n}$ AQ; & sic deinceps.» }

Demonstratione.

Nos considera valore absoluto de ratione de termine de gradu $n+1$ ad praecedente. Suo limite, pro $n=\infty$, vale mod$x$, et es, per hypothesi, minore de uno:

Hp .⊃. lim mod{[C(m,n+1)x^{n+1}]/[C(m,n)x^n]} | n = lim mod[x(m—n)/n] |n
= mod$x$ < 1 (1)

Tunc, per theorema «si in uno serie ad terminos positivo, ratione de uno termine ad praecedente habe limite minore de uno, serie es convergente», nos deduce que serie formato per valores absoluto de terminos de serie dato es convergente:

Hp . (1) . P25·4 .⊃. Σ[modC(m,n)x^n |n, N_0] εQ (2)

Et per theorema «si serie formato per valore absoluto de termines de serie dato es convergente, tunc et serie dato es convergente», seque convergentia de serie binomiale:

Hp . (2) . P26·1 .⊃. Σ[C(m,n)x^n |n, N_0] εq (3)

Summa de serie binomiale depende de valore de m. Nos pone

$$fm = \Sigma[C(m,n)x_n \,|n, N_0]$$

vel nos voca f expressione considerato, in quo varia m; tunc pro omni valore de m, fm es quantitate (determinato et finito); et $f1 = 1+x$:

xεq . mod$x$ < 1 . f= Σ[C(m,n)x^n |n, N_0]|m . (3) .⊃. fε qfq . f1=1+x (4)

Nos conserva ad f valore dato per (4); si m et n es quantitate, tunc lice multiplica duo serie fm et fn, ambo convergente in valore absoluto, cum regula P27·1. Theorema de Vandermonde dice que coefficiente de x^r vale C$(m+n,r)$, unde producto vale $f(m+n)$:

Hp(4) . $m,n\varepsilon$q . P27·1 . §Σ 15·4 .Ɔ.
$$(fm)\times(fn) = \Sigma\{\Sigma[C(m,r)C(n,s-r)x^s \mid r, 0\cdots s] \mid s, N_0\}$$
$$= \Sigma[C(m+n, s)x^s \mid s, N_0]$$
$$= f(m+n) \qquad (5)$$

Si m varia inter 0 et 1, et x es positivo, serie, de secundo termine in post, habe termine de signo alterno, decrescente in modo continuo et indefinito ; ergo summa de serie es minore de $1+mx$:

$x\varepsilon$Q . $m\varepsilon\theta$. P25·1 .Ɔ. $fm < 1+mx$ (6)

Et si x es negativo, summa de serie es minore de 1 :

$x\varepsilon -$Q . $m\varepsilon\theta$.Ɔ. $fm < 1$ (7)

In omni casu, limite supero de valores de f in intervallo de 0 ad 1 es finito :

(6) . (7) .Ɔ. l'f·θ εq (8)

Ergo functione f satisfac ad relatione $f(m+n) = (fm)\times(fn)$, pro omni valore de m et n. De theorema noto super ce proprietate functionale nos deduce que, pro omni valore de m, es $fm = (f1)\uparrow m$:

Hp(4) . (5) . (8) . §q P12·4 . $m\varepsilon$q .Ɔ. $fm = (1+x)^m$ (9)

Demonstratione de Euler, a.1774 PetrNC. t.19 p.109.

Suo expressione in symbolo es dato per formula (1) (2) (3) (4) (5) ... (9)

·2 $m\varepsilon -1+$Q .Ɔ. $2^m = \Sigma[C(m,n) \mid n, N_0]$ {Abel a.1826 t.1 p.245}

·3 $m\varepsilon$Q .Ɔ. $0 = \Sigma[(-1)^n C(m,n) \mid n, N_0]$

·4 Hp·1 .Ɔ. $(1-x)^{-m} = \Sigma[C(m+n-1, n)x^n \mid n, N_0]$
[$(-m,-x)|(m,x)$P·1 .Ɔ. P]

·5 $m\varepsilon$q . $x\varepsilon$ Q$-$/2 .Ɔ. $(1+x)^m = \Sigma\{C(m+n,n)[x/(1+x)]^n \mid n, N_0\}$

·6 $m\varepsilon$q . $x\varepsilon$Q . $n\varepsilon N_1$. $n>m$.Ɔ.
$$(1+x)^m - \Sigma[C(m,r)x^r \mid r, 0\cdots n] \;\varepsilon\; \theta C(m,n+1)x^{n+1}$$

✻ 32·0 $u\varepsilon$ qf N_0 . $\Sigma(u,N_0)\varepsilon$q .Ɔ.
$$\Sigma(u,N_0) = \Sigma\{\Sigma[C(n,r)u_r \mid r,0\cdots n]/2^{n+1} \mid n,N_0\}$$

✻ 35. Producto infinito.

·0 $u\varepsilon$ qfN_0 .Ɔ. $\Pi(u,N_0) = u_0u_1 \ldots = \lim \Pi(u, 0\cdots n) \mid n$ Df

·1-·3 $(\Pi \mid \Sigma)$ P21·1-·3

Producto considerato es infinito per forma. In valore pote es finito, tum vocare « convergente ».

✻ 36·0 $u\varepsilon\, \mathrm{Qf}N_0 \,.\, \Pi(u,N_0)\, \varepsilon Q \;.\supset.\; \lim u = 1$

[Hp . $n\varepsilon N_1$ $.\supset.$ $u_n = \Pi(u, 0\cdots n)/\Pi[u, 0\cdots(n-1)]$ (1)

Hp . (1) $.\supset.$ $\lim u = \Pi(u,N_0)/\Pi(u,N_0) = 1$]

·01 $u\varepsilon\, \mathrm{Qf}N_0 \;.\supset.\; \Pi(1+u, N_0)\, \varepsilon\, Q \cup \iota\infty$

[Hp $.\supset.$ $\Pi(u, 0\cdots n)\, |n\, \varepsilon\, (\mathrm{Qf}N_0)$cres . P4·2 $.\supset.$ Ths]

·02 $u\varepsilon\, \theta\mathrm{f}N_0 \;.\supset.\; \Pi(1-u, N_0)\, \varepsilon\, Q \cup \iota 0$

·1 $u\varepsilon\, \mathrm{Qf}N_0 \,.\, \Sigma(u,N_0) = \infty \;.\supset.\; \Pi(1+u, N_0) = \infty$

[Hp . §Π 4·1 . $n\varepsilon N_1+1$ $.\supset.$ $\Pi(1+u, 0\cdots n) > 1+\Sigma(u, 0\cdots n)$. P·01 $.\supset.$ P]

·11 $u\varepsilon\, \mathrm{Qf}N_0 \,.\, \Pi(1+u, N_0)\, \varepsilon Q \;.\supset.\; \Sigma(u,N_0)\, \varepsilon Q$ [P·1 $.\supset.$ P]

·2 $u\varepsilon\, \theta\mathrm{f}N_0 \,.\, \Sigma(u,N_0)\, \varepsilon Q \;.\supset.\; \Pi(1-u, N_0)\, \varepsilon Q$

[Hp $.\supset.$ $\exists\, N_0 \cap m\ni[\Sigma(u, m+N_0)\, \varepsilon\theta]$ (1)

Hp . $m\varepsilon N_0$. $\Sigma(u, m+N_0)\, \varepsilon\theta$. §Π 4·2 $.\supset.$ $\Pi(1-u, m+N_0) > 1-\Sigma(u, m+N_0)$ (2)

» » . (2) . P ·02 $.\supset.$ $\Pi(1-u, m+N_0)\, \varepsilon Q$ (3)

Hp . (3) $.\supset.$ Ths]

·21 $u\varepsilon\, \theta\mathrm{f}N_0 \,.\, \Pi(1-u, N_0) = 0 \;.\supset.\; \Sigma(u,N_0) = \infty$ [P·2 $.\supset.$ P]

·3 $u\varepsilon\, \mathrm{Qf}N_0 \,.\, \Sigma(u,N_0)\, \varepsilon Q \;.\supset.\; \Pi(1+u, N_0)\, \varepsilon Q$

[$n\varepsilon N_0$. Hp $.\supset.$ $\Pi(1+u, 0\cdots n) = /\Pi[1-u/(1+u), 0\cdots n]$ (1)

Hp $.\supset.$ $u/(1+u)\, \varepsilon\, \theta\mathrm{f}N_0$. $\Sigma[u/(1+u), N_0]\, \varepsilon Q$ (2)

Hp . (2) . P·2 $.\supset.$ $\Pi[1-u/(1+u), N_0]\, \varepsilon Q$. (1) $.\supset.$ Ths]

·4 $u\varepsilon\, \theta\mathrm{f}N_0 \,.\, \Pi(1-u, N_0)\, \varepsilon Q \;.\supset.\; \Sigma(u,N_0)\, \varepsilon Q$

[Hp . $n\varepsilon N_0$ $.\supset.$ $\Pi(1-u, 0\cdots n) = /\Pi[1+u/(1-u), 0\cdots n]$ (1)

Hp $.\supset.$ $u/(1-u)\, \varepsilon\mathrm{Qf}N_0$. $\Sigma[u/(1-u), N_0]\, \varepsilon Q$ (2)

(2) . P·3 $.\supset.$ $\Pi[1+u/(1-u), N_0]\, \varepsilon Q$. (1) $.\supset.$ Ths]

·41 $u\varepsilon\, \theta\mathrm{f}N_0 \,.\, \Sigma(u,N_0) = \infty \;.\supset.\; \Pi(1-u, N_0) = 0$ [P·4 $\supset$ P]

·5 $u\varepsilon\, \mathrm{Qf}N_0 \;.\supset:\; \Pi(1+u, N_0)\, \varepsilon Q \,.=.\, \Sigma(u,N_0)\, \varepsilon Q$ [P·11·3 $\supset$ P]

·6 $u\varepsilon\, \theta\mathrm{f}N_0 \;.\supset:\; \Pi(1-u, N_0)\, \varepsilon Q \,.=.\, \Sigma(u,N_0)\, \varepsilon Q$ [P·2·4 $\supset$ P]

·61 ———— $\Pi(1-u, N_0) = 0 \,.=.\, \Sigma(u,N_0) = \infty$ [P·21·41 $.\supset$ P]

·7 $u\varepsilon\, (-1+Q)\mathrm{f}N_0 \,.\, \Sigma(u,N_0),\, \Sigma(u^2,N_0)\, \varepsilon Q \;.\supset.\; \Pi(1+u, N_0)\, \varepsilon Q$

·71 ———————— . $\Sigma(u,N_0)\, \varepsilon Q \,.\, \Sigma(u^2,N_0) = \infty \;.\supset.\; \Pi(1+u, N_0) = 0$

{ ·1-·71 Cauchy a.1821 p.460 }

Dm. Weierstrass a.1856 t.1 p.173.

Pringsheim MA. a.1889 t.33 p.563.

✻ 37.

·1 $a,b\varepsilon\mathrm{q}\text{-}(-N_0) \,.\, a<b \;.\supset.\; \Pi[(a+r)/(b+r)\, |r, N_0] = 0$

·2 » $a>b$ » » » » $= \infty$

·3 $x \varepsilon \mathrm{q} \,.\, \mathrm{mod}x < 1 \,.\supset.\, /(1-x) = (1+x)(1+x^2)(1+x^4)(1+x^8) \ldots$
$= \Pi\{[1+x\uparrow(2n)] \,|\, n, \mathrm{N}_0\}$ \{ Euler a.1748 p.273 \}
[$x\varepsilon\mathrm{q} \,.\, n\varepsilon\mathrm{N}_1 \,.\supset.\, 1-x\uparrow(2^{n+1}) = (1-x)\Pi[1+x\uparrow(2^r) \,|\, r, 0\cdots n] \,.\supset.\, \mathrm{P}$]

·4 $(1-/4)(1-/9)(1-/16)\ldots = \Pi[(1-n^{-2}) \,|\, n, \mathrm{N}_1+1] = /2$
[$n\varepsilon\mathrm{N}_1 \,.\supset.\, \Pi[(1-n^{-2}) \,|\, n, 2\cdots 2n] = (2n+1)/(4n) \,.\supset.\, \mathrm{P}$]

·5 $p\varepsilon\, 1+\mathrm{N}_1 \,.\supset.\, \Pi[(1-p^{-n}) \,|\, n, \mathrm{N}_1]\, \varepsilon\, \mathrm{Q}-\mathrm{R}$
\{ Eisenstein JfM. a.1844 t.28 p.39 \}

·6 $m\varepsilon\, 1+\mathrm{Q}_0 \,.\supset.\, \Sigma \mathrm{N}_1^{-m} = \Pi[/(1-n^{-m}) \,|\, n, \mathrm{Np}]$
\{ Euler a.1744 PetrC. t.9 p.172 ; a.1748 p.225 \}

·7 $a\varepsilon\theta \,.\supset.\, \Pi[(1+a^n)|n,\mathrm{N}_1] \times \Pi[(1-a^n)|n, 2\mathrm{N}_0+1] = 1$

❋ 38. $x,y\varepsilon\mathrm{q} \,.\, \mathrm{mod}x < 1 \,.\supset.$
$\Pi[(1+x^n y)|n, \mathrm{N}_1] = 1+\Sigma\{x\uparrow[n(n+1)/2] \,/\Pi[(1-x^r)|r, 1\cdots n]\, y^n \,|n, \mathrm{N}_1\}.$
$/\Pi[(1-$ » » » $\Sigma\{$ x^n / » » » .
$\Pi[(1-x^n)|n, \mathrm{N}_1] = 1+\Sigma[(-1)^n\{x\uparrow[n(3n-1)/2]+x\uparrow[n(3n+1)/2]\}|n, \mathrm{N}_1]$
\{ Euler a.1748 t.1 p. 259-270 \}
$\Pi[(1-x^n)/(1+x^n)|n, \mathrm{N}_1] = 1-2\Sigma[(-)^n x\uparrow n^2|n, \mathrm{N}_1]$
$\Pi[(1-x^{2n})/(1-x^{2n-1})|n, \mathrm{N}_1] = 1+\Sigma\{x\uparrow[n(n+1)/2]|n, \mathrm{N}_1\}$
$\Pi[(1-x^n)^3|n, \mathrm{N}_1\} = 1+\Sigma\{(-)^n(2n+1)x\uparrow[n(n+1)/2]|n, \mathrm{N}_1\}$
\{ Jacobi, *Fundamenta* §66, Werke t.1 p.237 \}

❋ 40. LIMES ET LIMITE DE FUNCTIONE.

$u\varepsilon\, \mathrm{Cls'q} \,.\, x\varepsilon\, \delta u \,.\, f\varepsilon\, \mathrm{q}fu \,.\supset.$

·0 $\mathrm{Lm}(f,u,x) = a\,\ni\{v\varepsilon\,\mathrm{Cls'q} \,.\, x-\varepsilon\,\delta v \,.\supset_v.\, a\varepsilon\, \Lambda f`(u-v-\iota x)\}$ Df

u es classe de quantitates; x es elemento finito aut infinito de classe derivata des u; f es quantitate functio des u.

Tunc signo Lm (f,u,x), lege « limes de functio f, quando variabile varia in u, et tende ad x » indica omni elemento a tale que, si v es classe que non habe x ut elemento de classe derivata, semper a es limite generale de classe de valores de f in classe des u, non v ».

Si nos elimina signo Λ, definitio sume plure forma respondente ad differente casu particulare:

·1 $a,x\varepsilon\mathrm{q} \,.\supset\therefore\, a\varepsilon\, \mathrm{Lm}(f,u,x) \,.=:$
$h,k\varepsilon\mathrm{Q} \,.\supset_{h,k}.\, \exists\, u-\iota x \cap y\ni[\mathrm{mod}(y-x)<h \,.\, \mathrm{mod}(fy-a)<k]$ Dfp

« Si a et x es finito, tunc a es limes de functio f, pro variabile in classe u, tendente ad x, quando, si nos sume ad arbitrio duo quantitate positivo h et k, semper nos pote determina elemento y, pertinente ad classe u, diverso de x, differente de x de quantitate in valore absoluto minore de h, et que redde differentia $fy-a$ minore in valore absoluto de k ».

·2 $+\infty \varepsilon \mathrm{Lm}(f,u,x) .=:$
$h,k\varepsilon \mathrm{Q} .\supset_{h,k}. \exists\, u \frown y \ni [\mathrm{mod}(y-x) < h \,.\, fy > k]$ Dfp

·3 $+\infty \varepsilon \mathrm{Lm}(f,u,x) .=:$
$h\varepsilon \mathrm{Q} .\supset_h. \mathrm{l}' f\text{‘}\{u \frown y \ni [\mathrm{mod}(y-x) < h]\} = \infty$ Dfp
[Df Lm $.\supset.$ P·1·2·3]

Nota homogeneitate de omni definitione possibile de signo Lm. In P40·0, membro definiente contine literas variabile a,v,x,f,u, Signos ε,q,Cls',Δ,Λ es symbolo constante. Litera a es apparente, quare praecede signo $\ni$. Litera v es apparente, quare es indice ad signo $\supset$. Ergo secundo membro contine literas reale x,f,u, et nos pote repraesenta illo per membro definito, scripto sub forma de functione de f,u,x.

In P40·1 secundo membro contine literas apparente h,k,y, et literas reale u,x,f,a, que occurre in primo membro.

·31 $a,b\varepsilon \mathrm{q} \,.\, a -\!= b \,.\, f\varepsilon\, \mathrm{qf}a^{-}b .\supset.$
$\mathrm{Lm}(f, a^{-}b, a) = \bigcap [\Lambda f\text{‘}(x^{-}a) \mid x\text{‘}\, b^{-}a]$

·4 $\exists\, \mathrm{Lm}(f,u,x)$

« Omni functione semper habe limes ».

·5 $v\varepsilon \mathrm{Cls}'u \,.\, x\varepsilon\, \delta v .\supset. \mathrm{Lm}(f,v,x) \supset \mathrm{Lm}(f,u,x)$

$w\varepsilon$ Cls'q . Oper$\frown$ $.\supset.$		$v-w \supset$	$u-w$
Oper f‘ $.\supset.$		$f\text{‘}(v-w) \supset$	$f\text{‘}(u-w)$
Oper Λ $.\supset.$		$\Lambda f\text{‘}(v-w) \supset$	$\Lambda f\text{‘}(u-w)$
Oper $a\varepsilon$ $.\supset:$		$a\varepsilon\, \Lambda f\text{‘}(v-w) .\supset.$	$a\varepsilon\, \Lambda f\text{‘}(u-w)$
Syll $.\supset.\because$	$w\varepsilon$Cls'q $.\supset_w.$	$a\varepsilon\, \Lambda f\text{‘}(v-w) :\supset:$	$w\varepsilon$Cls'q $.\supset_w.$ $a\varepsilon\Lambda$ »
Oper $a\ni$ $.\supset:$ $a\ni[$	» »	» » $]\supset a\ni[$	» » » »
Df Lm $.\supset.$ P]			

·6 $u,v\varepsilon \mathrm{Cls}'\mathrm{q} \,.\, x\varepsilon\, \delta u \frown \delta v \,.\, f\varepsilon\, \mathrm{qf}(u \smile v) .\supset.$
$\mathrm{Lm}(f, u \smile v, x) = \mathrm{Lm}(f,u,x) \smile \mathrm{Lm}(f,v,x)$

P·5 et ·6 exprime variatione de $\mathrm{Lm}(f,u,x)$, cum variatione de campo u de variabilitate.

·7 $\text{Lm}(f,u,x) \supset \Lambda f`u$

·8 $a\varepsilon\, \dot{\delta} f`u \;.\supset.\; \exists\, \delta u \cap x\ni[\, a\varepsilon\, \text{Lm}(f,u,x)\,]$

[$x = \text{l}'\, \text{q} \cap z\ni\{a \mathrel{-\varepsilon} \delta f`[u\cap(z-\text{Q})]\} \;.\supset:$

$x\varepsilon\text{q} \,.\, h\varepsilon\text{Q} \;.\supset.\; a \mathrel{-\varepsilon} \delta f`[u\cap(x-h-\text{Q})] \,.\, a\varepsilon\, \delta f`[u\cap(x+h-\text{Q})] \;.\supset.$

$a\varepsilon\, \delta f`[u\cap(x-h)\overline{\ }(x+h)] \;.\supset.\; x\varepsilon\, \delta u \,.\, a\varepsilon\, \text{Lm}(f,u,x)$ (1)

$x = \pm\infty \;.\supset.$ Ths (1) (2)

(1) . (2) $.\supset.$ P]

·9 $f\varepsilon\, \text{qfN}_0 \;.\supset.\; \text{Lm}f = \text{Lm}(f, \text{N}_0, \infty)$ Dfp

Si x es successione de quantitates, tunc Lm f, definito per P1·0, coincide cum limes de functione f, quando variabile sume valores N_0, et tende ad ∞.

* 42. $u\varepsilon\, \text{Cls}'\text{q} \,.\, x\varepsilon\, \delta u \,.\, f\varepsilon\, \text{qf}u \;.\supset:$

·0 $\lim(f,u,x) = \iota\, \text{Lm}(f,u,x)$ Df

Si classe limes de functione f, in campo u, pro valore x, consta de uno solo individuo, nos voca illo: limite de functio f, in campo u, pro valore x.

Eliminatione de signo Lm porta ad definitiones sequente:

·1 $a,x\varepsilon\text{q} \;.\supset\therefore\; a = \lim(f,u,x) \;.=:$
$k\varepsilon\text{Q} \;.\supset_k.\; \exists\, \text{Q}\cap h\ni[y\varepsilon\, u-\iota x \,.\, \text{mod}(y-x)<h \;.\supset_y.\; \text{mod}(fy-a)<k]$ Dfp

Primo, in ordine de tempore, ennuntiato completo de definitione de limite. O. Bonnet BD. a.1871 t. 2 p.215:

« Étant donnée une fonction réelle .... bien déterminée, [fy] d'une variable réelle ... [y], on dit ... que cette fonction tend vers une limite finie et déterminée [a], à mesure que [y] tend vers une valeur particulière [x] (nous supposons x fini...), lorsqu'après avoir fixé arbitrairement un nombre réel et positif k aussi petit que l'on veut, il est possible de trouver un autre nombre réel et positif h, tel que, pour toute valeur de y, dont la différence avec x a un module différent de zéro, mais inférieur à h, la valeur correspondante de fy ait avec a une différence dont le module soit compris entre zéro et k. »

·2 $\infty = \lim(f,u,x) \;.=:\; k\varepsilon\text{Q} \;.\supset_k.$
$\exists\, \text{Q}\cap h\ni[\, y\varepsilon\, u-\iota x \,.\, \text{mod}(y-x)<h \;.\supset_y.\; fy>k]$ Dfp

·3 $\lim(f,u,x)\, \varepsilon\text{q} \;.=:\; h\varepsilon\text{Q} \;.\supset_k.\; \exists\, \text{Q}\cap h\ni[\, y,z\varepsilon\, u-\iota x \,.$
$\text{mod}(y-x)<h \,.\, \text{mod}(z-x)<h \;.\supset_{y,z}.\; \text{mod}(fy-fz)<k\,]$

·4 $\lim(f,u,x)\, \varepsilon\, \text{q}\cup\iota\infty\cup\iota-\infty \,.\, v\varepsilon\, \text{Cls}'u \,.\, x\varepsilon\, \delta v \;.\supset.$
$\lim(f,v,x) = \lim(f,u,x)$ [P40·5 $\supset$ P]

·5 $h,k \varepsilon \mathrm{Cls}'\mathrm{q} \,.\, f \varepsilon k\mathrm{f}h \,.\, g \varepsilon \mathrm{qf}k \,.\, x \varepsilon \,\overline{\mathrm{r}} h \,.\, \lim(f,h,x) \varepsilon \,\overline{\mathrm{r}} k \,.\supset.$
$\lim(gf,h,x) = \lim[g, k, \lim(f,h,x)]$

·6 $\lim(f,u,x) = \imath\, a \ni [v \varepsilon \mathrm{Cls}'u \,.\, x \varepsilon \,\overline{\mathrm{r}} v{-}v \,.\supset_v.\, a \varepsilon \Lambda f`v]$ Dfp

·7 $\lim(f,u,x) = \imath\, z \ni (y \varepsilon u\mathrm{f}\mathrm{N}_0 \,.\, \lim y = x \,.\supset_y.\, z = \lim fy)$ Dfp

Definitione possibile de limite de functione, ope limite de successione.

* 43·1 $u \varepsilon \mathrm{Cls}'\mathrm{q} \,.\, a \varepsilon \,\overline{\mathrm{r}} u \,.\, f \varepsilon \mathrm{qf}(u\mathrm{:}\mathrm{N}_0) :$
$n \varepsilon \mathrm{N}_0 \,.\supset_n.\, \lim\,[f(x,n)|x, u, a]\, \varepsilon \mathrm{q} :$
$\Sigma\{[\mathrm{l}' \mathrm{mod}\, f(x,n)|\, x`u]\,|n, \mathrm{N}_0] \,\varepsilon \mathrm{Q} :\supset.$
$\lim\{\Sigma[f(x,n)|n, \mathrm{N}_0]\,|x, u, a\} = \Sigma\{\lim[f(x,n)|x, u, a]\,|n, \mathrm{N}_0\}$
Comm(lim, Σ)

[$h = [\mathrm{l}'\mathrm{mod} f(x,u)|x`u]|n \,.\supset:$
$\Sigma(h,\mathrm{N}_0) \varepsilon \mathrm{Q} \,.\, \lim\Sigma(h, m+\mathrm{N}_0)|m \doteq 0$ (1)
$x\varepsilon u \,.\, n\varepsilon \mathrm{N}_0 \,.\supset.\, \mathrm{mod} f(x,n) \leqq h_n$ (2)
$n\varepsilon\mathrm{N}_0 \,.\, (2) \,.\supset.\, \mathrm{mod}\lim[f(x,n)|x, u, a] \leqq h_n$ (3)
$(3) \,.\, \mathrm{P}26{\cdot}1 \,.\supset.\, \Sigma\{\lim[f(x,n)|x, u, a]|n, \mathrm{N}_0\} \,\varepsilon \mathrm{q}$ (4)
$x\varepsilon u \,.\, m\varepsilon \mathrm{N}_1 \,.\supset.\, \Sigma[f(x,n)|n, \mathrm{N}_0] = \Sigma[f(x,n)|n, 0\cdots(m-1)]$
$+\Sigma[f(x,n)|n, m+\mathrm{N}_0]$ (5)
$m\varepsilon\mathrm{N}_1 \,.\, (5) \,.\, \mathrm{Oper\ Lm}|x \,.\supset.$
$\mathrm{Lm}\{\Sigma[f(x,n)|n,\mathrm{N}_0]|x, u, a\} = \Sigma\{\lim[f(x,n)|x, u, a]|n, 0\cdots(m-1)\} +$
$\mathrm{Lm}\{\Sigma[f(x,n)|n, m+\mathrm{N}_0]|x, u, a\}$ (6)
$m\varepsilon\mathrm{N}_1 \,.\, (2) \,.\supset.\, \mathrm{Lm}\Sigma[f(x,n)|n, m+\mathrm{N}_0] \supset \pm\Theta\Sigma(h, m+\mathrm{N}_0)$ (7)
$m\varepsilon\mathrm{N}_1 \,.\, (6) \,.\, (7) \,.\supset.\, \mathrm{Lm}\Sigma[f(x,n)|n, \mathrm{N}_0]|x, u, a\} \supset \Sigma\{\lim[f(x,n)|x, u, a]|n,$
$0\cdots(m-1)\} \pm \Theta\Sigma(h, m+\mathrm{N}_0)$ (8)
$(8) \,.\, \mathrm{Oper}\lim|m \,.\, (4) \,.\supset.$
$\mathrm{Lm}\{\Sigma f(x,n)|n, \mathrm{N}_0\}|x, u, a\} \supset \Sigma\{\lim[f(x,n)|x, u, a]|n, \mathrm{N}_0\} + \iota 0$ (9)
$(9) \,.\supset.\, \mathrm{Lm}\{\Sigma[f(x,n)|n, \mathrm{N}_0]|x, u, a\} = \iota\,\Sigma\{\lim[f(x, u, a]|u,\mathrm{N}_0\}$ (10)
$(10) \,.\, \mathrm{Oper}\,\imath \,.\supset.\, \mathrm{P}$]

Nos considera $f(x,n)$, functio reale de duo variabile x et n, ubi x varia in aliquo campo u, et n sume valores integro N_0. Tunc serie $\Sigma[f(x,n)|n,\mathrm{N}_0]$, si converge, habe summa que depende de x. Limite de ce summa, quando x varia in u, et tende ad aliquo valore a, prope alios u, vale, in generale, serie de limites, ut pro summa de numero finito de termine. Id es, es commutabile operatione lim cum Σ.

Existe ullo casu de exceptione. Pro elimina illos, nos suppone que serie formato per limites supero de valores absoluto de termines de serie dato, quando x sume omni valore in u, es convergente. Nos suppone etiam que omni termine de serie dato habe limite determinato et finito.

In vero, si nos voca h_0, h_1, h_2, ... serie formato per limites supero de valores absoluto de termines de serie dato, serie h es convergente, per hypothesi. (1)

Et omni termine de serie dato es inferiore in valore absoluto, ad termine correspondente de serie h, non excluso casu de aequalitate. (2)

Idem fi pro limites de termines de serie dato. (3)

Ergo serie de limites de termines es convergente. (4)

Nos decompone serie dato in summa de primos m termine, plus serie de termines sequente, vel resto. (5)

Tunc classe limes de summa de serie dato vale summa de limites de primos m termine, plus limes de serie resto. (6)

Serie resto es minore de resto in serie h; ergo serie resto es fractione positivo aut negativo de resto in serie h. (7)

Si nos substitue in formula (6), nos obtine (8); et si nos sume limite per m, resulta que classe limes de serie dato consta de solo numero summa de serie de limites. (9)

Ergo, limite de serie dato vale serie de limites.

* * *

Non in omni casu operatione « lim » es commutabile cum « Σ serie ». Per exemplo de propositione:

$$x \varepsilon \mathrm{q} \,.\, \mathrm{mod}x < 1 \,.\supset.\, 1/(1-x) = 1 + x + x^2 + x^3 + \ldots$$

seque:

$$x \varepsilon \mathrm{q} \,.\, \mathrm{mod}x < 1 \,.\supset.\, 1 = (1-x) + x(1-x) + x^2(1-x) + \ldots..$$

Limite de summa de serie in secundo membro quando x tende ad 1, vale 1. Serie formato per limites de termine de serie considerato vale

$$0 + 0 + 0 + \ldots = 0.$$

Ergo limite de summa de serie non vale summa de limites.

Serie $$\Sigma[f(x,n) | n, N_0]$$
es convergente, pro totos valore de x in campo u, quando (P3·4):

$$x \varepsilon u \,.\supset_x : h \varepsilon Q \,.\supset_h .\, \exists\, N_0 \frown p \ni \{ q \varepsilon p + N_0 \,.\supset_q .\, \mathrm{mod}\Sigma[f(x,r) | r, p \cdots q] < h \}$$

Serie es de convergentia « gleichmässig (Weierstrass a.1841 t.1 p.6,7) uniforme, aequabile », si satisfac conditione

$$h \varepsilon Q \,.\supset_h .\, \exists\, N_0 \frown p \ni \{ x \varepsilon u \,.\, q \varepsilon p + N_0 \,.\supset_{x,q} .\, \mathrm{mod}\Sigma[f(x,r) | r, p \cdots q] < h \}$$

que differ de praecedente per positione de x.

Abel (t.1 p.224) nota valore differente de duo conditione.

Vide Cauchy ParisCR. a.1853 t.36 p.454; Œuvres s.1 t.12 p.34.

Theorema praecedente, in quasi omni tractato de Analysi, consta de duo parte:

1.° Si serie formato per limites supero de termines de serie dato es convergente, tunc serie dato es de convergentia uniforme.

2.° Si serie es de convergentia uniforme, limite de serie vale serie de limites.

Aliquo Auctore voca « serie de convergentia uniforme simplice » (Dini), vel « de convergentia uniforme in sensu lato » (Tannery), serie que satisfac

ad conditione sequente:

$$h \varepsilon Q . \supset_h . \exists N_0 \cap p \ni \{ x \varepsilon u . \supset_x . \mathrm{mod}\Sigma[f(x,n)|u, p+N_0] < h \}$$

Progressio geometrico

$$x \varepsilon -1 \overline{\ } 1 . \supset . 1/(1-x) = 1+x+x^2+x^3+\ldots,$$

que nos considera in exemplo praecedente, nos es de convergentia uniforme in intervallo $-1 \overline{\ } 1$. In generale nullo serie ordinato secundo potestates de variabile x, es de convergentia uniforme in campo, ubi converge.

Substitutione ad convergentia uniforme, de conditione plus simplo de linea 3 de theorema es indicato per W e i e r s t r a s s a.1880 Werke t.2 p.202. Vide Form. t.4, pag.325.

lim Cx Subst

✻ 51. $m,n \varepsilon N_1 . u \varepsilon \mathrm{Cls}'\mathrm{Cx}n . x \varepsilon \delta u . f \varepsilon \mathrm{Cx}m \mathrm{f} u . \supset .$
§lim P40-43

Definitione de limite, et suo proprietates subsiste si nos considera numero complexo de ordine m functione de numeros complexo de ordine n in aliquo campo u.

·1 $a \varepsilon \mathrm{Cx}m . \supset : a \varepsilon \mathrm{Lm}(f,u,x) .=. 0 \varepsilon \mathrm{Lm}[\mathrm{mod}(fy-a)|y, u, x]$

·2 » $a = \lim$ » $0 = \lim$ » »

·3 $\infty \varepsilon \mathrm{Lm}(f,u,x) .=. \infty \varepsilon \mathrm{Lm}(\mathrm{mod}f,u,x)$

·4 $\infty = \lim(f,u,x) .=. \infty = \lim(\mathrm{mod}f, u, x)$

✻ 52·1 $n \varepsilon N_1 . u \varepsilon \mathrm{Cx}n \mathrm{f} N_0 . \Sigma(\mathrm{mod}u, N_0) \varepsilon Q . \supset . \Sigma(u,N_0) \varepsilon \mathrm{Cx}n$

[$r \varepsilon N_0 . s \varepsilon 1 \cdots n . \supset . \mathrm{mod}(u_r)_s \leqq \mathrm{mod}u_r$ (1)
$s \varepsilon 1 \cdots n . \supset . \Sigma[(u_r)_s | r, N_0] \varepsilon Q$ (2)
(2) $. \supset . \Sigma\{[\Sigma(u_r)_s \mathrm{unit}(n,s)|s, 1 \cdots n]|r, N_0\} \varepsilon Q$ (4)
(4) . §Cx P2·1 $. \supset .$ P]

« Si u es serie de numero complexo de ordine n, et serie de modulos es convergente, tunc serie u es convergente ».

·2 $m,n \varepsilon N_1 . m > n . \supset . \Sigma[(\mathrm{mod}x)^{-m}|x, \mathrm{Cx}n - \iota 0] \varepsilon Q$
{ Einsenstein JfM. t. 35 }

Super limite de determinante, et determinante de ordine infinito, vide Formul. t. 4 p. 211.

✻ 53. $a \varepsilon (\mathrm{Subst}n)\mathrm{f}N_0 . \supset .$

·0 $\lim a = \iota(\mathrm{Subst}\, n) \cap b \ni [x \varepsilon \mathrm{Cx}n . \supset_x . \lim(ax) = bx]$ Df

·1 $\lim a = 0 .=. \lim \mathrm{mod}a = 0$ Dfp

·2 $b \varepsilon \mathrm{Subst}n . \supset : \lim a = b .=. \lim(a-b) = 0$ Dfp

❋ 60. lim q′

·1 $u \varepsilon \mathrm{q'fN_0} \,.\, a \varepsilon \mathrm{q'} \,.\, \Sigma(u_n a^n \,|\, n, \mathrm{N_0})\, \varepsilon \mathrm{q'} \,.\, x \varepsilon \mathrm{q'} \,.\, \mathrm{mod}x < \mathrm{mod}a \;.\supset.$
$\Sigma(u_n x^n \,|\, n, \mathrm{N_0})\, \varepsilon \mathrm{q'}$ } Abel t.1 p.223 {

·2 $u \varepsilon \mathrm{q'fN_0} \,.\, a \varepsilon \mathrm{q'} \,.\, \infty \,\text{-}\varepsilon\, \mathrm{Lm}[\mathrm{mod}(u_n a^n)] \,|\, n \,.\, x \varepsilon \mathrm{q'} \,.\, \mathrm{mod}x < \mathrm{mod}a$
$.\supset.\; \Sigma(u_n x^n \,|\, n, \mathrm{N_0})\, \varepsilon \mathrm{q'}$

·3 $u \varepsilon \mathrm{q'fN_0} \,.\, x \varepsilon \mathrm{q'} \;.\supset:$
$\mathrm{mod}x < /\mathrm{maxLm}\, {}^n\!\sqrt{}(\mathrm{mod}u) \;.\supset.\; \Sigma(u_r x^r \,|\, r, \mathrm{N_0})\, \varepsilon \mathrm{q'}$.
» $>$ » » » $= \infty$

} Cauchy Œuvres s.1 t. 5 p.360:
« Une série ordonnée suivant les puissances ascendantes et entières d'une variable x, soit réelle, soit imaginaire, est convergente ou divergente suivant que le module de la variable est inférieur ou supérieur à l'unité divisée par la plus grande des limites vers lesquelles converge la racine $n$ième du coefficient de x^n ». {

Radio de convergentia de serie $\Sigma(u_n a^n \,|\, n, \mathrm{N_0})$
$= 1'\mathrm{mod}\{\mathrm{q'} \frown a\ni[\Sigma(u_n a^n \,|\, n, \mathrm{N_0})\, \varepsilon \mathrm{q'}]\}$
circulo de convergentia $= \mathrm{q'} \frown x\ni(\mathrm{mod}x <$ radio de convergentia).

·4 $u \varepsilon \mathrm{q'fN_0} \,.\, a \varepsilon \mathrm{q'} \,.\, \Sigma(u_n a^n | n, \mathrm{N_0})\, \varepsilon \mathrm{q'} \;.\supset.$
$\lim[\Sigma(u_n x^n | n, \mathrm{N_0}) \,|\, x, \theta a, a] = \Sigma(u_n a^n \,|\, n, \mathrm{N_0})$ } Abel t.1 p.223 {

·5 $a, b \varepsilon \mathrm{q} \,.\, \mathrm{real}a < \mathrm{real}b \,.\, b \,\text{-}\varepsilon\, -\mathrm{N_0} \;.\supset.\; \Pi[(a+r)/(b+r) \,|\, r, \mathrm{N_0}] = 0$

·6 $u \varepsilon \mathrm{q'}\text{-}(\iota-1)\, \mathrm{FN_0} \,.\, \Sigma\mathrm{mod}u\, \varepsilon\, \mathrm{Q} \;.\supset.\; \Pi[(1+u_r) \,|\, r, \mathrm{N_0}]\, \varepsilon\, \mathrm{q'} \,\text{-}\iota 0$
} Weierstrass a.1856 t.1 p.176 {

lim vct

❋ 70. (v | Cx) P51

(p | Cx) ——

Nos extende Df de limite ad vectores et ad punctos.

❋ 71. $k \varepsilon \mathrm{Cls'}(\mathrm{q} \cup \mathrm{Cx} \cup \mathrm{p} \cup \mathrm{v}) \,.\, x \varepsilon \Delta k \,.\, f \varepsilon (\mathrm{Cls'p})\mathrm{f}k \,.\supset.$

·1 $\lim(f,k,x) = \mathrm{p} \cap a \ni \{ \lim[\, \mathrm{d}(a, fz) \,|\, z, k, x] = 0 \}$ Df

Si f es classe de punctos, vel figura, functione de numeros reale, vel complexo, vel puncto, vel vectore, pertinente ad aliquo classe k, proximo ad x, tunc limite de figura variabile f, in classe k, pro valore x, es omni puncto a tale que limite de distantia de a ad figura fz, quando z varia, in classe k, et tende ad x, vale zero.

Ex. § rectaT, planO, ...

·2 $a \varepsilon \mathrm{p} \,.\, u \varepsilon (\mathrm{v} - \iota 0)\mathrm{f}k \,.\, \lim(u,k,x) \,\varepsilon\, \mathrm{v} - \iota 0 \,.\supset.$
$\lim[\mathrm{recta}(a,ux) | x, k, x] = \mathrm{recta}[a, \lim(u,k,x)]$

[$\mathrm{Hp} \,.\, v = \lim(u,k,x) \,.\supset.\, \lim[\mathrm{recta}(a,ux) | x,k,x] =$
$\mathrm{p} \cap z \ni (\lim \{ \mathrm{d}[z, \mathrm{recta}(a,ux)] | x,k,x \} = 0) =$
$\mathrm{p} \cap z \ni \{ \lim[\, (\sqrt{\{(z-a)^2 - [(z-a) \times \mathrm{U}u]^2\}})/(\mathrm{mod}\,ux) | x,k,x] = 0 \} =$
$\mathrm{p} \cap z \ni [\, \sqrt{\{(z-a)^2 - [(z-a) \times \mathrm{U}v]^2\}}/(\mathrm{mod}\,v) = 0] = \mathrm{p} \cap z \ni \{ \mathrm{d}[z, \mathrm{recta}(a, v)] = 0$
$= \mathrm{recta}(a,v)$]

Limite de recta passante per punto fixo et parallelo ad vectore variabile, es recta passante per puncto fixo et parallelo ad limite de vectore variabile. Nos suppone que vectore variabile habe valore non nullo, in campo determinato, et que suo limite es determinato et non nullo.

In modo simile, nos determina limite de alio figura; usu de vectores, vel de coordinatas, reduce illos ad limite de numero.

·3 $a \varepsilon \mathrm{p} \,.\, l \varepsilon \mathrm{v} - \iota 0 \,.\, u \varepsilon (\mathrm{v} - \mathrm{q}l)\mathrm{f}k \,.\, \lim(u,k,x) \,\varepsilon\, \mathrm{v} - \mathrm{q}l \,.\supset.$
$\lim[\mathrm{plan}(a, l, ux) | x, k, x] = \mathrm{plan}[a, l, \lim(u,k,x)]$

·4 $\mathrm{Lm}(f,k,x) = \mathrm{p} \cap a \ni \{ 0 \varepsilon \mathrm{Lm}[\mathrm{d}(a, fz) | z, k, x] \}$ Df

Generalizatione de Df·1. Ex. §Tang.

·5 $a \varepsilon \mathrm{Cls'p} \,.\, x \varepsilon \mathrm{p} \,.\supset.\, \lim[\mathrm{d}(y, a) \,|\, y, \mathrm{p}, x] = \mathrm{d}(x, a)$

❋ 72. $k \varepsilon \mathrm{Cls'q} \,.\, x \varepsilon \delta k \,.\, r \varepsilon 1 \cdots 3 \,.\, u \varepsilon \varphi^{r}\mathrm{f}k \,.\supset.$
$\lim(u,k,x) = \imath\, \varphi^{r} \cap b \ni \{ c \varepsilon \mathrm{p}^{4-r} \,.\supset_c.\, \lim[(uxac) | x,k,x] = bac \}$ Df

Df de limite de forma geometrico, analogo ad Df §α 9·5.

§2 cont

❋ 1. $m,n \varepsilon \mathrm{N}_1 . u \varepsilon \mathrm{Cls'} \mathrm{Cx}n . u \supset \delta u . \supset .$

·0 $f \varepsilon (\mathrm{Cx}m\, \mathrm{f}\, u)\mathrm{cont} .=: f \varepsilon (\mathrm{Cx}m \mathrm{f} u) : x \varepsilon u . \supset_x . \lim(f,u,x) = fx$ Df

·01 » $.=:$ » $: k \varepsilon \mathrm{Q} . x \varepsilon u . \supset_{k,x} .$
$\exists\, \mathrm{Q} \frown h \ni [y \varepsilon u . \mathrm{mod}(y-x) < h . \supset_y . \mathrm{mod}(fy-fx) < k]$ Dfp

Si u es classe de numeros reale aut complexo de ordine quocumque, et si omni u es proximo ad alio u, tunc nos dice que f es complexo functio des u continuo, $f\varepsilon$ (Cxm fu)cont, si f es complexo functio des u, et si pro omni x pertinente ad classe u, semper limite de f, in campo u, pro valore x, es æquale ad valore de f pro valore x de variabile ».

Si nos elimina signo « lim », Df sume forma :

« f es functio continuo in campo u, quando, fixato ad arbitrio quantitate positivo k, si x es individuo de classe u, nos semper pote inveni quantitate positivo h, tale que, si y es individuo de classe u, differente de x minus que h in valore absoluto, semper fi differentia inter fy et fx minore in valore absoluto de k ».

Definitione P·0 occurre in Abel t.1 p.223.

465. ***continuo*** HI, AF continue. ⊂ con- (47) + tene — -e + -uo (214).
Nota. Vocale -e- de thema, ante uno solo consonante, fi -i- in compositione :
tene, con-t*i*ne, at-t*i*ne, abs-t*i*ne, ŏb-t*i*ne, re-t*i*ne, per-t*i*ne,... ;
preme, op-pr*i*me ; ...

466. ***tene,*** F tien-t, HI tiene. || L ten-de, S tan, G teine, hypo-tein-usa, D dehne.
L ten-ue = A thin = D dünn = G tany- = R ton-cĭj = S tanu.
In L con-tin-uo, per-tine, ... « tene » = « se extende ».

·1 $u = \nabla u . f \varepsilon (\mathrm{Cx}m\, \mathrm{f} u)\mathrm{cont} . \supset . \Lambda f`u = f`u$

[$a \varepsilon \nabla f`u$. §lim P41·8 $. \supset . \exists\, \nabla u \frown x \ni [a \varepsilon \mathrm{Lm}(f,u,x)] . \supset .$
$\exists\, u \frown x \ni [a = \lim(f,u,x)] . \supset . \exists\, u \frown x \ni (a = fx) . \supset . a \varepsilon f`u$ (1)
(1) $. \supset . \nabla f`u \supset f`u . \supset . \Lambda f`u = f`u$]

·2 $u = \delta u \,.\, f \varepsilon (\mathrm{q f} u)\mathrm{cont} \,.\supset.\, \max f\text{‘}u, \min f\text{‘}u \,\varepsilon \mathrm{q}$

[$\mathrm{l}' f\text{‘}u, \mathrm{l}_{,} f\text{‘}u \,\varepsilon\, \Lambda f\text{‘}u \,.\, \mathrm{P}{\cdot}1 \,.\supset.\, \mathrm{P}$]

Functio continuo in aliquo campo u, coincidente cum suo classe derivata, sume valore maximo et valore minimo. Si $u = \delta u$, campo u es finito, et contine classe limite.

·3 $u = \delta u \,.\, f \varepsilon (\mathrm{Cx}m\, \mathrm{f} u)\mathrm{cont} \,.\, k \varepsilon \mathrm{Q} \,.\supset.$

$\exists\, \mathrm{Q} \cap h \ni [\, x,y \varepsilon u \,.\, \mathrm{mod}(y-x) < h \,.\supset_{x,y}.\, \mathrm{mod}(fy-fx) < k \,]$

[$x \varepsilon u \,.\, gx = \mathrm{l}'\, \Theta \cap h \ni \{ y,z \varepsilon u \,.\, \mathrm{mod}(y-x) < h \,.$

$\mathrm{mod}(z-x) < h \,.\supset_{y,z}.\, \mathrm{mod}(fy-fz) < k \} \,.\supset:$

$x,y \varepsilon u \,.\, \mathrm{mod}(y-x) < gx+gy \,.\supset.\, gx \leqq gy + \mathrm{mod}(y-x) \,.\, gy \leqq gx +$

$\mathrm{mod}(y-x) \,.\supset.\, \mathrm{mod}(gy-gx) \leqq \mathrm{mod}(y-x)$ (1)

(1) $.\supset.\, g \varepsilon (\mathrm{Qf} u)\mathrm{cont}$ (2)

(2) . P·2 $.\supset.\, \min g\text{‘}u \,\varepsilon \mathrm{Q}$ (3)

$h = \min g\text{‘}u \,.\, x,y \varepsilon u \,.\, \mathrm{mod}(y-x) < h \,.\supset.\, \mathrm{mod}(fx-fy) < k$]

Theorema de « continuitate uniforme ». Differ de Df de functio continuo, P·01, per positione de $y \varepsilon u$.

Vide Heine, JfM. a.1870 t.71 p.361 JfM. a.1871 t.74 p.188, Lüroth, MA. t.6 a.1873 p.319.

·4 $f \varepsilon \mathrm{Cx}n\, \mathrm{f}(u : \mathrm{N}_0) : n \varepsilon \mathrm{N}_0 \,.\supset_n.\, f(x,n)|x \;\varepsilon\; (\mathrm{Cx}n\, \mathrm{f}\, u)\mathrm{cont} :$

$\Sigma \{ \mathrm{l}'[\mathrm{mod} f(x,n)|x\text{‘}u]|n, \mathrm{N}_0 \} \,\varepsilon \mathrm{Q} :\supset.\, \Sigma[f(x,n)|n, \mathrm{N}_0]|x \;\varepsilon\; (\mathrm{Cx}n\, \mathrm{f} u)\mathrm{cont}$

[§lim 43·1 $.\supset.$ P]

Nos considera serie, que habe $f(x,n)$ pro termine de ordine n. x es numero complexo in aliquo campo u. Si omni termine de serie es functio continuo de x, et si serie formato per limites supero de valores absoluto de termines de serie dato, dum x varia in suo campo, es convergente, tum summa de serie es functio continuo de x. Seque de theorema super limite de serie.

✻ 2. $a,b \varepsilon \mathrm{q} \,.\, a \neq b \,.\, f \varepsilon (\mathrm{q f}\, a \overline{} b)\mathrm{cont} \,.\supset:$

·1 $fa < 0 \,.\, fb > 0 \,.\supset.\, 0 \varepsilon f\text{‘}(a \overline{} b)$ { Cauchy a.1821 note 3 }

[$y = \mathrm{l}'\, a \overline{} b \cap x \ni (fx < 0) \,.\supset.\, y \varepsilon a \overline{} b \,.\, fy = 0$]

·2 $fa \overline{} fb \supset f\text{‘}(a \overline{} b)$

✻ 3. $n \varepsilon \mathrm{N}_1 \,.\supset.\, \exists\, (\mathrm{Cx}n\, \mathrm{f\, q})\mathrm{cont} \cap f \ni (f\text{‘}\mathrm{q} = \mathrm{Cx}n)$

Existe complexo de ordine n, vel puncto in spatio ad n dimensiones, functio continuo de variabile reale, vel de tempore, tale que trajectoria de puncto mobile ple toto spatio. Id es, existe linea continuo, que transi per omni punto de plano, et existe linea, que transi per omni puncto de spatio, etc. Ce resultatu habe interesse in studio de principio de Geometria; nam non existe charactere specifico, que distingue linea ab superficie.

Si nos vol que, dum variabile t varia de 0 ad 1, puncto de coordinatas x et y, functiones de t, describe toto quadrato $(\Theta : \Theta)$, nos evolve t in fractione decimale, vel analogo ad decimale, in aliquo basi:

$$t = 0{\cdot}a_1 a_2 a_3 \ldots$$

ubi $a_1, a_2, a_3 \ldots$ es cifras. Si cum cifras de ordine pari nos forma numero x, et cum cifras de ordine dispari nos forma numero y, nos habe correspondentia reciproco inter uno fractione decimale et duo alio fractione decimale. Sed duo fractione decimale de forma differente, ut 0·0999... et 0·1000... pote habe idem valore; et correspondentia inter numero t et numeros x et y non es continuo. Si nos decompone quadrato de latere 1 in 100 quadratos de latere 1/10, tunc si t transi de valores 0·0900...⌒0·0999... ad valores 0·1000...⌒0·1999..., puncto (x,y) transi de ultimo quadrato in primo columna ad primo quadrato de secundo columna, et ce duo quadrato non es adjacente.

Nos pone quadratos partiale, ut illo fi adjacente. In basi 2 de numeratione, nos sume 4 quadratos partiale in ordine ut in figura (a), et in basi 3 ut in figura (b).

Tunc me divide omni quadrato partiale in alios quadrato, et ita ad infinito. Fig. (c) repraesenta successione de 16 quadratos in basi 2; fig. (d) successione de 8 quadratos in basi 3.

Si nos repraesenta per signo $\cap$ successione $\begin{matrix} 1 & 2 \\ 0 & 3 \end{matrix}$, vel figura ($a$), tunc figura ($e$) repraesenta successione de 64 quadratos in basi 2.

$$\begin{matrix} 1 & 2 \\ 0 & 3 \end{matrix}$$

(a)

$$\begin{matrix} 2 & 3 & 8 \\ 1 & 4 & 7 \\ 0 & 5 & 6 \end{matrix}$$

(b)



(c)



(d)



(e)

In scripto *Sur une courbe qui remplit toute une aire plane,* MA. a.1890 t.37 p.132, me da expressione analytico de correspondentia continuo inter numero reale t, et numero complexo $(x;y)$.

Vide Hilbert a.1891 MA. t.38 p.459, Cesàro, DarbouxB. a.1897 t.21 p.257, Moore AmericanT. a.1900 p.72, Lebesgue, *Leçons sur l'intégration*, Paris a.1904 p.45.

§3 e

❋ 1·0 $\mathrm{e} = \mathrm{l}'[(1+/m)^m | m \text{‘} Q]$ Df e

«e» indica limite supero de valores de $(1+1/m)^m$, quando varia m, in campo de quantitate positivo.

·1 $\mathrm{e} = \mathrm{l}_{,}[(1+/m)^{m+1} | m \text{‘} Q]$ Dfp

[$m, n \varepsilon Q$. §Q 31·8 .⊃. $(1+/m) \wedge m < (1+/n) \wedge (n+1)$ (1)

(1) .⊃. $\mathrm{l}'[(1+/m) \wedge m | m \text{‘} Q] \leqq \mathrm{l}_{,}[(1+/m) \wedge (m+1) | m \text{‘} Q]$ (2)

$m \varepsilon Q$.⊃. $(1+/m) \wedge (m+1) = [(1+/m) \wedge m](1+/m)$ (3)

(3) .⊃. $(1+/m) \wedge (m+1) \leqq \mathrm{l}'[(1+/m) \wedge m | m \text{‘} Q] \times (1+/m)$ (4)

(4) .⊃. $\mathrm{l}_{,}[(1+/m) \wedge (m+1) | m \text{‘} Q] \leqq \mathrm{e}$ (5)

(2) . (5) .⊃. P.]

Idem numero es limite infero de valores de $(1+1/m) \wedge (m+1)$, pro m positivo. In vero, omni valore de $(1+1/m) \wedge m$ es minore de omni valore de $(1+1/m) \wedge (m+1)$. Ergo limite supero de primos es inferiore vel aequale ad limite infero de secundos. Ratione de uno numero de secundo classe ad correspondente numero de primo classe vale $1+1/m$, que es proximo ad 1 ad arbitrio. Tunc limite supero de primo classe aequa limite infero de secundo.

·2 $m \varepsilon Q$.⊃. $(1+/m)^m < \mathrm{e} < (1+/m)^{m+1}$ [P·0·1 ⊃ P]

·3 $2 < \mathrm{e} < 3$ [$(1|m)$P·2 . $(5|m)$P·2 .⊃ P]

Si nos pone, in loco de m, 1 et 5, nos deduce parte integro de numero e. Pro $m = 20$, nos habe:

$$(1{,}05)^{20} < \mathrm{e} < (1{,}05)^{21},$$

id es, numero e supera valore de 1 franco ad interesse composito de 5 pro 100, per 20 anno, et es superato ab valore de 1 franco ad idem taxu de interesse, per 21 anno. Pro $m = 25$, nos deduce, ab tabulas de interesse:

$$2{\cdot}665 < \mathrm{e} < 2{\cdot}772.$$

e = 2·71828 18284 59045 23536 02874 71352 66249 77572
47093 69995 95749 66967 62772 40766 30353 54759
45713 82178 52516 64274 27466 39193 20030 59921
81741 35966 29043 57290 03342 95260 59563 07381
32328 62794 34907 63233 82988 07531 95251 01901
15738 34187 93070 21540 89126 94937 99405 34631
93819 87250 90567 36251 50082 37715 27509 03586
67692 05047 15575 85094 92906 45748 86005 84299
93465 94757 59371 00435 26480 0...

Nepero inveni numero e, ut basi de systema de logarithmo, que habe suo nomen; et calcula illo cum 7 figura decimale, cum positione $m = 10^7$.

R. Cotes, *Logometria*, a.1714 p.11, calcula 12 cifra decimale de e, quem voca « Ratio Modularis ».

Euler, PetrC. a.1739 p.187, indica illo per « e », et calcula E(eX↑23).

Vega, *Thesaurus logarithmorum*, a.1794, p.309 » E(eX↑42).

W. Shanks, LondonP. t.6 a.1854 p.397 » E(eX↑188).

M. Boorman, *Math. Magaz.*, t.1 a.1884 p.204 » E(eX↑346).

e = !. · !.!!.!!! !!!....! .!.!...! (expressione de e in systema binario).

·4 $x \varepsilon\, q - \iota 0 \,.\supset.\, e^x > 1 + x$

[$x \varepsilon Q \,.\, (/x)|m$ P·2 $.\supset.\, (1+x)\uparrow/x < e \,.\supset.$ Ths

$x \varepsilon -Q \,.\, x > -1 \,.\, (-/x-1)|m$ P·2 $.\supset.$ Ths

$x \varepsilon -Q \,.\, 1+x < 0 \,.\supset.$ Ths]

·5 $x \varepsilon\, \theta \cup -Q \,.\supset.\, e^x < /(1-x)$ [$(-x)|x$ P·4 $\supset$ P]

·6 $e, e^2 \varepsilon$ Q$-$R { Euler a.1737 PetrC. t.9 p.98 }

·7 $x \varepsilon\, r - \iota 0 \,.\supset.\, e^x - \varepsilon R$ { Lambert a.1761 p.265 }

·8 $e - \varepsilon\, R \pm \sqrt{R}$ { Liouville JdM. a.1840 t.5 p.193 }

※ 2·0 $x \varepsilon Q \,.\supset.\, \log x = {}^e\mathrm{Log}\, x$ Df log

log, lege: logarithmo naturale, (neperiano, hyperbolico), es logarithmo in basi e.

·01 $(\log, Q) = (e^x|x, q)^{-1}$

Logarithmo neperiano es functio inverso de exponentiale.

·1 $a \varepsilon Q - \iota 1 \,.\, x \varepsilon Q \,.\supset.\, {}^a\mathrm{Log}\, x = ({}^a\mathrm{Log}\, e) \log x = (\log x)/(\log a)$

·2 ${}^{10}\mathrm{Log}\, e = /(\log 10) =$

·43429 44819 03251 82765 11289 18916 60508 22943 97005 80366
65661 14453 78316 58646 49208 87077 47292 24949 33843 17483
18706 10674 47663 03733 64167 92871 58963 90656 92210 64662
81226 58521 27086 56867 03295 93370 86965 88266 88331 16360
77384 90514 28443 48666 76864 65860 85135 56148 21234 87653
43543 43573 17247 48049 05993 55353 05...

Ce numero, dicto « modulo de logarithmos decimale », es calculato cum 282 cifra per Adams, LondonP. a.1878 p.93.

·3 $x \varepsilon (Q-1) - \iota 0 \,.\supset.\, \log(1+x) < x$ [P1·4 $\supset$ P]

·31 » » $> x/(1+x)$

[$\{-x/(1+x)\}|x$ P·3 $\supset$ P]

·4 $x \varepsilon Q - \iota 1 \,.\, m \varepsilon N_1 \,.\supset.\, m(\sqrt[m]{x} - 1) > \log x > m(1 - \sqrt[m]{/x})$

[$(\sqrt[m]{x} - 1)|x$ P·3·31 $\supset$ P]

·5 $x \varepsilon R - \iota 1 \,.\supset.\, \log x - \varepsilon\, r$ { Lambert a.1761 p.265 }

✻ 3·1 $\mathrm{e} = \lim[(1+/m)^m \,|m, \mathrm{q} - -\Theta, \pm\infty]$ Dfp

$[\ \S\mathrm{Q}\ 31{\cdot}2\ .\supset.\ (1+/m)^m|m\ \varepsilon\,(\mathrm{QfQ})\mathrm{cres}\ .\ \S\lim\ 4{\cdot}1\ .\supset.$

$\lim[(1+/m)\upharpoonright m|m, \mathrm{Q}, \infty] = \mathrm{e}$ (1)

$\lim[(1+/m)\upharpoonright m|m, -1-\mathrm{Q}, \infty] = \lim[(1-/m)^{-m}|m, 1+\mathrm{Q}, \infty] =$

$\lim\{[1-/(m+1)]^{-(m+1)}|m, \mathrm{Q}, \infty\} = \lim\{[m/(m+1)]\upharpoonright-(m+1)|m, \mathrm{Q}, \infty\}$

$= \lim[(m+1)/m]\upharpoonright(m+1)|m, \ldots] = \lim[(1+/m)^m(1+/m)|m, \ldots] = \mathrm{e}$ (2)

$(1)\ (2) \supset \mathrm{P}\ ]$

Numero e, que es definito ut limite supero, vel infero de classe, P1·0·1, pote es definito ut limite de functione. Nota que idea de limite supero, indicato per l′, pertine ad algebra elementare, et es multo plus simplice que idea de limite, indicato per lim.

·2 $\mathrm{e} = \lim[(1+x)\upharpoonright/x\ |x, -1+\mathrm{Q}, 0]$ $[=\mathrm{P}{\cdot}1]$

·3 $x\varepsilon\mathrm{q}\ .\supset.\ \lim\,(1+x/m)^m|m = \mathrm{e}^x$

{ EULER Berol. Misc. a.1743 t.7 p.177:

«$\mathrm{e}^z = \left(1+\dfrac{z}{n}\right)^n$ existente n numero infinito.» }

$[\ \mathrm{Hp}\ .\supset.\ \lim\,(1+x/m)^m\,|m = \lim\,[(1+x/m)\upharpoonright(m/x)]\upharpoonright x\ |m =$

$\lim\,[(1+/n)\upharpoonright n]\upharpoonright x\,|n = [\,\lim\,(1+/n)\upharpoonright n\ |n\,]\upharpoonright x = \mathrm{e}\upharpoonright x\ ]$

Functio e^x, que vocare «functio exponentiale», es limite de functio algebrico $(1+x/m)^m$, quando m varia (et tende ad infinito, per valore integro).

Ce propositio demontra-re ab Eulero ut seque:

Expressio considerato vale $(1+x/m)\upharpoonright(m/x)$, toto ad x; sed $(1+x/m)\upharpoonright(m/x)$ habe forma $(1+1/n)^n$, que tende ad e; ergo...

✻ 4.

·1 $x\varepsilon\mathrm{q}\ .\supset.\ \mathrm{e}^x = \Sigma(x^n/n!\ |n, \mathrm{N}_0)$

$[\ \mathrm{P}2{\cdot}3\ .\supset.\ \mathrm{e}^x = \lim\,(1+x/m)\upharpoonright m\ |m$

$\S\Sigma\,15{\cdot}1\ .\supset.$ » $= \lim\Sigma[\mathrm{C}(m,r)x^r\,/m^r\,|r, \mathrm{N}_0]\ |m$ (1)

$\lim \mathrm{C}(m,r)\,x^r\,/m^r\,|m = x^r\,/r!\,\lim\{\Pi[(1-s/m)|s, 0\cdots(r-1)]\ |m = x^r\,/r!$ (2)

$\mathrm{l}'\,\mathrm{mod}$ » » » $`\mathrm{N}_1 = (\mathrm{mod}\,x)^r\,/\,r!\ .\ \Sigma[(\mathrm{mod}x)^r\,/r!\ |r, \mathrm{N}_0]\ \varepsilon\mathrm{Q}_0$ (3)

$(1)\,.\,(2)\,.\,(3)\ .\ \mathrm{Comm}(\lim, \Sigma)\ .\supset.\ \mathrm{P}\ ]$

Pro omni valore de quantitate x:

$$\mathrm{e}^x = 1+x+x^2/2!+x^3/3!+\ldots$$

Demonstratio: e^x vale limite de $(1+x/m)\upharpoonright m$, quando m varia, per propositio praecedente.

Formula de binomio da evolutione de $(1+x/m)^m$, secundo formula (1).

Termine de gradu r, in ce evolutione quando m varia, habe pro limite $x^r\,/r\,!$ (2)

Es satisfacto conditione, ita ut limite de serie vale serie de limite. (3)

Ergo subsiste propositio.

| NEWTON, 13 junii, a.1676:

« (Area hyperbolae) $= \frac{z}{b} + \frac{zz}{2abb} + \frac{z^3}{6aab^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5}$ etc. ubi coefficientes denominatorum prodeunt multiplicando terminos hujus arithmeticae progressionis, 1, 2, 3, 4, 5 etc. in se continuo; et hinc ex logarithmo dato potest numerus ei competens inveniri. » |

·2 $e = 1 + \Sigma /(N_1!) = \Sigma(/n! | n, N_0)$ [P·1 . $x=1$.⊃. P] Dfp

·21 P4·2 .⊃. P4·1

[$f = \Sigma(x^n / n! | n, N_0)$. $x,y \varepsilon$ q . §limP27·1 .⊃. $fx \times fy = f(x+y)$. $f1 = e$.
§ q P13·4(pag.118) .⊃. $fx = e^x$]

Si in serie ·1 nos pone $x = 1$, ori serie multo convergente pro numero e. Vice-versa, si nos sume ce serie pro definitione de e, seque serie generale ·1.

·3 $n \varepsilon N_1$.⊃. $e \varepsilon \Sigma(/r! | r, 0 \cdots n) + \theta/(n!n)$

| FOURIER; Vide Stainville, *Mélanges d'Analyse*, a.1815 p.339; CAUCHY a.1821 p.118 |

[P·2 . $n \varepsilon N_1$.⊃. $e = \Sigma(/r! | r, 0 \cdots n) + \Sigma(/r! | r, n+N_1)$ (1)

$\Sigma(/r! | r, n+N_1) = /(n+1)! \; |1 + /(n+2) + /[(n+2)(n+3)] + \ldots|$

$< /(n+1)! \; [1 + /(n+1) + /(n+1)^2 + /(n+1)^3 + \ldots]$

$= /(n+1)! \; (n+1)/n = /(n!n)$ (2)

(1) . (2) .⊃. P]

Resto in serie ·2, truncato post $n+1$ termine, es minore de ultimo termine scripto, diviso per n.

Si, per exemplo, nos pone $n = 10$, et calcula primos termine, cum 8 cifra decimale, in defectu, nos habe:

1 + 1	=	2· 00000000 ,
1/2 !	=	0· 50000000 ,
1/3 !	=	0· 16666666 ,
1/4 !	=	0· 04166666 ,
1/5 !	=	0· 00833333 ,
1/6 !	=	0· 00138888 ,
1/7 !	=	0· 00019841 ,
1/8 !	=	0· 00002480 ,
1/9 !	=	0· 00000275 ,
1/10 !	=	0· 00000027
Summa	=	2· 71828177

Ergo e > 2·71828177, nam nos calcula termines in defectu, et supprime resto. Si nos calcula termines in excessu, id es, si nos adde 8×10⌃−8, et si nos adde 3×10⌃−8, que supera resto, id es ultimo termine scripto diviso per 10, seque e < 2·71828188.

Unde, cum errore minore de 10⌃−7:

e = 2·7182818.

Si in serie ·1 nos pone $x=1/2$, $x=1/3$,... , nos obtine:

$\sqrt{e} = 1 + 1/2 + 1/(2\times4) + 1/(2\times4\times6) + \ldots$,

$\sqrt[3]{e} = 1 + 1/3 + 1/(3\times6 + 1/(3\times6\times9) + \ldots$,

serie multo commodo pro calculo numerico.

Si nos muta x in $-x$, resulta:

$e\curvearrowright -x = 1 - x + x^2/2! - x^3/3! + \ldots$

serie cum termines de signo alterno (per $x>0$). Resto es minore de primo termine relicto:

·4 $x \varepsilon Q . n \varepsilon N_0 .\supset. e^{-x} - \Sigma[(-x)^r/r! | r, 0\cdots n] \varepsilon \theta(-x)^{n+1}/(n+1)!$

Quod resulta de §lim P25·1, si termines de serie es decrescente, id es si $x<1$. Sed ce limitatione non es necessario.

·5 $n \varepsilon N_1 . a \varepsilon \text{rf } 1\cdots n .\supset. e^n + \Sigma(a_r e^{n-r} | r, 1\cdots n) -= 0$

{ Hermite a.1873 ParisCR. t.77 ; cfr. Gordan a.1893 MA. t.43 }

Eulero proba que e, et suo quadrato, es irrationale (P1·6). Lambert demonstra que, si x es rationale tunc e^x es irrationale (P1·7). In fine Hermite proba que e non es irrationale algebrico, id es, que numero e es radice de nullo aequatio algebrico, ad coefficientes rationale. Numero que non es «algebrico» vocare «transcendente».

·6 $n \varepsilon N_1 . x \varepsilon q - \iota 0 .\supset.$

$(e^{nx}-1) / (e^x - 1) = \Sigma\{x^r \Sigma[0\cdots(n-1)]^r / r! | r, N_0\}$

·7 $\lim n / \sqrt[n]{(n!)} | n = e$ Dfp

$[\ldots = \lim \sqrt[n]{(n^n / n!)} | n$

§limP11·5 $.\supset. \ldots = \lim[(n+1)^{n+1} / (n+1)! / n^n \times n!] | n$

$= \lim[(n+1)/n]^n | n = \lim(1+/n)^n | n = e$]

·8 $\lim n / \sqrt[n]{[(2n)!/n!]} | n = e/4$

❋ 5·1 $a \varepsilon q .\supset. \lim [(e^{ax}-1)/x | x, q, 0] = a$

$[\ldots = \lim(1 + ax + (ax)^2/2! + \ldots - 1)/x | x, q, 0]$

$= a + \lim[x \times (a^2/2! + a^3x/3! + \ldots) | x, q, 0]$

$=$ » » » $| x, \pm\theta, 0]$

$= a + \lim[\pm\theta x \times (\text{mod}a^2/2! + \text{mod}a^3/3! + \ldots) | x, \ldots]$

$= a + \lim[\pm\theta x \times (e\curvearrowright \text{mod}a - 1 - \text{mod}a) | x, \ldots] = a$]

·2 $a \varepsilon Q .\supset. \log a = \lim [(a^x - 1)/x | x, q, 0]$ Dfp

[$(\log a \mid a)$P·1 $\supset$ P]

·3 $a \varepsilon Q .\supset. \log a = \lim n(\sqrt[n]{a} - 1) | n$

[P·2 $\supset$ P] [P2·4 $\supset$ P]

·4 $\lim(\log, Q, \infty) = \infty$

·5 $\lim(\log x / x | x, Q, \infty) = 0$

·6 $m, n \varepsilon \mathrm{Q} .\supset. \lim[(\log x)^m / x^n \,|x, \mathrm{Q}, \infty] = 0$

·7 $x \varepsilon \mathrm{Q} .\supset. \lim(\log, \mathrm{Q}, x) = \log x$ Comm(lim, log)

·8 $\log \; \varepsilon \; (\mathrm{qfQ})\mathrm{cont}$ [P·7 . Df cont $.\supset$ P]

✻ 6·1 $x \varepsilon \mathrm{q} . -1 < x \leqq 1 .\supset. \log(1+x) = \Sigma[(-1)^{r-1} x^r / r \,|r, \mathrm{N}_1]$
$= x - x^2/2 + x^3/3 - \ldots$

{ MERCATOR a.1668 p.32:

« Hinc posito ..... 0[1 = numero terminorum : invenio, ...
Versione: Me pone 0·1 = x et inveni
aream ...= numero terminorum = 0[1, minus summa eorundem
$\log(1+x)$ = x = 0·1 — $\int x \mathrm{d}x = x^2/2$
terminorum = 0|005, plus summa quadratorum ab iisdem =
= 0·005 + $\int x^2 \mathrm{d}x = x^3/3$ — etc.
0|000333333, minus summa cuborum = 0|000025, plus summa quadrato - quadratorum = 0|000002, minus summa quadrato cuborum = 0|000000116, plus summa cubo - cuborum = 0|000000013, &c. » }

[$x \varepsilon \mathrm{q} . \mathrm{mod}x < 1$. P3·1 $.\supset. \log(1+x) = \lim n[\sqrt[n]{(1+x)} - 1] \,|n$
§lim 41·1 $.\supset.$ » $= \lim n\{\Sigma[\mathrm{C}(/n, r) x^r \,|r, \mathrm{N}_0] - \mathbf{1}\} \,|n$
» $= \lim \Sigma[n\,\mathrm{C}(/n, r) x^r \,|r, \mathrm{N}_1] \,|n$ (1)
$r \varepsilon \mathrm{N}_1 .\supset. \lim n\mathrm{C}(/n, r) x^r \,|n = (-\mathbf{1})^{r-1} x^r / r$ (2)
» l′mod » $|n‘\mathrm{N}_1 = (\mathrm{mod}x)^r / r$ (3)
$\mathrm{mod}x < 1 .\supset. \Sigma[(\mathrm{mod}x)^r / r \,|r, \mathrm{N}_1] \,\varepsilon\, \mathrm{Q}_0$ (4)
» .(**1**).(2).(3).(4) . Comm(lim, Σ) $.\supset.$ Ths (5)
§lim25·4 $.\supset. \Sigma[(-1)^{r-1}/r \,|r, \mathrm{N}_1] \varepsilon \mathrm{Q}$. §lim60·4 $.\supset.$
$\Sigma[(-1)^{r-1}/r \,|r, \mathrm{N}_1] = \lim\{\Sigma[(-1)^{r-1} x^r / r \,|r, \mathrm{N}_1] \,|x, \theta, 1\}$
$= \lim[\log(1+x) \,|x, \theta, 1] = \log 2$ (6)
(5) . (6) $.\supset.$ P]

·2 $x \varepsilon \theta .\supset. \log[(1+x)/(1-x)] = 2(x + x^3/3 + x^5/5 + \ldots)$

[P·1 $.\supset. \log(1+x) = x - x^2/2 + x^3/3 - \ldots . \log(1-x) = -x - x^2/2 - \ldots .\supset.$ P]

·3 $x \varepsilon \mathrm{Q} .\supset. \log(x+1) - \log x = 2\{/(2x+1) + /[3(2x+1)^3] + \ldots\}$
$= 2\Sigma\{/[(2n+1)(2x+1)^{2n+1}] \,|n, \mathrm{N}_0\}$

{ GREGORIO a.1668 p.12 }

[$/(2x+1)] \,|x$ P·2 $.\supset.$ P]

·4 $\log 2 = 1 - /2 + /3 - /4 + \ldots$
$\log 2 = /2 + /(2 \times 2^2) + /(3 \times 2^3) + \ldots$
$\log 2 = 2[/3 + /(3 \times 3^3) + /(5 \times 3^5) + /(7 \times 3^7) + \ldots$

[$1|x$ P·1 . $(-/2)|x$ P·1 . $1|x$ P·3 $.\supset$ P]

P·1-·3 permitte calculo numerico de logarithmo naturale de dato numero, P. ex. pro calculo de log 2, nos deduce tres serie convergente; primo converge in modo lento. Ultimo es rapido. Me calcula 8 termine:

2/3	= 0· 666 666 666
2/81	= 0· 024 691 358
2/1215	= 0· 001 646 090
2/15309	= 0· 000 130 642
2/177147	= 0· 000 011 290
2/1948617	= 0· 000 001 026
2/20726199	= 0· 000 000 096
2/215233605	= 0· 000 000 009
Summa	= 0· 693 147 177

que es valore de log2 per defectu.

Nunc me calcula termines per excessu, id es me adde $8\times 10\uparrow -3$. Calcula resto per excessu:

$$2/(17\times 3\uparrow 17)+2/(19\times 3\uparrow 19)+\ldots..$$
$$< 2/(15\times 3\uparrow 17)+2/(15\times 3\uparrow 19)+\ldots..$$
$$= 2/(15\times 3\uparrow 15)\times(1/9+1/9\uparrow 2+\ldots..)$$
$$= 2/(15\times 3\uparrow 15)\times 1/8 = 0 \cdot 000\ 000\ 001$$

Valore per defectu, plus $10\times 10\uparrow -9$ es valore per excessu de log 2. Unde, cum 8 cifra decimale: $\log 2 = 0{\cdot}69314718\ldots..$

Si in P·3, $x = 4$, seque

$$\log 5 = 2\log 2 + 2[1/9+1/(3\times 9^3)+1/(5\times 9^5)+\ldots..] = 1{\cdot}60943791$$

et $\quad \log 10 = \log 2 + \log 5 = 2{\cdot}30258509.$

Suo reciproco es modulo de logarithmos decimale P2·2, id es factore que transforma logarithmos naturale in decimale.

Pro ultimo exemplo, me quaere logarithmo decimale de 11.

$$\mathrm{Log}11 = \mathrm{Log}(10\times 1{\cdot}1) = \mathrm{Log}10+\mathrm{Log}1{\cdot}1 = 1+\mathrm{Log}1{\cdot}1.$$

Si me voca m modulo de logarithmos decimale:

$$m = /\log 10 = 0{\cdot}43429\ldots..$$

et si me evolve in serie log1·1, secundo formola 6·1, seque

$$\mathrm{Log}1{\cdot}1 = m\log 1{\cdot}1 = m\times 0{\cdot}1 - m\times 0{\cdot}01/2 + m\times 0{\cdot}001/3 - m\times 0{\cdot}0001/4+\ldots$$

Me calcula cum 8 cifra primos termine:

1⁰ termine	= 0·04342944,
2⁰ »	= — 0·00217147,
3⁰ »	= 0·00014476,
4⁰ »	= — 0·00001085,
5⁰ »	= 0·00000086,
6⁰ »	= — 0·00000007.
Summa	= 0·04139267.

Errore, vel differentia inter Log1·1 et summa praecedente, es summa de classes:

$\theta\times 10\uparrow -8$ (errore in 1⁰ termine) $- \theta\times 10\uparrow -8$ (errore in 2⁰ termine) $+ \theta\times 10\uparrow -8 - \theta\times 10\uparrow -8 + \theta\times 10\uparrow -8 - \theta\times 10\uparrow -8$ (errore in ultimo termine)

$+\theta\times 10\upharpoonleft -8$ (errore in suppressione de resto de serie; nam per §lim P25·1, resto es fractione de primo termine relicto, que vale $10\upharpoonleft -8) =$ $(-3^{-}+4)10\upharpoonleft -8$. Ergo, summa praecedente minus 3, aut plus 4 unitate de ultimo ordine es valore per defectu et per excessu de Log1·1:

$$0{\cdot}04139264 < \mathrm{Log}1{\cdot}1 < 0{\cdot}04139271$$

unde $$\mathrm{Log}11 = 1{\cdot}0413927$$

cum errore $< 10\upharpoonleft -7$.

·5 $a,b\varepsilon Q .\supset.$
$\log[(a+b)/2] = (\log a+\log b)/2+\Sigma\{[(a-b)/(a+b)]^{2n}/(2n) \mid n, N_1\}$
[$(a+b)/2=\surd(ab)/\surd\{1-[(a-b)/(a+b)]^2\}$. P4·1 .⊃. P]

Si es noto logarithmo de numeros 2, 3, 7, 11, 13, ce formula, ubi serie es reducto ad suo primo termine, da logarithmo di omni numero cum errore inferiore ad $(10\upharpoonleft -7)/2$, (Koralek a. 1851, Cauchy s.1 t.11 p.383).

·6 $x\varepsilon Q . n\varepsilon N_0 .\supset.$
$\log(1+x) - \Sigma[(-1)^{r-1}x^r/r \mid r, 1\cdots n] \;\varepsilon\; (-1)^n\theta x^{n+1}/(n+1)$

·7 $x\varepsilon q . \mathrm{mod}x<1 .\supset. \log[x+\surd(1+x^2)] =$
$\Sigma[(-1)^r C(r-/2, r)x^{2r+1}/(2r+1) \mid r, N_0]$

·8 $x\varepsilon Q . n\varepsilon N_0 .\supset.$
$\log[x+\surd(1+x^2)] - \Sigma[(-1)^r C(r-/2, r)x^{2r+1}/(2r+1) \mid r, 0\cdots n]$
$\varepsilon\; \theta(-1)^{n+1} C(n+/2, n+1)x^{2n+3}/(2n+3)$

* 7·1 $m\varepsilon N_1 .\supset. \lim \Sigma / (n\cdots mn) \mid n = \log m$

·2 $m,n\varepsilon N_1 . m<n .\supset. \lim \Sigma/(mp\cdots np) \mid p = \log(m/n)$

{ Joh. Bernoulli II a.1729 CorrM. t.2 p.300:

« Si l'on coupe la progression harmonique $1/x$ ... en deux parties ... soit la raison du nombre des termes dans la première et seconde partie comme m à n, la somme de tous les termes de cette seconde partie sera $= \log[(m+n)/n]$. »

·3 $a\varepsilon\ (e\upharpoonleft -/e)^{-}1 .\supset.$
$\lim(a\upharpoonleft)^n 1 \mid n = \Sigma\{(n+1)^{n-1}(\log a)^n/n! \mid n, N_0\}$

{ Euler PetrA. a.1777 t.1 }

·4 $x\varepsilon Q .\supset.$
$\log x = [(x-x^{-1})/2] / \Pi\, [(x\upharpoonleft 2^{-r}+x\upharpoonleft -2^{-r})/2 \mid r, N_1]$
$\log x = [(x-1)/\Pi\, [(1+x\upharpoonleft 2^{-r})/2 \mid r, N_1] =$
$(1-x^{-1})/\Pi\, [(1+x\upharpoonleft -2^{-r})/2 \mid r, N_1]$

{ Seidel JfM. a.1871 t.73 p.276, 278 }

* 8·1 $\min(x^x \mid x\ {}^{\backprime}Q) = e\upharpoonleft(-/e)$

·2 $\max[(x\upharpoonleft/x) \mid x\ {}^{\backprime}Q] = e\upharpoonleft/e$

E β ✽ 9·1 $\text{E}\,\text{e} = 2$. $\text{E}\,/\beta\,\text{e} = 1$

$n \varepsilon \text{N}_1 .\supset. \text{E}(/\beta)^{3n-1}\text{e} = 2n . \text{E}(/\beta)^{3n}\text{e} = \text{E}(/\beta)^{3n+1}\text{e} = 1$

{ Cotes a.1714 *Logometria*, p.7:

« Dividatur ... 2,71828 &c. per 1, ... & rursus minor per numerum qui reliquus est, & hic rursus per ultimum residuum, atque ita porro pergatur: & prodibunt quotientes 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12. 1, 1, 14, 1, 1, 16, 1, 1, &c. » }

·2 $n \varepsilon \text{N}_1 .\supset. \text{E}\,(/\beta)^n[(\text{e}-1)/(\text{e}+1)] = 2+4(n-1)$

{ Euler a.1748 p.319 }

Super alio fractio continuo relativo ad numero e, vide §Fc P11.

Np ✽ 10·1 $a \varepsilon \theta .\supset.$

$\lim \{[\text{Num}(\text{Np} \frown 2\cdots n) - \Sigma/\log`(2\cdots n)]\times n\,\rceil(a+/2)\} \mid n = 0$

{ Jensen AM. a.1899 t.22 p.364 }

·2 $n \varepsilon \text{N}_1 .\supset.$

$\lim\{[\text{Num}\ \text{Np}\frown 1\cdots x]/x - \Sigma[r!/(\log x)^{r+1} \mid r, 0\cdots n]\}(\log x)^{n+2} \mid x = (n+1)!$

{ Tchebychef JdM. a.1848 t.17 p.384, Œuvres t.1 p.44 }

Subst e

✽ 11. $n \varepsilon \text{N}_1 . a,b \varepsilon \text{Subst}n .\supset.$

·0 $\text{e}^a = \Sigma[(a^n/n!) \mid n, \text{N}_0]$ Df

Si a es substitutione de ordine n, e$\rceil a$ indica summa de serie P5·1, semper convergente.

·1 $\text{e}^a \,\varepsilon\, \text{Subst}\,n$

[$r\varepsilon\text{N}_1$. §lin 2·4 $.\supset.$ $\text{mod}(a^r/r!) \leqq (\text{mod}a)^r / r!$ (1)
P5·1 $.\supset.$ $\Sigma[(\text{mod}a)^r/r! \mid r, \text{N}_0] \,\varepsilon\text{Q}$ (2)
(1) . (2) $.\supset.$ $\Sigma[\text{mod}(a^r/r!) \mid r, \text{N}_0] \,\varepsilon\text{Q}$ (3)
(3) . §lim 51·4 $.\supset.$ $\Sigma[(a^r/r!) \mid r, \text{N}_0] \,\varepsilon\, \text{Subst}\,n$ (4)
(4) . P·0 $.\supset.$ P]

·2 $\text{e}^a = \lim(1+a/n)^n \mid n$ [(Subst|q)P5·1 $\supset$ P]

·3 $ab = ba .\supset. \text{e}^{a+b} = \text{e}^a\text{e}^b$

[P·0 $.\supset.$ $\text{e}^a\,\text{e}^b = \Sigma(a^r/r! \mid r, \text{N}_0)\times\Sigma(b^r/r! \mid r, \text{N}_0)$
§lim 35·1 $.\supset.$ » $= \Sigma\{\Sigma[a^{n-s}b^s/[(n-s)!\,s!] \mid s, 0\cdots n] \mid n, \text{N}_0\}$
§Σ 15·1 $.\supset.$ » $= \Sigma[(a+b)^n/n! \mid n, \text{N}_0]$
P·0 $.\supset.$ » $= \text{e}^{a+b}$]

·4 Dtrm e$\rceil a$ = e$\rceil$Invar$a$

q′ e

❋ 12. $x,y\varepsilon\mathrm{q} \,.\, a,b\varepsilon\mathrm{q}' \,.\, m\varepsilon\mathrm{n} \,.\supset.$

·0 $\mathrm{e}^a = 1 + a + a^2/2! + a^3/3! + \ldots$ Dfp [P 11·0 $\supset$ P]

·1 $\mathrm{e}^{a+b} = \mathrm{e}^a\mathrm{e}^b$ [P 11·3 $\supset$ P]

·2 $\mathrm{e}^{-a} = /\mathrm{e}^a$ ·3 $(\mathrm{e}^a)^m = \mathrm{e}^{ma}$

·4 $\mathrm{K}\,\mathrm{e}\frown(\mathrm{i}x) = \mathrm{e}\frown(-\mathrm{i}x)$

[$\mathrm{K}\,\mathrm{e}\frown(\mathrm{i}x) = \mathrm{K}(1 + \mathrm{i}x - x^2/2! - \mathrm{i}x^3! + x^4/4! - \ldots)$
$= (1 - \mathrm{i}x - x^2/2! + \mathrm{i}x^3/3! - \ldots)$
$= \mathrm{e}\frown(-\mathrm{i}x)$]

·5 $\mathrm{mod}\,\mathrm{e}^{\mathrm{i}x} = 1$ [$\mathrm{mod}\,\mathrm{e}^{\mathrm{i}x} = \sqrt{}(\mathrm{e}^{\mathrm{i}x}\,\mathrm{K}\,\mathrm{e}^{\mathrm{i}x}) = \sqrt{}(\mathrm{e}^{\mathrm{i}x}\,\mathrm{e}^{-\mathrm{i}x}) = 1$]

·6 $\mathrm{mod}\,\mathrm{e}^a = \mathrm{e}\frown(\mathrm{real}\,a)$

[$\mathrm{mod}\,\mathrm{e}^a = \mathrm{mod}[\mathrm{e}\frown(\mathrm{real}\,a + \mathrm{i}\,\mathrm{imag}\,a)] = \mathrm{mod}(\mathrm{e}\frown\mathrm{real}\,a)\,\mathrm{mod}(\mathrm{e}\frown\mathrm{i}\,\mathrm{imag}\,a)$
$= \mathrm{e}\frown\mathrm{real}\,a$]

sin cos tng e

❋ 13. $x\varepsilon\mathrm{q} \,.\supset.$

·1 $\cos x = \mathrm{c}x = \mathrm{real}\,\mathrm{e}^{\mathrm{i}x}$ Df

·2 $\sin x = \mathrm{s}x = \mathrm{imag}\,\mathrm{e}^{\mathrm{i}x}$ Df

·3 $x,y\varepsilon\mathrm{q} \,.\, m\varepsilon\mathrm{N}_1 \,.\supset.\, \mathrm{s}x+y = (\mathrm{s}x)+y \,.\, \mathrm{s}xy = (\mathrm{s}x)y \,.\, \mathrm{s}x^m = (\mathrm{s}x)^m$ Df

·31 $(\mathrm{c}|\mathrm{s})\mathrm{P}·3$ Df

Parte reale et coefficiente de i in $\mathrm{e}^{\mathrm{i}x}$, ubi x es numero reale, es cosinu et sinu de numero x. Relatione importante inter Analysi et Trigonometria, exposito in modo claro per Eulero. Omni formula de Trigonometria deriva in modo simplice de proprietates de exponentiale. Vide Dem de P13·7 , 14·1, etc.

Abbreviationes s, t, per sin, tng es comune ad mathematicos anglo de seculo XVII (Oughtred, Trigon. a.1657 p.4, Jones a.1706 p.240). Illos es de novo introducto per Gudermann in functiones elliptico. Idem abreviatione s, c... es et in Le Verrier a.1848 JdM. t.8 p.294; illos es indispensabile in longo serie de formula.

Secundo conventione §f 9, $\sin^2 x$ significa $\sin(\sin x)$, et non $(\sin x)^2$, que nos indica per $\mathrm{s}x^2$, secundo Euler, Stolz, etc.

Lagrange seque notatione exacto (mutato ab editores de OEuvres!).

Notatione $\sin^2 x$ pro indica $(\sin x)^2$ es in Legendre a.1811 Exercices t.1 p.9; adoptato per Jacobi a.1827.

Gauss, que adopta et notatione de Formul., dice: einige neuere mathematische Schrifsteller für das Quadrat von cosA, $\cos^2 A$ gebrauchen, *ganz gegen alle Analogien* ».

$x \varepsilon \mathrm{q} \,.\supset:$

·4 $\mathrm{c}x = (\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x})/2 \;.\; \mathrm{s}x = (\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x})/(2\mathrm{i})$ Dfp

·5 $\mathrm{e}^{\mathrm{i}x} = \mathrm{c}x + \mathrm{i}\,\mathrm{s}x \;.\; \mathrm{e}^{-\mathrm{i}x} = \mathrm{c}x - \mathrm{i}\,\mathrm{s}x$

} Euler a.1748 p.104 { [Df s,c $\supset$ P·4·5]

·6 $\mathrm{s}0 = 0 \;.\; \mathrm{c}0 = 1 \;.\; \mathrm{s}-x = -\mathrm{s}x \;.\; \mathrm{c}-x = \mathrm{c}x$ [P·4 $\supset$ P]

·7 $\mathrm{c}x^2 + \mathrm{s}x^2 = 1$ [P·5 . Oper$\times$ $.\supset.$ P]

·8 $-1 \leq \mathrm{s}x \leq 1 \;.\; -1 \leq \mathrm{c}x \leq 1$ [P·7 $\supset$ P]

✻ 14. $x,y,z,t \varepsilon \mathrm{q} \,.\supset.$

·1 $\mathrm{s}(x+y) = \mathrm{s}x\,\mathrm{c}y + \mathrm{c}x\,\mathrm{s}y \;.\; \mathrm{c}(x+y) = \mathrm{c}x\,\mathrm{c}y - \mathrm{s}x\,\mathrm{s}y$

» — » — » . » — » + »

[$\mathrm{e}\wedge(\mathrm{i}x)\,\mathrm{e}\wedge(\mathrm{i}y) = \mathrm{e}\wedge[\mathrm{i}(x+y)]$. Oper real . Oper imag $.\supset.$ P]

} Abû'lwéfa a.998; *Journal Asiatique* a.1892 s.8 t.19 p.419 {

·2 $\mathrm{s}(x-y)\,\mathrm{s}(z-t) + \mathrm{s}(y-z)\,\mathrm{s}(x-t) + \mathrm{s}(z-x)\,\mathrm{s}(y-t) = 0$

} Ptolemæo t.1 p.36 {

[§n 6·2 $.\supset.$ $(\mathrm{e}^{2x\mathrm{i}} - \mathrm{e}^{2y\mathrm{i}})(\mathrm{e}^{2z\mathrm{i}} - \mathrm{e}^{2t\mathrm{i}}) + (\mathrm{e}^{2y\mathrm{i}} - \mathrm{e}^{2z\mathrm{i}})(\mathrm{e}^{2x\mathrm{i}} - \mathrm{e}^{2t\mathrm{i}}) + (\mathrm{e}^{2z\mathrm{i}} - \mathrm{e}^{2x\mathrm{i}})(\mathrm{e}^{2y\mathrm{i}} - \mathrm{e}^{2t\mathrm{i}}) = 0 \,.\supset.$

$\mathrm{e}^{(x+y)\mathrm{i}}(\mathrm{e}^{(x-y)\mathrm{i}} - \mathrm{e}^{-(x-y)\mathrm{i}})\,\mathrm{e}^{(z+t)\mathrm{i}}(\mathrm{e}^{(z-t)\mathrm{i}} - \mathrm{e}^{-(z-t)\mathrm{i}}) + \ldots = 0$.

$\supset.$ $(\mathrm{e}^{(x-y)\mathrm{i}} - \mathrm{e}^{-(x-y)\mathrm{i}})(\mathrm{e}^{(z-t)\mathrm{i}} - \mathrm{e}^{-(z-t)\mathrm{i}}) + \ldots = 0 \,.\supset.$ P]

✻ 15·1 $x \varepsilon \mathrm{q} \,.\, m \varepsilon \mathrm{N}_1 \,.\supset.$

$\mathrm{c}(mx) = \mathrm{real}(\mathrm{c}x + \mathrm{i}\,\mathrm{s}x)^m$

$= \Sigma\{(-1)^r \mathrm{C}(m, 2r)(\mathrm{c}x)^{m-2r}(\mathrm{s}x)^{2r} \mid r, 0 \cdots \mathrm{E}(m/2)\}$.

$\mathrm{s}(mx) = \mathrm{imag}(\mathrm{c}x + \mathrm{i}\,\mathrm{s}x)^m$

$= \Sigma\{(-1)^r \mathrm{C}(m, 2r+1)(\mathrm{c}x)^{m-2r-1}(\mathrm{s}x)^{2r+1} \mid r, 0 \cdots \mathrm{E}[(m-1)/2]\}$

[$\mathrm{e}\wedge(m\mathrm{i}x) = [\mathrm{e}\wedge(\mathrm{i}x)]\wedge m$. Oper real . Oper imag $.\supset.$ P]

} Vieta a.1615 p.11 : « Si fuerint duo triangula quorum angulus acutus primi [x], sit submultiplus ad angulum acutum secundi [mx] ...

Ad similitudinem laterum circa rectum ... efficitur a base [cos$x$] et perpendiculo [sin$x$] primi ut binomia radice potestas æque-alta [$(\cos x + \sin x)^m$], et singularia facta homogenea distribuuntur in duas partes successive, utrobique primum affirmata, deinde negata, et harum primæ parti similis fit basis secundi [cosmx], perpendiculum [sinmx] reliquæ.» {

P·1 vocare saepe « formula de Moivre ».

·2 $x \varepsilon \mathrm{q} \,.\, m \varepsilon \mathrm{N}_1 + 1 \,.\supset.$

$\mathrm{s}(2mx) = \mathrm{c}x\, \Sigma\{(-1)^r 2^{2r+1} \mathrm{C}(m+r, 2r+1)(\mathrm{s}x)^{2r+1} \,|\, r, 0 \cdots (m-1)\}$

$\mathrm{s}[(2m+1)x] = (2m+1) \Sigma\{[(-1)^r 2^{2r}/(2r+1)] \mathrm{C}(m+r, 2r)(\mathrm{s}x)^{2r+1} \,|\, r, 0 \cdots m\}$

$\mathrm{s}(mx) = \mathrm{s}x\, \Sigma\{(-1)^r \mathrm{C}(m-r-1, r)(2\mathrm{c}x)^{m-2r-1} \,|\, r, 0 \cdots \mathrm{E}[(m-1)/2]\}$

$\mathrm{c}(2mx) = 1 - 2m \Sigma\{[(-1)^r 2^{2r}/(r+1)] \mathrm{C}(m+r, 2r+1)(\mathrm{s}x)^{2r+2} \,|\, r, 0 \cdots (m-1)\}$

$\mathrm{c}[(2m+1)x] = \mathrm{c}x\, \Sigma\{(-1)^r 2^{2r} \mathrm{C}(m+r, 2r)(\mathrm{s}x)^{2r} \,|\, r, 0 \cdots m\}$

$2\mathrm{c}(mx) = (2\mathrm{c}x)^m -$

$\quad m \Sigma\{[(-1)^r/(r+1)] \mathrm{C}(m-r-2, r)(2\mathrm{c}x)^{m-2r-2} \,|\, r, 0 \cdots \mathrm{E}[(m-2)/2]\}$

{ Euler a.1748 t.1 p.206 }

·3 $m \varepsilon 2\mathrm{N}_1 \,.\, x \varepsilon \mathrm{q} \,.\supset.$

$(\mathrm{c}x)^m = \Sigma\{\mathrm{C}(m, r)\, \mathrm{c}[(m-2r)x] \,|\, r, 0 \cdots (m-2)/2\}/2^{m-1} + \mathrm{C}(m, m/2)/2^m$

$(\mathrm{s}x)^m = (-1)^{(m/2)} \Sigma\{(-1)^r \mathrm{C}(m, r)\, \mathrm{c}[(m-2r)x] \,|\, r, 0 \cdots (m-2)/2\}/2^{m-1}$

$\quad + \mathrm{C}(m, m/2)/2^m$

·4 $m \varepsilon 2\mathrm{N}_0 + 1 \,.\, x \varepsilon \mathrm{q} \,.\supset.$

$(\mathrm{c}x)^m = \Sigma\{\mathrm{C}(m, r)\, \mathrm{c}[(m-2r)x] \,|\, r, 0 \cdots (m-1)/2\}/2^{m-1}$

$(\mathrm{s}x)^m = (-1)^{(m-1)/2} \Sigma\{(-1)^r \mathrm{C}(m, r)\, \mathrm{s}[(m-2r)x] \,|\, r, 0 \cdots (m-1)/2\}/2^{m-1}$

·5 $x, y \varepsilon \mathrm{q} \,.\, \sin y -= 0 \,.\, m \varepsilon \mathrm{N}_1 \,.\supset.$

$\Sigma[\mathrm{s}(x+2ry) \,|\, r, 0 \cdots m] = \mathrm{s}(x+my)\, \mathrm{s}[(m+1)y] / \mathrm{s}y$

$\Sigma[\mathrm{c}(x+2ry) \,|\, r, 0 \cdots m] = \mathrm{c}(x+my)\, \mathrm{s}[(m+1)y] / \mathrm{s}y$

[$\Sigma\{[\mathrm{c}(x+2ry) + \mathrm{i}\,\mathrm{s}(x+2ry)] \,|\, r, 0 \cdots m\} =$

$\Sigma[\mathrm{e}\wedge[\mathrm{i}(x+2ry)] \,|\, r, 0 \cdots m] = \mathrm{e}\wedge(\mathrm{i}x)\{\mathrm{e}\wedge[2(m+1)\mathrm{i}y] - 1\}/[\mathrm{e}\wedge(2y\mathrm{i}) - 1]$

$= \mathrm{e}\wedge[\mathrm{i}(x+my)]\{\mathrm{e}\wedge[(m+1)\mathrm{i}y] - \mathrm{e}\wedge[-(m+1)\mathrm{i}y]\}/[\mathrm{e}\wedge(\mathrm{i}y) - \mathrm{e}\wedge(-\mathrm{i}y)]$

$= \mathrm{e}\wedge[\mathrm{i}(x+my)]\, \mathrm{s}[(m+1)y] / \mathrm{s}y$]

{ Archimede, *de Sphaera et cylindro* I 21 }

·6 $x \varepsilon \mathrm{q} \,.\, \sin x -= 0 \,.\, m \varepsilon \mathrm{N}_1 \,.\supset.$

$2\, \Sigma[\mathrm{s}(rx)^2 \,|\, r, 0 \cdots m] = m - \mathrm{c}[(m+1)x]\, \mathrm{s}(mx) / \mathrm{s}x$

$2\, \Sigma[\mathrm{c}(rx)^2 \,|\, r, 0 \cdots m] = m + \mathrm{c}(mx)\, \mathrm{s}[(m+1)x] / \mathrm{s}x$

* 16. $x \varepsilon \mathrm{q} \,.\supset.$

·1 $\mathrm{s}x = x - x^3/3! + x^5/5! - \ldots = \Sigma\{[(-1)^n x^{2n+1}/(2n+1)!] \,|\, n, \mathrm{N}_0\}$

·2 $\mathrm{c}x = 1 - x^2/2! + x^4/4! - \ldots = \Sigma\{[(-1)^n x^{2n}/(2n)!] \,|\, n, \mathrm{N}_0\}$

[$\mathrm{e}\wedge(\mathrm{i}x) = 1 + \mathrm{i}x - x^2/2 - \mathrm{i}\, x^3/3! + x^4/4! + \ldots$. Oper real . Oper imag $.\supset.$ P·1·2]

{ Newton a.1676 }

·3 $n \varepsilon \mathrm{N}_1 \,.\supset.$

$\mathrm{s}x - \Sigma[(-1)^r x^{2r+1}/(2r+1)! \,|\, r, 0 \cdots n] \;\varepsilon\; (-1)^{n+1} \theta\, x^{2n+3}/(2n+3)!$

$\mathrm{c}x - \Sigma[(-1)^r x^{2r}/(2r)! \,|\, r, 0 \cdots n] \;\varepsilon\; (-1)^{n+1} \theta\, x^{2n+2}/(2n+2)!$

Hp. $x \varepsilon \theta\pi/2$, que occurre in plure tractato, es inutile.

Dm in Peano *Lezioni* a.1893 t.1 p.83.

·4 $\lim(\sin x/x|x, q, 0) = 1$

[$\lim[(e^{ix}-1)/x|x, q, 0] = i$. Oper imag .⊃. P]

·5 $\lim[\mathrm{sgn}\ \sin x\ |x, Q, 0] = 1$

[$x\varepsilon Q$.⊃. $\mathrm{sgn}\ \sin x = \mathrm{sign}(\sin x/x)$. P·4 .⊃. P]

·6 $x\varepsilon q\ .\ r\varepsilon Q\ .\ \mathrm{mod}r<1$.⊃.

$(1-r\cos x)/(1-2r\cos x+r^2) = \Sigma[r^n\cos(nx)|n, N_0]$.

$r\sin x/(1-2r\cos x+r^2) = \Sigma[r^n\sin(nx)|n, N_1]$

[$\Sigma[(re^{ix})^n|n,N_0] = 1/(1-re^{ix})$. Oper real . Oper imag .⊃. P]

·7 $a,x,n\varepsilon q\ .\ \mathrm{mod}a<1$.⊃.

$(1+2a\cos x+a^2)^n = \Sigma[C(n,r)^2a^{2r}|r, N_0]$

$+2\Sigma\{a^r\cos(rx)\Sigma[a^{2s}C(n,s)C(n,r+s)|s, N_0]|r, N_1\}$

❋ 17. $x\varepsilon q\ .\ cx -= 0$.⊃. ·0 $\mathrm{tng}x = tx = sx/cx$ Df

·1 $tx = (e^{2ix}-1)/[i(e^{2ix}+1)]$. $t0 = 0$. $t-x = -tx$

Nos non introduce alio functione trigonometrico :

cotang = /tng, sec = /cos, cosec = /sin.

TABULA DE e^x

$x =$	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
$e^x =$	1·00	1·10	1·22	1·34	1·49	1·64	1·82	2·01	2·22	2·45
$x =$	1	2	3	4	5	6	7	8	9	10
$e^x =$	2·71	7·38	20·0	54·5	148	403	1096	2980	8103	22026

§4 π

❋ 1·0 $\pi = 2 \min Q \frown x \ni (cx = 0)$ Dfπ

Numero $\pi/2$ pote es definito, in modo analytico, ut minimo radice positivo de aequatione $\cos x = 0$, id es de aequatione $1 - x^2/2! + x^4/4! - \ldots = 0$.

Numero π se praesenta ut ratione de circumferentia ad diametro. Ce signo, introducto per Jones, adoptato per Euler, es hodie de usu generale. Illo es litera initiale de περίμετρος.

·1 $\cos(\pi/2) = 0$ [Df $\pi \supset$ P]

·2 $x \varepsilon \theta\pi/2 . \supset . \cos x > 0$ [Df$\pi \supset$ P]

·3 $x \varepsilon \theta\pi . \supset . \sin x > 0$

[$\sin x = 2\sin(x/2)\cos(x/2) . \cos(x/2) > 0 . \supset . \operatorname{sgn} \sin x = \operatorname{sgn} \sin(x/2)$ (1)

(1) . $n \varepsilon N_1 . \supset . \operatorname{sgn} \sin x = \operatorname{sgn} \sin(x/2^n)$ (2)

(2) $. \supset . \operatorname{sgn} \sin x = \lim \operatorname{sgn} \sin(x/2^n) | n$

P16·5 $. \supset . \operatorname{sgn} \sin x = 1$]

·4 $\sin(\pi/2) = 1$

[$\sin(\pi/2)^2 + \cos(\pi/2)^2 = 1 . \cos(\pi/2) = 0 . \sin(\pi/2) > 0 . \supset .$ P]

·5 $e^{\pi i/2} = i$ [P·1·4 $\supset$ P]

·6 $e^{\pi i} = -1 \quad . \quad e^{2\pi i} = 1$

❋ 2·1 $\pi/4 \, \varepsilon \, (8/9)^2 - \theta X^{-2}$ }Ahamesu a.—2000:

N.41. « 9 . 9/9 = 1 . 9—1 = 8 . 8×8 = 64 ».

N.42. « 10 . 10/9 = 1+/9 . 10—(1+/9) = 8+2/3+/6+/18 .

(8+2/3+/6+/18)² = 79+/108+/324 ». {

Papyro de mathematico aegyptio Ahamesu, de anno circa 2000 ante aera vulgare, es toto composito ut duo linea transcripto, ubi nos da ad cifra forma actuale. Interpretatione, que nos debe adde, es:

N. 41: Si nos suppone que 9 es diametro de circulo, tunc divide illo per 9, et habe 1; subtrahe ce nono de illo, et habe 8; eleva ad potestate 2, et habe (circa) area de circulo.

N. 42: Si 10 es diametro de circulo, tunc divide illo per 9, et habe 1+1/9; subtrahe ce nono de illo, et habe 8+2/3+/6+/18 (secundo methodo aegyptio, ubi omni fractione de denominatore >3, habe 1 ut numeratore). Eleva ce fractione ad quadrato, et resulta fractio scripto, que exprime area.

·2 $3+1/7>\pi>3+10/71$

} ARCHIMEDE, *Κύκλου μετρήσις* P3:

Παντὸς κύκλου ἡ περίμετρος τῆς διαμέτρου τριπλασίων ἐστί, καὶ ἔτι ὑπερέχει ἐλάσσονι μὲν ἢ ἑβδόμῳ μέρει τῆς διαμέτρου, μείζονι δὲ ἢ δέκα ἑβδομηκοστομόνοις. {

Versione: «Perimetro de panto cyclo es triplice de diametro, et quod super es minore que septimo parte de diametro, majore que decem diviso 71».

·3 $\pi \varepsilon 3+8\times60^{-1}+30\times60^{-2}-\theta\, 60^{-2}$ } PTOLEMAEO t.1 p.512:

...τοῦ λόγου τῶν περιμέτρων πρὸς τὰς διαμέτρους ὄντος, ὃ ἔχει τὰ $\overline{\gamma}\ \overline{\eta}\ \overline{\lambda}$ *πρὸς τὸ ἕν.* {

·4 $\pi \varepsilon 62832/20000-\theta X^{-5}$

} ARYABHATA p.399 (Versione de Rodet):

«Ajoutez 4 à 100, multipliez par 8, ajoutez encore 62000, voilà pour un diamètre de deux myriades (ayutâs) la valeur approximative de la circonférence du cercle» {

·5 $377/120>\pi>333/106$

} A. ANTHONISZ, voir BM. a.1888 p.36; a.1889 p.84 {

$\pi \varepsilon 355/113-\theta X^{-6}$ } A. METIO a.1625 p.88 {

·8 $\pi \varepsilon \sqrt{2}+\sqrt{3}-5\theta X^{-3}$ } IdM. a.1901 p.269 {

·81 $\pi \varepsilon \sqrt{(1+\sqrt{6})}+\sqrt{(9-3\sqrt{6})}-\theta X^{-3}$ } MASCHERONI a.1798 p.248 {

·82 $\pi \varepsilon 9/5+3/\sqrt{5}-\theta X^{-4}$ } VIETA a.1593 Opera p.393 {

·83 $\pi \varepsilon \sqrt{(40/3-2\sqrt{3})}+7\theta X^{-5}$ } KOCHANSKI AErud. a.1685 p.398 {

·84 $\pi \varepsilon (13\sqrt{146})/50+\theta X^{-6}$ } SPECHT JfM. a.1828 t.3 p.83 {

·85 $\pi \varepsilon (501+80\sqrt{10})/240-\theta X^{-6}$

} GERGONNE Ann. a.1817 t.8 p.252 {

·86 $\pi \varepsilon 4-2\sqrt{2}+2\sqrt{3}/3+\sqrt{6}/3-2\theta X^{-3}$

} George PEIRCE, AmericanB. a.1901 p.426 {

$\pi = 3\cdot$

14159 26535 89793 23846 26433 83279 50288 41971 69399 37510
58209 74944 59230 78164 06286 20899 86280 34825 34211 70679
82148 08651 32823 06647 09384 46095 50582 23172 53594 08128
48111 74502 84102 70193 85211 05559 64462 29489 54930 38196
44288 10975 66593 34461 28475 64823 37867 83165 27120 19091
45648 56692 34603 48610 45432 66482 13393 60726 02491 41273
72458 70066 06315 58817 48815 20920 96282 92540 91715 36436
78925 90360 01133 05305 48820 46652 13841 46951 94151 16094
33057 27036 57595 91953 09218 61173 81932 61179 31051 18548
07446 23799 62749 56735 18857 52724 89122 79381 83011 94912
98336 73362 44065 66430 86021 39501 60924 48077 23094 36285
53096 62027 55693 97986 95022 24749 96206 07497 03041 23668
86199 51100 89202 38377 02131 41694 11902 98858 25446 81639
79990 46597 00081 70029 63123 77381 34208 41307 91451 18398
05709 85 . . .

Vieta, *Canon mathematicus*, Lutetiæ, a.1579 p.15,	calcula	E(10 ↑ 9 π)
Adriano Romano, *Ideæ Math.*, Anvers, a.1613	»	»	15 »
Ludolpho a Ceulen (D. Cöln, L. Colonia) a.1615 p.144	»	»	32 »
Grienberger, *Elementa Trigonometrica*, Romæ a.1630	»	»	39 »
Sharp a.1699 (edito per H. Sherwin, *Mathematical tables* a.1705 p.59)	»	»	71 »
Machin, (edito per Jones, *Synopsis Palmarium Matheseos* a.1706 p.243, que designa illo per π)	»	»	100 »
Lagny, *Hist. de l'Acad. des Sc. de Paris*, a.1719 p.144	»	»	112 »
Vega, *Thesaurus Logarithmorum*, a.1794 p.633	»	»	136 »
Thibaut, *Grundriss der reinen Math.*, 4. ed. a.1822 p.312	»	»	156 »
Dahse a.1840; JfM. a.1844 t.27. p.198	»	»	200 »
Clausen a.1847 (edito per Schuhmacher, *Astronomische Nachrichten* t.25 col.207)	»	»	248 »
Richter, *Archives Math. de Grunert*, a.1853, t.21, p.119	»	»	330 »
Rutherford, LondonP. a.1853	»	»	440 »
Shanks, » »	»	»	530 »
» » « » a.1874, t. 23, p.45	»	»	707 »

π = !! · ..!..!.. ..!!!!!! .!!.!.!.

Calculo de π in basi 2, proposito per Leibniz (Opera a.1768 t.3 p.521, 547,...), facto per Jacob Bernoulli (a.1705, Leibniz MathS. t.3 p.97).

✻ 3·1 $\pi \varepsilon$ Q-R } LAMBERT BerlinM. a.1768 p.265-322 {

·2 π^2-εR } LEGENDRE *Géométrie* a.1794 note 4 {

·3 $n \varepsilon \mathrm{N}_1 \,.\, x \varepsilon \mathrm{rf}\, 1 \cdots n \;.\supset.\; \pi^n + \Sigma(x_r \pi^{n-r} | r, 1 \cdots n) \;-\!=\! 0$
} LINDEMANN a 1882 MA. t.20 p.213 ;
cfr. GORDAN a.1893 MA. t.43 p.222 {

✻ 4. $x \varepsilon \mathrm{q} \;.\supset:$ ·1 $\mathrm{s}x = 0 \;.=.\; x \varepsilon \mathrm{n}\pi$

·2 $\mathrm{s}(\pi - x) = \mathrm{s}x \quad . \quad \mathrm{s}(2\pi + x) = \mathrm{s}x$

·3 $\mathrm{c}x = 0 \;.=.\; x \varepsilon \pi/2 + \mathrm{n}\pi$

·4 $\mathrm{c}(\pi - x) = -\mathrm{c}x \quad . \quad \mathrm{c}(2\pi + x) = \mathrm{c}x$

·5 $\mathrm{c}x = \mathrm{s}(\pi/2 - x) \quad . \quad \mathrm{s}x = \mathrm{c}(\pi/2 - x)$ Dfp

✻ 5·1 $e^{\pi i/3} = (1 + i\sqrt{3})/2 \quad . \quad e^{\pi i/4} = (1+i)/\sqrt{2}$

$e^{\pi i/5} = [1 + \sqrt{5} + i\sqrt{(10 - 2\sqrt{5})}]/4 \quad . \quad e^{\pi i/6} = (\sqrt{3} + i)/2$

$e^{\pi i/8} = [\sqrt{(2+\sqrt{2})} + i\sqrt{(2-\sqrt{2})}]/2$

$e^{\pi i/10} = [\sqrt{(10 + 2\sqrt{5})} + i(\sqrt{5} - 1)]/4$

$e^{\pi i/12} = [\sqrt{6} + \sqrt{2} + i(\sqrt{6} - \sqrt{2})]/4$

$e^{\pi i/15} = e^{\pi i/6} \times e^{-\pi i/10}$

$= \{\sqrt{(30 + 6\sqrt{5})} + \sqrt{5} - 1 + i[\sqrt{(10+2\sqrt{5})} + \sqrt{3} - \sqrt{15}]\}/8$

$e^{\pi i/16} = \{\sqrt{[2 + \sqrt{(2+\sqrt{2})}]} + i\sqrt{[2 - \sqrt{(2+\sqrt{2})}]}\}/2$

Constructione de polygonos regulare, correspondente ad formulas praecedente, es in Euclide IV P6-16.

$e^{\pi i/17} = \{[15 + \sqrt{17} + \sqrt{(34 - 2\sqrt{17})} + 2\sqrt{[17 + 3\sqrt{17} - \sqrt{(170 + 38\sqrt{17})}]}]$
$+ 4i\sqrt{[34 - 2\sqrt{17} - 2\sqrt{(\text{———})} - 4\sqrt{[\text{——————————}]}]}\}/32$
} GAUSS a.1801 t.1 p.462 {

$e^{\pi i/20} = \{\sqrt{(3+\sqrt{5})} + \sqrt{(5 - \sqrt{5})} + i[\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}]\}/4$

$e^{\pi i/30} = \{\sqrt{(18 + 6\sqrt{5})} + \sqrt{(10 - 2\sqrt{5})} + i[\sqrt{(30 - 6\sqrt{5})} - \sqrt{(6 + 2\sqrt{5})}]\}/8$

$e^{\pi i/60} = \{\sqrt{(5+\sqrt{5})} + \sqrt{(9 - 3\sqrt{5})} + \sqrt{(15 + 3\sqrt{5})} - \sqrt{(3 - \sqrt{5})}$
$+ i[\;\text{»}\quad\text{»}\quad-\quad\text{»}\quad\text{»}\;]\}/8$

✻ 6·0 $n \varepsilon \mathrm{N}_1 \;.\supset.\; {}^n\sqrt{}^{*}1 = [e \upharpoonright (2m\pi i/n)] \,|\, m \,\text{‘}\, 0 \cdots (n-1)$

·1 $n \varepsilon \mathrm{N}_1 \,.\, x \varepsilon \mathrm{q}' \;.\supset.$
$x^n - 1 = \Pi\{[x - e \upharpoonright (2m\pi i/n)] | m, 0 \cdots (n-1)\} \,.$
$x^n + 1 = \Pi[\{x - e \upharpoonright [(2m+1)\pi i/n]\} | m, 0 \cdots (n-1)]$
} COTES Rogero a.1722 p.114 {

·2 $n\varepsilon\, 2\mathrm{N}_1 \,.\, x\varepsilon \mathrm{q}' \,.\supset.$
$x^n-1 = (x^2-1)\Pi\{[x^2-2x\, \mathrm{c}(2m\pi/n)+1]\, |m, 1\cdots(n-2)/2\}\,.$
$x^n+1 = \Pi\{[x^2-2x\, \mathrm{c}[(2m+1)\pi/n]+1]\, |m, 0\cdots(n-2)/2\}$

·3 $n\varepsilon\, 2\mathrm{N}_1+1 \,.\, x\varepsilon \mathrm{q}' \,.\supset.$
$x^n-1 = (x-1)\Pi\{[x^2-2x\, \mathrm{c}(2m\pi/n)+1]\, |m, 0\cdots(n-1)/2\}\,.$
$x^n+1 = (x+1)\Pi\{[x^2-2x\, \mathrm{c}[(2m+1)\pi/n]+1]\, |m, 0\cdots(n-3)/2\}$
[= P·1]

·4 $n\varepsilon \mathrm{N}_1 \,.\supset.\, \Pi\{\mathrm{s}[r\pi/(2n)]|r, 1\cdots n\} = \sqrt{n}\,/\,2^{n-1}$
Dem. in Heinrich Weber, Lehrbuch der Algebra, Braunschweig, 1895, t. I, p. 578.

·5 $n\varepsilon \mathrm{N}_1 \,.\, a\varepsilon\, \mathrm{qF}\, 0\cdots n \,.\supset.$
$\mathrm{Dtrm}\{a[\mathrm{rest}(r+s, n)]\, |(r;s), (0\cdots n \,\dot{:}\, 0\cdots n)\} =$
$\Pi\{\Sigma(a_r x^r|r, 0\cdots n)|x, {}^{n+1}\sqrt{}*1\}$
{ J. W. Glaisher a.1879 QJ. t.16 p.31 }
Determinante (tabula) que figura in ce P dicere « circulante ».

❋ 7·1 $\pi/4 = \Sigma\{(-1)^n/(2n+1)\, |n, \mathrm{N}_0\}$ [§tang⁻¹ P7·1.$y=1$.⊃. P]
{ Leibniz a.1682 MathS. t.5 p.120:
« Quadrato Diametri existente 1,
Circuli aream fore $\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}$ etc.,
nempe quadratum diametri integrum demtą (ne nimius fiat valor) ejus tertia parte, addita rursus (quia nimium demsimus) quinta, demtaque iterum (quia nimium re-adjecimus) septima, et ita porro. »}

·2 $\pi/4 = /\Sigma\{[\mathrm{C}(/2, n)]^2\, |n, \mathrm{N}_0\} = /[1+/4+/(2\times 4)^2+\ldots]$
{ Gauss, Werke, t.3 }

·3 $\pi^2/6 = \Sigma\, \mathrm{N}_1^{-2}$
{ Euler a.1736, PetrC. a.1734-35 t.7 (edito a.1740) }

·4 $\pi^4/90 = \Sigma\mathrm{N}_1^{-4}$ ·41 $m\varepsilon\mathrm{N}_1 \,.\supset.\, \Sigma\mathrm{N}_1^{-2m}\, \varepsilon\, \mathrm{R}\pi^{2m}$
{ Joh. Bernoulli t.4 p.24 }

·5 $\pi^2/8 = 1+3^{-2}+5^{-2}+7^{-2}+\ldots$

·6 $\pi^3/32 = 1-3^{-3}+5^{-3}-\ldots$ $5\pi^5/1536 = 1-3^{-5}+5^{-5}-\ldots$
{ Euler a.1748 p.137 }

·7 $-\log(\pi/4) = \Sigma(2\mathrm{N}_1+1)^{-2}+\Sigma(2\mathrm{N}_1+1)^{-4}/2+\Sigma(2\mathrm{N}_1+1)^{-6}/3+\ldots$
{ Euler a.1748 p.150 }

·8 $\log(\pi^2/6) = \Sigma\mathrm{Np}^{-2}+\Sigma\mathrm{Np}^{-4}/2+\Sigma\mathrm{Np}^{-6}/3+\ldots$
{ Euler a.1748 p.235 }

·9 $\pi = 1 + /2 + /3 + /4 - /5 + /6 + /7 + /8 + /9 - /10 + \ldots\ldots$
$= \Sigma\{(/n)\times(-1)\wedge\Sigma[\mathrm{mp}(r,n)|r,\mathrm{Np}\frown(4\mathrm{N}_1-1)]|n,\mathrm{N}_1\}$
} Euler CM. t.1 p.557 {

✻ 8·1 $x\varepsilon\, \mathrm{q-n}$ $.\supset.$ $\pi^2/[\mathrm{s}(\pi x)]^2 = \Sigma[/(x+n)^2\ |n,\mathrm{n}]$

·2 $x\varepsilon\, \mathrm{q-n}$ $.\supset.$ $\pi/\mathrm{t}(\pi x) = /x + 2x\Sigma[/(x^2-n^2)\ |n,\mathrm{N}_1]$
$= \lim[\Sigma/(x-r)\,|r, -n\cdots n]\,|n$ } Euler a.1748 p.159 {

·3 $x\varepsilon\, \mathrm{q'-n\,i}$ $.\supset.$ $\pi[\mathrm{e}\wedge(\pi x) + \mathrm{e}\wedge(-\pi x)]/[\mathrm{e}\wedge(\pi x) - \mathrm{e}\wedge(-\pi x)] =$
$/x + \Sigma\{2x/(n^2+x^2)\ |n,\mathrm{N}_1\}$

·4 $a\varepsilon\, \theta\pi/2$ $.\supset.$ $/a = /\mathrm{t}a + \Sigma\{2^{-r}\mathrm{t}(2^{-r}a)\ |r,\mathrm{N}_1\}$
} Euler a.1746 CorrM. t.1 p.371 {

✻ 9.

·1 $x\varepsilon\, \mathrm{q}-2\mathrm{n}\pi\ .\ a\varepsilon\,(\mathrm{Qf\,N_0})\mathrm{decr}\ .\ \lim a = 0$ $.\supset.$ $\Sigma(a_r\mathrm{e}^{\mathrm{i}rx}\,|r,\mathrm{N}_0)\ \varepsilon\mathrm{q}'$

·2 $a\varepsilon\,(\mathrm{Qf\,N_0})\mathrm{decr}\ .\ \lim a = 0$ $.\supset:$
$x\varepsilon\mathrm{q}-2\mathrm{n}\pi$ $.\supset.$ $\Sigma[a_r\mathrm{c}(rx)|r,\mathrm{N}_0]\ \varepsilon\mathrm{q}$: $x\varepsilon\mathrm{q}$ $.\supset.$ $\Sigma[a_r\mathrm{s}(rx)|r,\mathrm{N}_0]\ \varepsilon\mathrm{q}$
[= P·1]

✻ 10.

·1 $\pi = 2(2/1)\,(2/3)\,(4/3)\,(4/5)\,(6/5)\,(6/7)\ldots$
$= 4\Pi[(1-n^{-2})\,|n, 2\mathrm{N}_1+1] = 2/\Pi[(1-n^{-2})\,|n,2\mathrm{N}_1]$
$= 2\Pi\{2\mathrm{E}[(n+1)/2]\,/\,[2\mathrm{E}(n/2)+1]\ |n,\mathrm{N}_1\}$
} Wallis a.1655 t.1 p.469:

« Dicimus, fractionem illam $\dfrac{3\times3\times5\times5\times7\times7\times\ \&c.}{2\times4\times4\times6\times6\times8\times\ \&c.}$ seu $\dfrac{9\times25\times49\times81\ \&c.}{8\times24\times48\times80\ \&c.}$ in infinitum continuatam, esse ipsissimum quaesitum numerum □ præcise ad quem ita se habet 1, ut Circulus ad Quadratum Diametri » {

·2 $n\varepsilon\mathrm{N}_1$ $.\supset.$ $\mathrm{C}(2n,n)\ < 2^{2n}/\surd(n\pi)$
$> 2^{2n}/\surd[(n+/2)\pi]$ [P·1 .⊃. P·2]

·3 $x\varepsilon\mathrm{q}$ $.\supset.$ $\mathrm{s}x = x\Pi[(1-x^2n^{-2}\pi^{-2})\,|n,\mathrm{N}_1]$ } Euler a.1748

·4 » $\mathrm{c}x = \Pi[(1-4x^2n^{-2}\pi^{-2})\,|n, 2\mathrm{N}_0+1]$ p.120{

·5 $x\varepsilon\, \theta\pi/2$ $.\supset.$ $\Pi\{\cos(x/2^n)\ |n,\mathrm{N}_1\} = \sin x\,/x$
} Euler a.1737 PetrC. t.9 p.235 {

·6 $x\varepsilon\mathrm{q}'$ $.\supset.$ $(\mathrm{e}^x-\mathrm{e}^{-x})/2 = x(1+x^2\pi^{-2})(1+x^2 2^{-2}\pi^{-2})\ldots$
———————— $= x\Pi[(1+x^2\pi^{-2}n^{-2})\,|n,\mathrm{N}_1]$

·7 $x \varepsilon q' . \supset . (e^x+e^{-x})/2 = (1+4x^2\pi^{-2})(1+4x^2 3^{-2}\pi^{-2})\ldots$
———————— $= \Pi[(1+4x^2\pi^{-2}n^{-2}) \mid n, 2N_0+1]$

} EULER a.1748 p.119,120 {

Functiones in P·6 et ·7 dicere sinu et cosinu «hyperbolico» et indicare per Shx, Chx (Riccati a.1757).

·8 $\lim(n!\, n^{-n} e^n/\surd n) \mid n = \surd(2\pi)$ } STIRLING a.1730 p.137 {

B π

❋ 11·1 $n \varepsilon N_1 . \supset . \Sigma(N_1^{-2n}) = 2^{2n-1}\pi^{2n} B_n/(2n)!$.
$2\Sigma(2N_0+1)^{-2n} = -\Sigma[(-1)^r/r^{2n} \mid r, N_1] = (2^{2n}-1)\pi^{2n} B_n/(2n)!$

} JOH. BERNOULLI t.4 p.21 {

·2 $\lim(n^2 B_n/B_{n+1}) \mid n = \pi^2$

·3 $\lim B_n(\pi e/n)^{\frown}(2n+/2) \mid n = 4\pi\surd e$

·4 $x \varepsilon q' - \iota 0 . \operatorname{mod} x < 2\pi . \supset .$
$x/(e^x-1) = 1 - x/2 + \Sigma[(-1)^{r+1} B_r x^{2r}/(2r)! \mid r, N_1]$

·5 $a \varepsilon N_1 . n \varepsilon N_1+1 . \supset . \log a! \ \varepsilon\ (\log 2\pi)/2 - a + (a+/2)\log a$
$+\Sigma\{(-1)^r B_{r+1}/[(2r+1)(2r+2)]\, a^{-2r+1} \mid r, 0\cdots(n-1)\} +$
$\theta(-1)^n B_{n+1}/[(2n+1)(2n+2)]\, a^{-2n+1}$ } STIRLING a.1730 {

❋ 12·1 $x \varepsilon q - \iota 0 . \operatorname{mod} x < \pi . \supset .$
$/\sin x = /x + \Sigma[2(2^{2n+1}-1) B_{n+1} x^{2n+1}/(2n+2)! \mid n, N_1]$

·2 $x \varepsilon q . \operatorname{mod} x < \pi/2 . \supset .$
$\operatorname{tng} x = \Sigma\{2^{2n}(2^{2n}-1) B_n x^{2n-1}/(2n)! \mid n, N_1\}$
$= x + x^3/3 + 2x^5/(3\times5) + 17x^7/(5\times7\times9) + \ldots$

·3 $\mathrm{Hp}·1 . \supset . x/\operatorname{tng} x = 1 - \Sigma[2^{2n} B_n x^{2n}/(2n)! \mid n, N_1]$

·4 $\mathrm{Hp}·1 . \supset . \log(\sin x/x) = -\Sigma\{2^{2n-1} B_n x^{2n}/[n(2n)!] \mid n, N_1\}$

·5 $x \varepsilon q . \operatorname{mod} x < \pi/2 . \supset .$
$\log\cos x = -\Sigma\{(2^{2n}-1)2^{2n-1} B_n/[n(2n)!]x^{2n} \mid n, N_1\}$
$= -x^2/2! - 2x^4/4! - 16\times17x^8/8! - \ldots$

·6 $x \varepsilon q - \iota 0 . \operatorname{mod} x < \pi/2 . \supset .$
$\log(\operatorname{tng} x/x) = \Sigma\{(2^{2n-1}-1)2^{2n} B_n/[n(2n)!]x^{2n}\ n, N_1\}$

} EULER a.1748 p.152 {

§5 $\log^*$ $\sin^{-1}$ $\cos^{-1}$ tng^{-1}

✱ 1·1 $x \varepsilon q' - \iota 0 . \supset . \log^* x = q' \frown y \ni (e^y = x)$ Df

·2 ——————— $\log x = \imath (\log^* x) \frown y \ni (-\pi < \text{imag} y \leqq \pi)$ Df

·3 $\sin^{-1} = s^{-1} = [s, (-\pi/2) \urcorner (\pi/2)]^{-1}$ Df

·4 $\cos^{-1} = c^{-1} = [c, \Theta\pi]^{-1}$ Df

·5 $\text{tang}^{-1} = t^{-1} = [t, (-\pi/2) \text{-} (\pi/2)]^{-1}$ Df

$\log^* x$ indica classe de solutiones de equatione $e^y = x$.

$\log x$ = « valore principale de logarithmo », indica solutione que habe coefficiente de unitate imaginario inter $-\pi$ et $+\pi$.

Si y es quantitate inter -1 et $+1$, $\sin^{-1} y$ indica quantitate, minimo in valore absoluto, de que sinu es y. Idem per $\cos^{-1} y$, quem nos sume inter limite 0 et π; et $\text{tng}^{-1} y$, inter limite $-\pi/2$ et $\pi/2$.

Ce functiones es inverso de sin, cos, tang.

Notatione que nos adopta es de usu generale in Anglia. Vige et notatione arc sin$y$, arc(sin $= y$).

Euler, BerolMisc. a.1743 t.7 p.167, adopta notatione: sinAx, Asin$x$, per: sin$x$, $\sin^{-1} x$; A es litera initiale de Arcu.

Logarithmo imaginario occurre in:

{ Cotes a.1714 LondonT. t.29 p.32:

« ... si quadrantis circuli quilibet arcus x], radio CE [1] descriptus sinum habeat CX [sin$x$], sinumque complementi ad quadrantem XE [cos$x$]: sumendo radio CE pro Modulo, arcus erit rationis inter EX+XC$\sqrt{-1}$ et CE [cos$x$ + i sin$x$] mensura ducta in $\sqrt{-1}$. » }

{ Euler a.1728 (Vide BM. a.1899 p.46):

« Sit radius circuli a ... habebis quadrans circuli $= \frac{aa}{4\sqrt{-1}} \log(-1)$ »

✱ 2·1 $r \varepsilon Q . x \varepsilon q . -\pi < x \leqq \pi . \supset . \log(r e^{ix}) = \log r + ix$

imag log x vocare « argumento, amplitudo, azimuth, anomalia,... » de x

Numero imaginario, de modulo r, et de argumento x, vale $r e^{ix}$.

·2 $x \varepsilon q' - \iota 0 . \supset . \log^* x = \log x + 2n\pi i$

[$y \varepsilon \log^* x . \supset . e \nwarrow (y - \log x) = 1 . \supset . y - \log x \,\varepsilon\, 2n\pi i$]

·3 $\log i = i\pi/2 \ . \ \log(-1) = \pi i$

✱ 3. $y \varepsilon (-1) \text{-} 1 . \supset$:

·1 $s^{-1} y = \imath q \frown x \ni (sx = y . -\pi/2 \leqq x \leqq \pi/2)$ Dfp

·11 $c^{-1} y = \imath q \frown x \ni (cx = y . 0 \leqq x \leqq \pi)$ Dfp

·2 $q \frown x\ni(sx = y) = (2n\pi + s^{-1}y) \cup (\pi + 2n\pi - s^{-1}y)$

·21 $q \frown x\ni(cx = y) = (2n\pi + c^{-1}y) \cup (2n\pi - c^{-1}y)$

·3 $s^{-1}y + c^{-1}y = \pi/2$ ·31 $cs^{-1}y = \surd(1-y^2)$

·4 $s^{-1}y = -i \log[\surd(1-y^2) + iy]$ Dfp

·41 $c^{-1}y = -i \log[y + i\surd(1-y^2)]$ Dfp

·5 $p,q\varepsilon q \,.\, q^2 < p^3 \,.\supset.$
$q \frown x\ni(x^3 - 3px + 2q = 0) = 2\surd p\; c\{[c^{-1}(q\,p^{-3/2}) + (0\cdots2)\pi]/3\}$
{ VIETA Opera a.1615 p.159 }

Resolutio trigonometrico de aequatione de gradu 3.

✻ 4. $y\varepsilon q \,.\supset:$

·1 $t^{-1}y = \iota\, q \frown x\ni(tx = y \,.\, -\pi/2 < x < \pi/2)$ Dfp

·2 $t^{-1}y = \log[(1+yi)/(1-yi)]/(2i)$ Dfp
{ Joh. BERNOULLI t.1 p.409; EULER a.1748 p.105 }

·3 $q \frown x\ni(tx = y) = n\pi + t^{-1}y$

·4 $y\varepsilon Q \,.\supset.\, t^{-1}y + t^{-1}/y = \pi/2$

·5 $x\varepsilon Q \,.\, y\varepsilon q \,.\supset.\, \log(x+iy) = \log\surd(x^2+y^2) + i\, t^{-1}(y/x)$

✻ 5.

·1 $x,y\varepsilon q \,.\, \mathrm{mod}(xy) < 1 \,.\supset.\, t^{-1}[(x+y)/(1-xy)] = t^{-1}x + t^{-1}y$

·2 $a\varepsilon N_1 \,.\, x,y\varepsilon N_1 \,.\, xy = a^2+1 \,.\, b = a+x \,.\, c = a+y \,.\supset.$
$t^{-1}/a = t^{-1}/b + t^{-1}/c$

[$a,b,c\varepsilon N_1 \,.\supset:\, t^{-1}/a = t^{-1}/b + t^{-1}/c \,.=.\, bc = a(b+c)+1 \,.=.$
$(b-a)(c-a) = a^2+1$]

·3 $\pi = 4t^{-1}1$ ·4 $\pi = 4(t^{-1}/2 + t^{-1}/3)$

·41 $\pi = 8t^{-1}/3 + 4t^{-1}/7$ { EULER a.1737 PetrC. t.9 p.231 }

[$(/3\,,\,/7)\,|\,(x,y)$ P·1 $.\supset.\, \mathrm{tng}^{-1}/2 = \mathrm{tng}^{-1}/3 + \mathrm{tng}^{-1}/7$ (1)
(1) . P·3 $.\supset.$ Ths]

·42 $\pi = 8t^{-1}/2 - 4t^{-1}/7$ { BERTRAND a.1855 p.301 }

·43 $\pi = 16\,t^{-1}/5 - 4\,t^{-1}/239$ { MACHIN; v. JONES a.1706 p.263 }

[$(/5,\,/5)\,|\,(x,y)$P·1 $.\supset.\, \mathrm{tng}^{-1}\,5/12 = 2\mathrm{tng}^{-1}/5$ (1)
$[(5/12\,,\,5/12)]\,|\,(x,y)$P·1 . (1) $.\supset.\, \mathrm{tng}^{-1}120/119 = 4\mathrm{tng}^{-1}/5$ (2)
$(120/119\,,\,-1)\,|\,(x,y)$P·1 $.\supset.\, \mathrm{tng}^{-1}/239 = \mathrm{tng}^{-1}120/119 - \mathrm{tng}^{-1}1$ (3)
(2) . (3) . P·6 $.\supset.$ Ths]

·5 $\pi = 4(t^{-1}/2 + t^{-1}/5 + t^{-1}/8)$ { DAHSE a.1844 p.198 {

·6 $\pi = 20t^{-1}/7 + 8t^{-1}3/79$ { VEGA a.1794 p.633 {

·7 $\pi = 16\,t^{-1}/5 - 4\,t^{-1}/70 + 4\,t^{-1}/99$
{ EULER a.1737 PetrC. t.9 p.232 {

·8 $(m,n,x,y)\ni[m,n\varepsilon n\,.\,x,y\varepsilon N_1\,.\,x<y\,.\,m\,t^{-1}/x + n\,t^{-1}/y = \pi/4]$
$= \iota(1,1,2,3) \cup \iota(2,-1,2,7) \cup \iota(2,1,3,7) \cup \iota(4,-1,5,239)$
{ STÖRMER BsF. a.1899 t.27 p.170 {

✻ 6·1 $x\varepsilon q'\,.\,\mathrm{mod}x \leqq 1\,.\,x -= -1\,.\supset.$
$\log(1+x) = x - x^2/2 + x^3/3 - \ldots$ [(q'|q) Dem §e P6·1]

Serie de $\log(1+x)$ considerato in §e P6·1, subsiste pro x imaginario, si suo modulo es minore de 1, et si suo modulo vale 1, exceptuato $x = -1$; da valore principale de logarithmo.

Contine, ut casu particulare P6·2 ... ·9 et P7·1 . P6·1 es successivo es noto ad Eulero a.1748.

·2 $x\varepsilon$ q$-$nπ $.\supset.$ $\log[2c(x/2)] = cx - c(2x)/2 + c(3x)/3 - \ldots$

·3 $x\varepsilon (-\pi)^{-}\pi$ $.\supset.$ $x/2 = sx - s(2x)/2 + s(3x)/3 - \ldots$
{ EULER PetrNC. a.1760 t.5 p.204 {

[$e^{ix}|x$ P·1. Oper real . Oper imag $.\supset.$ P·2·3]

·4 $x\varepsilon\, 2\theta\pi$ $.\supset.$ $sx + s(2x)/2 + s(3x)/3 + \ldots = (\pi - x)/2$
[$(\pi-x)|x$ P·3 $.\supset.$ P]
{ EULER PetrNC. a.1774 t.19, *Calc. Int.* a.1794 t.4 p.235 {

·5 $x\varepsilon\, \theta\pi$ $.\supset.$ $sx + s(3x)/3 + s(5x)/5 + \ldots = \pi/4$
[P·3·4 . Oper+ $.\supset.$ P]
{ EULER, *Calc. Diff.*, Pars II, Art. 57-93 {

·6 $x\varepsilon$ q$-$nπ $.\supset.$ $\log[2s(x/2)] = -cx - c(2x)/2 - c(3x)/3 - \ldots$
[$(\pi-x)|x$ P·2 $.\supset.$ P]

$r\varepsilon q\,.\,\mathrm{mod}r < 1\,.\,x\varepsilon q\,.\supset.$

·7 $\log\surd(1 + 2rcx + r^2) = rcx - r^2c(2x)/2 + r^3c(3x)/3 - \ldots$

·8 $t^{-1}[r\,sx/(1 - r\,cx)] =$
$rsx + r^2s(2x)/2 + r^3s(3x)/3 + \ldots = \Sigma[r^n s(nx)/n\,|n, N_1]$
[$re^{ix}|x$ P·1 $\supset$ P·7·8]

Delambre a.1808 applica P·7·8 ad quaestiones de Geodaesia.

✻ 7·1 $y\varepsilon q\,.\,-1 \leqq y \leqq 1\,.\supset.$ $t^{-1}y = y - y^3/3 + y^5/5 - \ldots$
{ LEIBNIZ a.1673-74, MathS. t.5 p.401 {
[$(y,\pi/2)\,|\,(r,x)$ P6·8·3 $.\supset.$ P]

Commercio epistolico, etc., publicato in 1722, tribue ce P ad Jac. Gregory, secundo copia de litera de 1671 ad Collins.

Sed, secundo Joh. Bernoulli, (Leibniz MathS. t.3 p.917,934), authenticitate de ce litera es dubio.

·2 $y \varepsilon \mathrm{q} \,.\, n \varepsilon \mathrm{N}_1 \,.\supset.$
$\mathrm{t}^{-1}y \;\varepsilon\; y - y^3/3 + y^5/5 - \ldots \mp y^{2n-1}/(2n-1) \pm \theta y^{2n+1}/(2n+1)$

✻ 8.

·1 $y \varepsilon \Theta \,.\supset.\, \mathrm{s}^{-1}y = y + 1/(2\times3)\, y^3 + (1\times3)/(2\times4\times5)\, y^5 + (1\times3\times5)/(2\times4\times6\times7)\, y^7 + \ldots$ } NEWTON a.1676 {

·2 $y \varepsilon \mathrm{q} \,.\, \mathrm{mod}\,y < 1 \,.\, m \varepsilon \mathrm{q} \,.\supset.$
$\mathrm{s}(m\mathrm{s}^{-1}y) = my + \Sigma\{m\Pi\,[[(2r+1)^2 - m^2]\,|r, 0\cdots n]\,y^{2n+3}/(2n+3)!\,|n, \mathrm{N}_1\}$

·3 $\mathrm{Hp}\cdot2 \,.\supset.\, \mathrm{c}(m\mathrm{s}^{-1}y) =$
$1 + \Sigma\{(-1)^{n+1}\,\Pi\,[(m^2 - 4r^2)\,|r, 0\cdots n]\,y^{2n+2}/(zn+2)!\,|n, \mathrm{N}_0\}$
} EULER Calc. integr. p.101-106 {

·4 $m \varepsilon \mathrm{Q} \,.\, x \varepsilon \theta\pi/2 \,.\supset.\, \mathrm{tang}^{-1}(m\,\mathrm{tang}x) =$
$x + \Sigma\{[(m-1)/(m+1)]^n \sin(2nx)/n\,|n, \mathrm{N}_1\}$

✻ 9·1 $\pi = 4\Sigma\{\mathrm{t}^{-1}/(2n^2)\,|n, \mathrm{N}_1\}$
} PEACOCK a.1820 *Ex. of the diff. Calc.*, Cambridge p.67 {

·2 $\pi = 4\Sigma\{\mathrm{t}^{-1}/(2^n - 1)\,|n, \mathrm{N}_1 + 1\} = 4(\mathrm{t}^{-1}/3 + \mathrm{t}^{-1}/7 + \mathrm{t}^{-1}/15 + \ldots)$
} PEACOCK a.1820 p.68 {

·3 $\pi = 4\Sigma\{\mathrm{t}^{-1}/(n^2+n+1)\,|n, \mathrm{N}_1\}$ } EULER PetrNC. a.1764 t.9 {

§6 ang

✻ 1. $u, v, w \varepsilon \mathrm{v}\text{-}\iota 0 \,.\supset.$ ·0 $\mathrm{ang}(u,v) = \cos^{-1}[\cos(u,v)]$ Df

·1 $\mathrm{ang}(u,v) \;\varepsilon\; \Theta\pi$ ang

·2 $\mathrm{ang}(u,v) = \mathrm{ang}(v,u) = \mathrm{ang}(-u, -v) = \mathrm{ang}(\mathrm{U}u, \mathrm{U}v)$
$= \pi - \mathrm{ang}(u, -v)$ }EUCLIDE I P15 13{

·3 $\mathrm{ang}(u,w) \leqq \mathrm{ang}(u,v) + \mathrm{ang}(v,w)$ } » XI P20{

·4 $\mathrm{ang}(u,v) + \mathrm{ang}(v,w) + \mathrm{ang}(w,u) \leqq 2\pi$ } » 21{

[P·3 $.\supset.\, \mathrm{ang}(v,w) \leqq \mathrm{ang}(-u,v) + \mathrm{ang}(-u,w) \,.\supset.$
$\mathrm{ang}(u,v) + \mathrm{ang}(u,w) + \mathrm{ang}(v,w) \leqq [\mathrm{ang}(u,v) + \mathrm{ang}(-u,v)] + [\mathrm{ang}(u,w) + \mathrm{ang}(-u,w)] = \pi + \pi$]

·5 $\cos(u,v) = \cos \text{ang}(u,v)$. $\sin(u,v) = \sin \text{ang}(u,v)$ Dfp

ang(u,v) es angulo des vectore u et v, rato ut numero.

Angulo de vertice puncto o, rato ut loco de punctos, es $o+\text{Q}u+\text{Q}v$. .

Geometria et considera angulos ut systema de grandore homogeneo. Nostro ang(u,v) es ratione de angulo de vectores u et v, ut grandore, ad angulo dicto « radiante », que es angulo clauso per arcu de circulo aequale ad radio. Unde:

(angulo recto) $= \pi/2$(radiante)

gradu sexagesimale: $1^0 =$ (angulo recto)/90 , $1' = 1^0/60$, $1'' = 1'/60$

gradu centesimale $1^g =$ (angulo recto)/100

seque: radiante $= 57^0{,}29577951308$ (Calculato per Cotes, a.1722 p.95)

$= 3437'{,}7467707849$

$= 206264''{,}806247$

❋ 2. $p,q \varepsilon \text{p} . p -= q . r \varepsilon \text{p} - \text{recta}(p,q) . a = \text{d}(q,r) . b = \text{d}(r,p)$.
$c = \text{d}(p,q) . a' = \text{ang}(p-q,p-r) . b' = \text{ang}(q-r,q-p)$.
$c' = \text{ang}(r-p,r-q) . s = (a+b+c)/2 .\supset.$

·1 $a=b .=. a'=b'$ } Euclide I P 5, 6 {

·2 $a<b .=. a'<b'$ } » 18, 19 {

·3 $a'+b'+c' = \pi$ [P1·4] } Euclide I P32 {

·4 $a^2 = b^2+c^2-2bc\cos a'$ [= IV §vct P13·1]

·5 $\sin a'/a = \sin b'/b = \sin c'/c$

} Nasîr Eddin Attûsi a.1260 l.III {

[$p-r = (p-q)+(q-r) .\supset. [\text{cmp}\perp(p-q)](p-r) = [\text{cmp}\perp(p-q)](q-r)$
Oper mod $.\supset.$ P]

·6 $bc \sin a' = 2\sqrt{[s(s-a)(s-b)(s-c)]}$ }Herone a.—150 p.286{

$(bc \sin a')/2$ es area de triangulo pqr.

·7 $\sin(a'/2) = \sqrt{[(s-b)(s-c)/(bc)]}$. $\cos(a'/2) = \sqrt{[s(s-a)/(bc)]}$.
$\text{tng}(a'/2) = \sqrt{\{(s-b)(s-c)/[s(s-a)]\}}$

·8 $\text{tng}[(a'-b')/2] / \text{tng}[(a'+b')/2] = (a-b)/(a+b)$

❋ 3. $u \varepsilon \text{v} - \iota 0 . v \varepsilon \text{v} - \text{q}u . w \varepsilon \text{v} - (\text{q}u + \text{q}v) .\supset.$

·0 $\text{ang}(u;v,w) = \text{ang}[(\text{cmp}\perp u)v, (\text{cmp}\perp u)w]$ Df

= angulo dihedro determinato per planos uv et uw.

$a = \text{ang}(v,w) . b = \text{ang}(w,u) . c = \text{ang}(u,v) . a' = \text{ang}(u;v,w) . b' = \text{ang}$
$(v;w,u) . c' = \text{ang}(w;u,v) . s = (a+b+c)/2 . s' = (a'+b'+c')/2 .\supset.$

·01 $a < b+c . a+b+c < 2\pi$ [= P53·5·6]

·02 $a=b .=. a'=b'$ ·03 $a<b .=. a'<b'$

·04 $a'>b'+c'-\pi$. $\pi<a'+b'+c'<3\pi$

·1 $\cos a=\cos b \cos c+\sin b \sin c \cos a'$

[$\cos(v,w) = \mathrm{U}v\times\mathrm{U}w$

$= [(\mathrm{cmp}\|u)\mathrm{U}v+(\mathrm{cmp}\perp u)\mathrm{U}v]\times[(\mathrm{cmp}\|u)\mathrm{U}w+(\mathrm{cmp}\perp u)\mathrm{U}w]$

$= [(\mathrm{cmp}\|u)\mathrm{U}v]\times[(\mathrm{cmp}\|u)\mathrm{U}w]+[(\mathrm{cmp}\perp u)\mathrm{U}v]\times[(\mathrm{cmp}\perp u)\mathrm{U}w]$.⊃. P]

} Al battâni a.929 ; vide BD. a.1892 p.147 {

} Regiomontanus a.1533 p.127 :

« In omni triangulo sphærali ex arcubus circulorum magnorum constante, proportio sinus versi anguli cuiuslibet [$1-\cos a'$] ad differentiam duorum sinuum versorum, quorum unus est lateris eum angulum subtendentis [$1-\cos a$], alius vero differentiæ duorum arcuum ipsi angulo circumiacentium [$1-(\cos b \cos c+\sin b \sin c)$] est tanquam proportio quadrati sinus recti totius [1] ad id, quod sub sinibus arcuum dicto angulo circumpositorum continetur rectangulum [$\sin b \sin c$]. »{

Demonstratione de ce P, dicto «theorema fundamentale de trigonometria de sphaera», es traductione de illo dato per Cauchy (s.1 t.9 p.264), per methodo de projectiones. Substitutione de projectiones per productos ×, reduce duo pagina de Cauchy ad duo linea.

·11 Dm P1·3 [P·1 .⊃. $\cos a \gtreqless \cos(b+c)$. Oper $\cos^{-1}$.⊃P]

·2 $\sin a'/\sin a=\sin b'/\sin b=\sin c'/\sin c$

} Abû'lwéfa a.940–998 ; vide *Journal Asiatique* a.1892 s.8 t.19 p.423 {

} Regiomontano a.1533 p.95 :

« In omni triangulo ... sinus laterum ad sinus angulorum eis oppositorum eandem habent proportionem ». {

·3 $\cos a'=-\cos b'\cos c'+\sin b'\sin c'\cos a$

·4 $\cos b \cos c'=\sin b/\mathrm{tng}\, a-\sin c'/\mathrm{tng}\, a'$

·5 $\sin b \sin c \sin a'=2\sqrt{[\sin s \sin(s-a) \sin(s-b) \sin(s-c)]}$.

·6 $\sin(a'/2) =\sqrt{\{\sin(s-b) \sin(s-c) / (\sin b \sin c)\}}$. }Neper

$\cos(a'/2) =\sqrt{\{\sin s \sin(s-a) / (\sin b \sin c)\}}$. a.1614 p.48{

$\sin(a/2) =\sqrt{\{-\cos s' \cos(s'-a') / (\sin b' \sin c')\}}$.

$\cos(a/2) =\sqrt{\{\cos(s'-b') \cos(s'-c') / (\sin b' \sin c')\}}$.

$\sin[(a'-b')/2] \sin(c/2)=\sin[(a-b)/2] \cos(c'/2)$.

$\cos[(a'-b')/2] \sin(c/2)=\sin[(a+b)/2] \sin(c'/2)$.

$\sin[(a'+b')/2] \cos(c/2)=\cos[(a-b)/2] \cos(c'/2)$.

$\cos[(a'+b')/2] \cos(c/2)=\cos[(a+b)/2] \sin(c'/2)$.

} Delambre *Connaiss. des temps*, a.1807 {

$\text{tng}[(a'+b')/2] = /\text{tng}(c'/2)\cos[(a-b)/2]/\cos[(a+b)/2]$.
$\text{tng}[(a'-b')/2] =$ » sin » sin » .
{ NEPER a.1614 p.48 }
$\text{tng}[(a+b)/2] = \text{tng}(c/2)\cos[(a'-b')/2]/\cos[(a'+b')/2]$.
$\text{tng}[(a-b)/2] =$ » sin » sin »

·7 $[\sin(u,v,w)]^2 = 4\sin s\sin(s-a)\sin(s-b)\sin(s-c)$

Vide alios formula de trigonometria de sphaera in Formul. t.4 p.336.

❋ 4·1 $a,b,c\varepsilon\text{p} \,.\, \text{d}(a,b)=\text{d}(a,c)=\text{d}(b,c)=1 \,.\supset.$
$\cos(b-a, c-a) = \sin[a-b, a-(b+c)/2] = /2$.
sin » $= \cos$ » » $= \surd 3/2$.

·2 $a,b,c,d\varepsilon\text{p} \,.\, \text{d}(a,b)=\text{d}(a,c)=\text{d}(a,d)=\text{d}(b,c)=\text{d}(b,d)=\text{d}(c,d)$
$=1 \,.\supset.\, \cos[(a+b)/2-c, (a+b)/2-d] = /3$
sin » » » » $= 2\surd 2/3$
$\cos[(a+b)/2-c, (a+b)/2-(c+d)/2] = \surd(2/3)$
sin » » » » » $= \surd/3$

Angulos de triangulo et de tetrahedro regulare.

❋ 5. $a,b\varepsilon\text{p}_2 \,.\supset.$

·0 $\text{ang}(a,b) = \min x\ni[p,q\varepsilon a \,.\, p-=q \,.\, r,s\varepsilon b \,.\, r-=s \,.\, x=\text{ang}(p-q,r-s)]$ Df

·1 $\text{ang}(a,b)\ \varepsilon\ \Theta\pi/2$

❋ 6. $a\varepsilon\text{p}_2 \,.\, b\varepsilon\text{p}_3 \,.\supset.$ P5·0·1

·2 $\text{ang}(a,b) = \text{ang}[a, (\text{proj}\,b)`a]$ Dfp

❋ 7. $a,b\varepsilon\text{p}_3 \,.\supset.$

·0 $\text{ang}(a,b) = \max x\ni[r\varepsilon\text{p}_2 \,.\, r\supset a \,.\, x=\text{ang}(r,b)]$ Df

·1 $=$ P5·1

P5-7. Angulo de duo recta, de recta cum plano, et de duo plano.

TABULA DE SIN ET TANG

Gradu	*0*	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*	*90*
sin	·000	·173	·342	·500	·642	·766	·866	·939	·984	1·000
tang	·000	·176	·363	·577	·839	1·191	1·732	2·747	5·671	∞

§7 Rotat vel μ

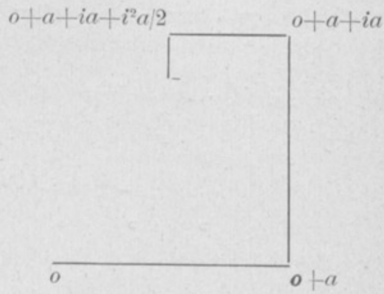
Si a,b es vectore unitario et orthogonale, et nos voca i rotatione que fer a in b: et si t es quantitate reale, et u es vectore in plano a,b, tunc $e^{it}u$ repraesenta vectore u rotato in plano a,b, de angulo que habe t pro mensura in radiantes.

$e^i\, a$ indica vectore a rotato de uno radiante. Unde resulta constructione simplice de radiante.

Per definitione:

$e^i\, a = (1+i+i^2/2!+i^3/3!+...)a$

Contrue puncto $o+a$, adde ia, et resulta puncto $o+a+ia$. Adde $(ia)\times i/2$, id es termine praecedente rotato de angulo recto, et diviso per 2, et habe $o+a+ia+i^2a/2$. Adde $i^3/3!a = (i^2a/2)i/3$,



id es termine praecedente rotato de angulo recto et diviso per 3. Et ita porro. Successione de punctos o, $o+a$, $o+a+ia$, ... verge in modo rapido ad limite $o+e^i\, a$, extremo de arcu de centro o, et aequale ad radio a.

Signo e⌐(it) es multo importante in applicationes de Calculo ad Geometria. Signos Rotat, vel, μ occurre solo in praesente §, que es exercitio.

❋ 1.

$a,b\varepsilon \mathrm{v} \,.\, a^2=b^2=1 \,.\, a\times b=0 \,.\, i=b/a \,.\, o\varepsilon \mathrm{p} \,.\, p,q,r\varepsilon\, o+\mathrm{q}a+\mathrm{q}b \,.\, t,t'\varepsilon \mathrm{q} \,.$
$m\varepsilon \mathrm{N}_1 \,.\, u\varepsilon\, \mathrm{q}a+\mathrm{q}b \;.\supset.$

·0 $\mathrm{Rotat}(o,i,t) = [o+e^{it}(p-o) \mid p,\, o+\mathrm{q}a+\mathrm{q}b]$ Df

= rotatione de punctos in plano $o+\mathrm{q}a+\mathrm{q}b$, circa o, de angulo de radiante.

·1 $\mathrm{Rotat}(o,i,t')\, \mathrm{Rotat}(o,i,t) = \mathrm{Rotat}(o,i,\, t+t')$

·2 $\mathrm{Rotat}(o,i,t)^m = \mathrm{Rotat}(o,i,mt)$

·3 $\mathrm{Rotat}(q,i,-t)\, \mathrm{Rotat}(p,i,t) = \mathrm{Transl}[(1-e^{-it})(q-p)]$

·4 $\mathrm{Rotat}(q,i,t) = \mathrm{Transl}[(1-e^{it})(q-p)]\, \mathrm{Rotat}(p,i,t)$

·5 $e^{it} -= 1 \;.\supset.\; \mathrm{Transl}u\, \mathrm{Rotat}(p,i,t) = \mathrm{Rotat}[p+u/(1-e^{it}),\, i,t]$

·6 » $\mathrm{Rotat}(p,i,t)\, \mathrm{Transl}u = \mathrm{Rotat}[p-u/(1-e^{it}),\, i,t]$

·7 $e^{i(t+t')} -= 1 .\supset.$ Rotat(q,i,t') Rotat$(p,i,t) =$
Rotat$[p+(q-p)(1-e^{it'})/(1-e^{i(t+t')}), i, t+t']$

·3·7 Producto de duo rotatione, in plano considerato, vale translatione vel rotatione.

·5·6 Producto de translatione per rotatione vale rotatione.

✻ 2·0 $a\varepsilon\varphi^2 .\supset.$ vel$a = [\mathrm{I}\omega(aax)|x, \varphi^1]$

·1 $a\varepsilon\varphi^2 .\supset.$ vel$a$ ε (vfφ^1)lin

·2 $a,b\varepsilon\varphi^2 .\supset.$ vel$(a+b) =$ vel$a$ + vel$b$ Distrib(vel,+)

·3 $a\varepsilon\varphi^2 . h\varepsilon$q $.\supset.$ vel$(ha) = h$vel$a$

·4 $u\varepsilon$v $.\supset.$ (vel$a)u = \mathrm{I}[(\omega a)au]$ [Df vel . IV§aP18·3 $.\supset.$ P]

·5 $u\varepsilon$v $.\supset. u\times$(vel$a)u = 0$
[$u\times$(vel$a)u = u\times\mathrm{I}[(\omega a)au] = ua(\omega a)au/\psi = 0$]

·6 $x,y\varepsilon$p $.\supset. (x-y)\times[$(vel$a)x] = (x-y)\times[$(vel$a)y]$

·7 $n\varepsilon\mathrm{N}_1 . a\varepsilon$ p f$1\cdots n . u\varepsilon$ v f$1\cdots n . l\varepsilon\varphi^2 .\supset.$
$\Sigma[u_r\times$(vel$l)a_r|r,1\cdots n] = la\Sigma(a_r a u_r|r,1\cdots n)/\Psi$

·0 Si a es forma de gradu 2, nos indica per vel$a$, lege: «velocitate repraesentato per forma a», transformatione indicato. Determina motu infinitesimo de corpore rigido.

·2 Lege de compositione des motus infinitesimo.

·6 Resal, *Traité de Cinématique pure* a.1862 p.28: «Lorsqu'un [segment de longueur invariable] se meut dans l'espace, les projections des vitesses des deux [extremités sur le segment] sont égales et de même sens».

·7 Summa considerato in primo membro es labore de systema de fortia u applicato ad punctos a, quando ce puntos sume motu repraesentato per forma l. Peano, TorinoA. a.1895 d.125.

Königs, *Cinématique*, p.435: «Le travail virtuel est égal au produit du temps écoulé δt par le moment des deux systèmes de segments qui représentent, l'un le dyname des forces, et l'autre le système de rotations».

✻ 3·0 $a\varepsilon\varphi^2 .\supset. \mu a = \mathrm{e}\!\upharpoonleft\!(vela)$ Df

μa es «motu repraesentato per a, forma de gradu 2». Ce exponentiale es definito in §e P11.

·1 $a\varepsilon\mathrm{p}^2 .$ mod$\omega a = 1 .\supset. \mu(\pi a) =$ Sym(recta$a$)

Si $t\varepsilon$q, $\mu(ta)$ repraesenta rotatione de t radiante circa axi a.

·2 $a\varepsilon\mathrm{v}^2 . x\varepsilon$p $.\supset. (\mu a)x = x+\mathrm{I}a$

·3 $a \varepsilon \mathrm{v}^2 . \supset . (\mu a, \mathrm{p}) = \mathrm{Transl}(\mathrm{I}a)$

Si a es bivectore, substitutione μa, super punctos es translatione indicato per vectore Ia.

·4 $a \varepsilon \varphi^2 . \supset . (\mu a, \mathrm{p}) \varepsilon$ Motor

·5 $m \varepsilon$ Motor $. \supset . \exists \varphi^2 \frown a \ni [m = (\mu a, \mathrm{p})]$

Omni motu habe forma μa.

·6 $u,v \varepsilon \mathrm{v} . uav -= 0 . o \varepsilon \mathrm{p} . \supset .$
$(\mathrm{Sym}\, ou)(\mathrm{Sym}\, ov) = \mu[2 \mathrm{ang}(u,v)\, o \mathrm{UI}(uav)]$

·7 $o \varepsilon \mathrm{p} . u,v,w \varepsilon \mathrm{v} . uavaw -= 0 . \supset .$
$\mu[2\mathrm{ang}(u,v) o \mathrm{UI}(uav)]\ \mu[2\mathrm{ang}(v,w) o \mathrm{UI}(vaw)]\ \mu[2\mathrm{ang}(w,u) o \mathrm{UI}(wau)]$
$= 1$

Si o es puncto, u,v,w vectore non coplanare tunc: producto de tres rotatione circa axi per o, perpendiculare ad facie de trihedro (u,v,w), et de angulo duplo de angulo de facie, vale identitate.

§8 Fc (fractio continuo)

✻ 1. $a \varepsilon \mathrm{qfN}_1 . n \varepsilon \mathrm{N}_1 . \supset .$ ·0 $\mathrm{Fc}(a, 1 \cdots 1) = /a_1$ Df

·01 $\mathrm{Fc}[a, 1\cdots(n+1)] = /[a_1 + \mathrm{Fc}(a_{1+r}|r, 1\cdots n)]$ Df

·1 $\mathrm{Fc}(a, \mathrm{N}_1) = \lim \mathrm{Fc}(a, 1\cdots n)|n$ Df

a es successione de quantitate, et n es numero. $\mathrm{Fc}(a, 1\cdots n)$ indica fractio continuo formato per elementos $a_1, a_2, \ldots, a_n$. Df per inductione.

Notatione $\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$ non es commodo.

Cataldi, *Trattato del modo brevissimo di trouare la Radice quadra*... a.1613 p.70:

« Notisi, che non si potendo comodamente nella stampa formare i rotti, & rotti di rotti come andariano,... noi da qui inanzi gli formaremo tutti à questa similitudine

Di 18. la R sia $4. \& \frac{2}{8.} \& \frac{2}{8.} \& \frac{2}{8.} \& \frac{2}{8.}$

facendo vn punto all'8 denominatore di ciascun rotto, à significare, che il seguente rotto è rotto d'esso denominatore. »

J. Müller *Allg. Arithm.* a.1838 introduce notatione $\frac{1}{a_1} \dot{+} \frac{1}{a_2} \dot{+} \cdots \dot{+} \frac{1}{a_n}$,

modificato in $\frac{1}{a_1 +}\ \frac{1}{a_2 +} \ldots$ per alios.

✻ 2. $a \varepsilon \mathrm{N_1 f N_1} . n \varepsilon \mathrm{N_1} . \supset .$ ·1 $\mathrm{Fc}(a, 1 \cdots n) \, \varepsilon \, \mathrm{R} \frown (1 - \mathrm{R_0})$

·2 $\mathrm{nt\,Fc}[a, 1 \cdots (n+2)] = a_{n+2} \times \mathrm{nt\,Fc}[a, 1 \cdots (n+1)] + \mathrm{nt\,Fc}(a, 1 \cdots n)$

·3 dt » » » dt » » dt »

·4 $\mathrm{Fc}[a, 1 \cdots (n+1)] - \mathrm{Fc}[a, 1 \cdots n] =$
$(-1)^n / \{\mathrm{dt\,Fc}(a, 1 \cdots n) \times \mathrm{dt\,Fc}[a, 1 \cdots (n+1)]\}$

Si a es successione de numeros naturale, tunc differentia inter duo fractio continuo de ordine successivo vale $\pm$ 1 diviso per producto de denominatores de duo fractio.

Ce regula occurre in Schwenter Daniel, *Deliciae Physico-mathematicae*, Nürnberg a.1636 p.111.

·5 $\mathrm{dt\,Fc}(a, 1 \cdots n) = \mathrm{dt\,Fc}[a(n-r+1) | r, 1 \cdots n]$
{ Cayley, Phil. Magaz. a.1853 }

$\mathrm{dt\,Fc}(a, 1 \cdots n)$ vocare « continuante de ordine n de serie a » (Sylvester AJ. a.1878 t.1 p.344).

·6 $\mathrm{Fc}(a, 1 \cdots n) =$
$/a_1 + \Sigma\{(-1)^r / \mathrm{dtFc}(a, 1 \cdots r) \times \mathrm{dtFc}[a, 1 \cdots (r+1)] | r, 1 \cdots (n-1)\}$
{ Euler PetropC. t.9 p.104 }

·7 $m \varepsilon \mathrm{N_1} . m < n . \supset . \mathrm{Fc}(a, 1 \cdots n) - \mathrm{Fc}(a, 1 \cdots m) =$
$(-1)^m \mathrm{dtFc}[a_{m+r+1} | r, 1 \cdots (n-m-1)] / [\mathrm{dtFc}(a, 1 \cdots m) \times \mathrm{dtFc}(a, 1 \cdots n)]$

✻ 3·1 $x \varepsilon \mathrm{R} . \supset . \, \exists \, \mathrm{N_1} \frown n \ni \{ \exists \, \mathrm{N_1 f} 1 \cdots n \frown a \ni [x = \mathrm{E}x + \mathrm{Fc}(a, 1 \cdots n)]\}$

·2 $a, b, c \varepsilon \mathrm{n} . \mathrm{D}(a, b) = 1 . \mathrm{mod}a < \mathrm{mod}b : n \varepsilon \mathrm{N_1} . d \varepsilon \mathrm{N_1 F} 1 \cdots n .$
$\mathrm{mod}(a/b) = \mathrm{Fc}(d, 1 \cdots n) . u = (-1)^{n-1} \times \mathrm{sgn}a \times c \times \mathrm{dtFc}[d, 1 \cdots (n-1)] . v = (-1)^n \times \mathrm{sgn}b \times c \times \mathrm{ntFc}[d, 1 \cdots (n-1)] . \supset . u, v \varepsilon \mathrm{n} .$
$au + bv = c$ { Lagrange, BerlinM. a.1767 p.175 }

✻ 4. $a \varepsilon \mathrm{N_1 f N_1} . n \varepsilon \mathrm{N_1} . \supset .$

·0 $\mathrm{nt\,Fc}[a, 1 \cdots (n-1)] / \mathrm{nt\,Fc}(a, 1 \cdots n) = \mathrm{Fc}[a_{n-r+1} | r, 1 \cdots (n-1)]$

·1 dt » /dt » $= \mathrm{Fc}(a_{n-r+1} | r, 1 \cdots n)$

·2 $\mathrm{nt\,Fc}(a, 1 \cdots n) = \mathrm{dt\,Fc}[a, 1 \cdots (n-1)] . = .$
$\mathrm{Fc}(a, 1 \cdots n) = \mathrm{Fc}(a_{n-r+1} | r, 1 \cdots n)$

·3 $p, q \varepsilon \mathrm{N_1} . p < q . p^2 \, \varepsilon \, (q \times \mathrm{N_1} + 1) \cup (q \times \mathrm{N_1} - 1) . \supset .$
$\exists \mathrm{N_1} \frown n \ni \{ \exists \mathrm{N_1 f} \, 1 \cdots n \frown a \ni [p/q = \mathrm{Fc}(a, 1 \cdots n) = \mathrm{Fc}(a_{n-r+1} | r, 1 \cdots n)]\}$

·4 $\mathrm{Fc}(a, 1 \cdots n) = \mathrm{Fc}(a_{n-r+1} | r, 1 \cdots n) . \supset .$
$[\mathrm{nt\,Fc}(a, 1 \cdots n)]^2 \, \varepsilon \, [(\mathrm{dt\,Fc}(a, 1 \cdots n)) \times \mathrm{N_1} + 1] \cup [(\mathrm{dt\,Fc}(a, 1 \cdots n)) \times \mathrm{N_1} - 1]$
{ Serret JdM. t.13 s.1 a.1848 p.12 }

✻ 10. $a \varepsilon \mathrm{N_1 f N_1} \,.\, n \varepsilon \mathrm{N_1} \,.\supset.$ ·1 $\mathrm{Fc}(a, \mathrm{N_1}) \;\varepsilon\; \theta\text{-}\mathrm{R}$

·2 $\mathrm{Fc}(a, \mathrm{N_1}) - \mathrm{Fc}(a, 1\cdots n) \;\varepsilon\; \theta(-1)^n /\{[\mathrm{dt\, Fc}(a, 1\cdots n)]^2$

·3 $\mathrm{Fc}(a, 1\cdots 2n) < \mathrm{Fc}(a, \mathrm{N_1}) < \mathrm{Fc}[a, 1\cdots(2n-1)]$

·4 $b, c \varepsilon \mathrm{N_1} \,.\, \mathrm{mod}\, [\mathrm{Fc}(a, \mathrm{N_1}) - b/c] < \mathrm{mod}\, [\mathrm{Fc}(a, \mathrm{N_1}) - \mathrm{Fc}(a, 1\cdots n)]$
$.\supset.\, b > \mathrm{ntFc}(a, 1\cdots n) \,.\, c > \mathrm{dtFc}(a, 1\cdots n)$ {Euler a.1748 §382}

✻ 11·1 $x \varepsilon \mathrm{Q\text{-}R} \,.\supset.\, x = \mathrm{E}x + \mathrm{Fc}(\mathrm{E}(/\beta)^n x | n, \mathrm{N_1})$

·2 $(\sqrt{5}-1)/2 = \mathrm{Fc}\,(1, 1, 1, \ldots) = \mathrm{Fc}(\iota 1 : \mathrm{N_1})$

nt de ce fractio continuo forma « serie de Fibonacci »: 1,1,2,3,5,8,13,...

·3 $a \varepsilon \mathrm{N_1} \,.\supset.\, \sqrt{(a^2+1)} = a + \mathrm{Fc}\,(2a, 2a, 2a, \ldots)$

✻ 12·1 $a \varepsilon \mathrm{Q f N_1} \,.\supset:\, \mathrm{Fc}(a, \mathrm{N_1}) \varepsilon \mathrm{Q} \,.=.\, \Sigma(a, \mathrm{N_1}] = \infty$

{ Seidel, Untersuchungen über die Konvergenz und Divergenz der Kettenbrücke, München a.1846. }

✻ 13. $\mathrm{e} = 2 + \mathrm{Fc}(1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots)$

$(\mathrm{e}-1)/(\mathrm{e}+1) = \mathrm{Fc}(2, 6, 10, 14, 18, \ldots) = \mathrm{Fc}[(4x-2) | x, \mathrm{N_1}]$

$2/(\mathrm{e}^2-1) = \mathrm{Fc}(3, 5, 7, \ldots) = \mathrm{Fc}[(2x+1) | x, \mathrm{N_1}]$

$(\sqrt{\mathrm{e}} - 1)/2 = \mathrm{Fc}(3, 12, 20, 28, 54, \ldots)$

$\sqrt{\mathrm{e}} - 1 = \mathrm{Fc}(1, 1, 1, 5, 1, 1, 9, 1, 1, 13, \ldots)$

$(\sqrt[3]{\mathrm{e}} - 1)/2 = \mathrm{Fc}(5, 18, 30, 42, 54, \ldots)$

{ Euler a.1737 PetrC. t.9 (edito in 1744) p.121 }

$n \varepsilon \mathrm{N_1} \,.\supset.$

$(\sqrt[n]{\mathrm{e}}-1)/(\sqrt[n]{\mathrm{e}}+1) = \mathrm{Fc}(2n, 6n, 10n, \ldots) = \mathrm{Fc}[(4x-2)n \,|x, \mathrm{N_1}]$

$2/(\sqrt[n]{\mathrm{e}}-1) - 2n + 1 = \mathrm{Fc}(6n, 10n, \ldots) = \mathrm{Fc}[(4x+2)n \,|x, \mathrm{N_1}]$

{ Euler a.1737 PetrC. t.9 (edito in 1744) p.132 }

✻ 14. $\pi = 3 + \mathrm{Fc}\,[7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 1, \ldots]$ {Wallis t.2 p.51}

Reductione in fractio continuo de valore de π dato per Ludolff.

VI

CALCULO DIFFERENTIALE

VI. CALCULO DIFFERENTIALE.

§1 D (derivata)

Derivata de functione es limite de ratione de incremento de functione ad incremento de variabile, quando incremento de variabile tende ad 0.

Ergo nos considera functione f, que es dato pro valores de variabile pertinente ad aliquo classe u. Classe u pote coincide cum classe de numeros reale q, vel cum intervallo, vel habe forma plus complexo:

$$u \varepsilon \text{Cls'q} \ . \ f \varepsilon \ \text{qf}u.$$

Nos sume in classe u uno numero x, que nos suppone proximo ad alios valore de classe u:

$$x \varepsilon \ u \frown \delta u.$$

Ratione $(fy-fx)/(y-x)$ depende de y. Suo limite, quando y varia in u et tende ad x, es vocato « derivata ».

❋ 1. $u \varepsilon \text{Cls'q} \ . \ f \varepsilon \ \text{qf}u \ . \ x \varepsilon \ u \frown \delta u \ . \supset .$

·0 $D(f,u,x) = \lim[(fy-fx)/(y-x) \,|\, y, u, x]$ Df D

Si u es classe de quantitate, si f indica quantitate functione des u, et si x pertine ad classe u, et ad suo derivata, tunc $D(f,u,x)$, que nos lege « derivata de functione f, in campo u, pro valore x » vale limite de $(fy-fx)/(y-x)$ ubi varia y, in campo u, et tende ad x.

Membro definiente derivata, contine literas reale f, u, x et litera y apparente, nam seque signo de inversione |. Ergo nos indica derivata quæsito per signo $D(f,u,x)$, ubi figura omni litera variabile reale.

Campo u de variabilitate es necessario in notatione de derivata, nam derivata pote depende de campo u. Vide P6·6. In aliquo casu non tace illo, justa conventiones ·1·4:

·1 $D(f,u) = [D(f,u,x)|x, u]$ Df

Nos indica per $D(f,u)$ lege « derivata de f in campo u » functione $D(f,u,x)$ ubi x varia in campo u ».

·2 $D(f,u)x = D(f,u,x)$ [P·1⊃P]

Si nos indica per uno litera h systema (f,u) de functione et de campo de variabilitate relativo, tunc h es « functione definito », et suo derivata pro valore x resulta indicato per Dhx, que significa $(Dh)x$ « valore de derivata, pro valore x de variabile », et non $D(hx)$, « derivata de numero hx », que non habe sensu.

·3 $u,v \varepsilon$ Cls'q . $f \varepsilon$ qf$(u \cup v)$. $x \varepsilon$ in$u \cap$ in$v$.⊃.
$D(f,u,x) = D(f,v,x)$

·4 $u \varepsilon$ Cls'q . $f \varepsilon$ qf$u$. $x \varepsilon$ in$u$.⊃.
$D(f,x) = Dfx = \imath y \ni \{v \varepsilon$ Cls'u . $x \varepsilon$ in$v$.$\supset_v$. $y = D(f,v,x)\}$

Si x es interno ad campo u, tunc nos tace campo u, et per $D(f,x)$ vel Dfx nos intellige valore constante de $D(f,v,x)$, ubi v es classe arbitrario, que contine x in suo interno.

Nota.

Leibniz indica derivata de y relativo ad x, per signo $\frac{dy}{dx}$ ubi « recta aliqua pro arbitrio assumpta vocetur dx » (MathS. t.5 p.220) et « ipsas dx, dy, ut ipsarum x,y differentiis sive incrementis, vel decrementis momentaneis proportionales haberi posse » (p.169). Acta Erud. Lips. a.1684:

« *Additio et Subtractio*: si sit $z-y+w+x$ æqu. v, erit $d\overline{z-y+w+x}$ seu dv æqu. $dz-dy+dw+dx$.

Multiplicatio: $d\overline{xv}$ æqu. $xdv+vdx$

Potentiæ: $dx^a = a.x^{a-1}dx$.

Radices: $d\sqrt[b]{x^a} = \frac{a}{b}dx\sqrt[b]{x^{a-b}}$

Suffecisset autem regula potentiae integrae tam ad fractas tam ad radices determinandas. »

In aliquo casu ille pone $d\bar{x}=1$; tunc signo d indica derivata:

$$d\bar{x} = 1 \text{ , } d\bar{x}^2 = 2x \text{ , } d\bar{x}^3 = 3x^2 \text{ etc. } \quad d\sqrt{x} = \frac{1}{2\sqrt{x}} \text{ etc.}$$

(*Briefwechsel* t.1 p.226)

Newton indica derivata per uno puncto supra functione (vide Taylor in P15); Lagrange per uno accentu (vide P17); Arbogast per Dfx.

Cauchy (*Œuvres* s.1 t.4 p.255) indica derivatas per D_x , D_y ,... ubi indice designa variabile respectu que nos deriva.

Jacobi, Werke, t.3 p.396 a.1841:

« quando sine graviori incommodo licet, quanquam maxime affectanda sunt signa, quibus et omnis ambiguitas tollatur, et formulae sine omni interpretatione verbali adjecta, per se clarae et intelligibiles fiant, in hoc tamen casu... ».
distingue derivatas de functione de plure variabile, per signo ∂; ce derivatas vocare « partiale ».

Ce notatione ne suffice, quare si nos habe functione de 3 variabile $f\varepsilon$ qf(q⁝q⁝q), et si $u\varepsilon$ qfq, $w\varepsilon$ qf(q⁝q), es necesse plure specie de d pro indica 4 derivata respectu x:

D $f(x,y,z)|x$, D $f(x,ux,z)|x$, D $f[x,y,w(x,y)]|x$, D $f[x,ux,w(x,ux)]|x$.

467. ***derivata.*** Vide 280.

468. ***differentiale***, AD differentiale, F différentielle, H differencial, I differenziale, R differentsïal'. Es dx de Leibniz.

⊂ differentia (166) + -le (6)

De textu praecedente resulta que in scriptura dy/dx de Leibniz, hodie multo diffuso, dx et dy es quantitate finito, dx es arbitrario, non nullo; in casu particulare dx = 1 (Leibniz), dy = Dy. In applicationes de Calculo ad praxi, aliquo Auctore considera differentiale ut quantitate satis parvo, vel infinite parvo.

✱ 2.

$u\varepsilon$ Cls'q . $f,g\varepsilon$ qf$u$. $x\varepsilon$ u⁀δu . D(f,u,x), D(g,u,x) εq . $a\varepsilon$q .⊃.

·1 D[(a+fx)|x, u, x] = D(f,u,x)

« Derivata de summa de quantitate constante et de functione vale derivata de functione ».

·2 D[(fx+gx)|x, u, x] = D(f,u,x) + D(g,u,x) Distrib(D, +)

« Derivata de summa de duo functione vale summa de derivatas de functiones ».

[Df D .⊃. D[(fx+gx)|x,u,x] = lim{[(fy+gy)−(fx+gx)]/(y−x) |y, u, x}
Distrib(/,+).⊃. » = lim{ (fy−fx)/(y−x) + (gy−gx)/(y−x) »
Distrib(lim,+) .⊃. P]

·3 $x\varepsilon$q .⊃. D(a×x |x, x) = a

·4 D(a×fx |x, u, x) = a×D(f,u,x)

« Derivata de producto de quantitate constante per functione vale quantitate constante per derivata de functione ».

·5 $\mathrm{D}(fx \times gx \mid x, u, x) = fx \times \mathrm{D}(g,u,x) + gx \times \mathrm{D}(f,u,x)$

« Derivata de producto de duo functione vale primo factore per derivata de secundo, plus secundo factore per derivata de primo ».

$$[\ \mathrm{D}(fx \times gx \mid x, u, x) = \lim[(fy \times gy - fx \times gx)/(y-x) \mid y, u, x]$$
$$= \lim[fx \times (gy - gx)/(y-x) + gx \times (fy - fx)/(y-x) +$$
$$(fy-fx)/(y-x) \times (gy-gx)/(y-x) \times (y-x) \mid y, u, x]$$
$$= fx \times \mathrm{D}(g,u,x) + gx \times \mathrm{D}(f,u,x) + 0\ ]$$

❋ 3·1 $u \varepsilon \mathrm{Cls} \,.\, f,g \varepsilon \mathrm{qF}u \,.\supset.\, f+g = [(fx+gx) \mid x, u]$ Df

·2 $u \varepsilon \mathrm{Cls} \,.\, f \varepsilon \mathrm{qF}u \,.\, a \varepsilon \mathrm{q} \,.\supset.\, a \times f = (a \times fx \mid x, u)$ Df

·3 $u,v,w \varepsilon \mathrm{Cls} \,.\, f \varepsilon v\mathrm{F}u \,.\, g \varepsilon w\mathrm{F}v \,.\supset.\, gf = [g(fx) \mid x, u]$ Df

·4 $u \varepsilon \mathrm{Cls'q} \,.\, u \supset \delta u \,.\, f,g,\mathrm{D}f,\mathrm{D}g \ \varepsilon\ \mathrm{qF}u \,.\supset.\, \mathrm{D}(f+g) = \mathrm{D}f + \mathrm{D}g$
[=P2·2]

·5 $\mathrm{Hp}·4 \,.\, a \varepsilon \mathrm{q} \,.\supset.\, \mathrm{D}(a \times f) = a \times (\mathrm{D}f)$ [=P2·4]

Df ·1-·3 simplifica aliquo formula praecedente. Df ·1 jam occurre in III §20 Cx P1·2.

❋ 4·1 $u,v \varepsilon \mathrm{Cls'q} \,.\, f \varepsilon v\mathrm{F}u \,.\, g \varepsilon \mathrm{qF}v \,.\, x \varepsilon u \frown \delta u \,.\, fx \ \varepsilon\ \delta v \,.$
$\mathrm{D}fx, \mathrm{D}g(fx) \ \varepsilon \mathrm{q} \,.\supset.\, \mathrm{D}(gf)x = \mathrm{D}g(fx) \times \mathrm{D}fx$

u et v es classe de quantitates ; f es functio definito, que ad omni u fac corresponde aliquo v ; g es quantitate functio definito de v ; x pertine ad classe u et ad suo derivata ; nos suppone fx, que es v, proximo ad alios v. Functio f habe derivata pro valore x, et g habe derivata pro valore fx. Tunc derivata de functio gf (functio composito de g et f, producto functionale de g et de f) vale derivata de g pro valore fx multiplicato per derivata de f, pro valore x.

$$[\ \mathrm{D}(gf)x = \lim[(gfy - gfx)/(y-x) \mid y, u, x] \qquad (1)$$
$$\lim[(gfy-gfx)/(y-x) \mid y,\ u \frown y \ni (fy - = fx),\ x]$$
$$= \lim[(gfy-gfx)/(fy-fx) \times (fy-fx)/(y-x) \mid y,u,x]$$
$$= \mathrm{D}g(fx) \times \mathrm{D}fx \qquad (2)$$
$$y \varepsilon u \,.\, fy = fx \,.\supset.\, gfy = gfx \,.\supset.$$
$$(gfy-gfx)/(y-x) = \mathrm{D}g(fx) \times (fy-fx)/(y-x) \qquad (3)$$
$$(3) \,.\, \mathrm{Oper\ lim} \,.\supset.\, \lim[(gfy-gfx)/(y-x) \mid y,\ u \frown y \ni (fy=fx),\ x]$$
$$= \mathrm{D}g(fx) \times \mathrm{D}fx \qquad (4)$$
$$(1) \,.\, (2) \,.\, (4) \,.\supset.\, \mathrm{P}\ ]$$

·2 $u,v \varepsilon \text{Cls'q} \,.\, f \varepsilon (v\text{F}u)\text{rcp} \,.\, x \varepsilon u \frown \delta u \,.\, \text{D}fx \,\varepsilon\, \text{q}-\iota 0 \,.\, y = fx$
$.\supset.\ \text{D}f^{-1}y = /\text{D}fx$

Si u et v es classe de quantitates, et si f es v functio definito des u, et reciproco (III §1 P4·6), si x pertine ad classe u et ad suo classe derivata, et si derivata de functio f pro valore x, habe valore determinato finito non nullo, et si nos voca y valore fx, tunc derivata de functio inverso de f (definito in III §4 P3), pro valore y, vale reciproco de derivata de f pro valore x.

In modo plus breve, si nos tace plure conditione:

« Derivata de functio inverso es reciproco de derivata de functio directo ».

$[\ \text{D}(f^{-1},v,y) = \lim[(f^{-1}z - f^{-1}y)/(z-y)|z,v,y]$
$= \lim[(w-f^{-1}y)/(fw-y)|w,u,x]$
$= /\lim(fw-fx)/(w-x)|w,u,x]$
$= /\text{D}(f,u,x)\ ]$

❋ 5·1 $x \varepsilon \text{q}-\iota 0 \ .\supset.\ \text{D}(/, x) = -/x^2$
$[\ \text{D}(/, x) = \lim[(/y - /x)/(y-x)\ |y, \text{q}-\iota 0, x]$
$= \lim[-/(yx)$ » » $] = -/x^2\ ]$

·2 $\text{Hp P2} \,.\, 0 -\varepsilon f'u \ .\supset.\ \text{D}(/f,u,x) = -\text{D}(f,u,x)/(fx)^2$
$[\ (\text{q}-\iota 0, /) \mid (v,g)\ \text{P4·1} \ .\supset.$ » $= \text{D}(/,fx) \times \text{D}(f,u,x) \,.\, \text{P·1} \ .\supset \text{P}]$

·3 $\text{Hp·2} \ .\supset.\ \text{D}(gx/fx\,|x,u,x) =$
$[fx \times \text{D}(g,u,x) - gx \times \text{D}(f,u,x)]/(fx)^2$

Dem. 1 [P2·5 . P5·2 $.\supset.$ P]

Dem. 2 $[\ \text{D}(gx/fx \mid x,u,x) = \lim[(gy/fy - gx/fx)/(y-x) \mid y,u,x]$
$= \lim\{(fx \times gy - gx \times fy)/[(y-x) \times fx \times fy] \mid y, \ldots$
$= \lim\{fx \times (gy-gx) - gx \times (fy-fx)/[(y-x) \times fx \times fy] \mid y, \ldots$
$= \lim\{[fx \times (gy-gx)/(y-x) - gx \times (fy-fx)/(y-x)]/(fx \times fy) \mid y, \ldots$
$= (fx\text{D}gx - gx\text{D}fx)/(fx)^2\ ]$

« Derivata de quotiente de duo functio vale denominatore per derivata de numeratore, minus numeratore per derivata de denominatore, toto diviso per quadrato de denominatore ».

❋ 6·1 $m \varepsilon \text{N}_1 \,.\, x \varepsilon \text{q} \ .\supset.\ \text{D}(x^m|x, x) = mx^{m-1}$

Dem. 1 $[\ \text{D}(x^n|x, x) = \lim[(y^m - x^m)/(y-x)|y, \text{q}, x]$
$= \lim[\Sigma(y^{m-r}x^{r-1} \mid r, 1\cdots m)|y, \text{q}, x] = mx^{m-1}\ ]$

Dem. 2 $[\ m=1 \ .\supset \text{P}$ (1)

$m \varepsilon \text{N}_1 \,.\, \text{D}(x \urcorner m|x, x) = m\,x \urcorner (m-1) \,.\, \text{P2·5} \ .\supset.$

$\text{D}(x^{m+1}|x, x) = \text{D}[(x^m \times x)|x, x] = mx^{m-1} \times x + x^m = (m+1)x \urcorner m$ (2)

(1) . (2) . Induct $.\supset.$ P]

Si m es numero naturale, et x es quantitate, tunc derivata de x^m, ubi varia x, in campo de numeros reale, pro valore x de variabile, vale mx^{m-1}.

Nota que in formula $x^m \mid x$, litera x es apparente; formula vale $z^m|z$, et indica « potestate m »; litera x non es idem litera x, que occurre in Hp. Si nos pone $m = 2$, formola fi:

$$x\varepsilon q \,.\supset.\, D(x^2 \mid x, x) = 2x,$$

et si nos pone $x = 1$, formula fi:

$$D(x^2 \mid x, 1) = 2.$$

« derivata de quadrato, in campo de valores reale, pro valore 1, vale 2 ».

In formula incompleto $Dx^2 = 2x$, que occurre in plure libro, non lice substitutione materiale de valore numerico ad x.

Demonstratione 1. In vero, ratione de incremento de functione ad incremento de variabile es summa de m termine, que omni tende ad x^{m-1}.

Demonstratione 2. Potestate es casu particulare de producto, et regula deriva ex regula de derivatione de producto.

P·2·3·4·5 extende regula ad alio valore de exponente.

·2 $m\varepsilon n \,.\, x\varepsilon q - \iota 0 \,.\supset.\, D(x^m|x, x) = mx^{m-1}$

$[\; m\varepsilon N_0 \,.\, P{\cdot}1 \,.\supset.\, P$ (1)

$n\varepsilon N_1 \,.\, m = -n \,.\supset.\, D(x^{-n}|x, x) = D(/x^n\,|x, x) =$

$-nx^{n-1}/x^{2n} = -nx^{-n-1} = mx^{m-1}$ (2)

$(1) \,.\, (2) \,.\supset.\, P \;]$

·3 $x\varepsilon Q \,.\supset.\, D(\sqrt{\,}, x) = /(2\sqrt{x})$

$[\quad » \quad = \lim(\sqrt{y} - \sqrt{x})/(y-x) = \lim /(\sqrt{y} + \sqrt{x}) \;]$

·4 $m\varepsilon N_1 \,.\, x\varepsilon Q \,.\supset.\, D(\sqrt[m]{x}\,|x, x) = /[m\,\sqrt[m]{x^{m-1}}]$

Dm. 1 $[\; D(x^{/m}\,|x, x) = \lim[(y^{/m} - x^{/m})/(y-x)\,|y, Q, x]$

$= /\lim\{[(y^{/m})^m - (x^{/m})^m]/(y^{/m} - x^{/m})$ » »

$= /[m(x^{/m})^{(m-1)}] \;]$

Dm. 2 $[\; (\sqrt[m]{\,}, Q) = (z^m \mid z, Q)^{-1} \,.\supset.$

$D(\sqrt[m]{\,}, Q, x) = /D(z^m \mid z, Q, \sqrt[m]{x}) = /[(mz^{m-1} \mid z)|\sqrt[m]{x}] = /[m(\sqrt[m]{x})^{m-1}] \;]$

·5 $m\varepsilon r \,.\, x\varepsilon Q \,.\supset.\, D(x^m|x, x) = mx^{m-1}$

$[\; P{\cdot}1 \,.\, P{\cdot}4 \,.\supset.\, P \;]$

·6 $x\varepsilon q - \iota 0 \,.\supset.\, D(\mathrm{mod}, x) = \mathrm{sgn}\,x \,.$

$D(\mathrm{mod}, Q_0, 0) = 1 \quad.\quad D(\mathrm{mod}, -Q_0, 0) = -1$

Derivata de functione « mod » depende de campo de variabilitate.

·7 $a,b,c,d\varepsilon q \,.\, x\varepsilon q \,.\, a+bx -= 0 \,.\supset.$

$D[(c+dx)/(a+bx)|x, x] = (ad-bc)/(a+bx)^2$

·8 $x\varepsilon q \,.\supset.\, D[x/\sqrt{(1+x^2)}|x, x] = (1+x^2)^{(-3/2)}$

✻ 7. Theorema de maximo et minimo.

·1 $a,b \varepsilon \mathrm{q} \,.\, a < b \,.\, f \varepsilon \mathrm{qF} a \dashv b \,.\, x \varepsilon a^{-}b \,.\, fx = \max f`a \dashv b \,.\, \mathrm{D}fx \,\varepsilon \mathrm{q}$
$.\supset.\ \mathrm{D}fx = 0$

[Hp $.\supset.\ \mathrm{D}fx = \lim[\,(fy-fx)/(y-x)\,|y,\ a \dashv x,\ x\,]$ (1)
Hp $.\ y \varepsilon a \dashv b \frown (x-\mathrm{Q})\ .\supset.\ fy-fx \leqq 0$
» » » $.\supset.\ (fy-fx)/(y-x) \geqq 0$ (2)
(1) . (2) $.\supset.\ \mathrm{D}fx \geqq 0$ (3)
Hp $.\supset.\ \mathrm{D}fx = \lim[\,(fy-fx)/(y-x)\,|y,\ x \vdash b,\ x\,]$ (4)
$y \varepsilon a \dashv b \frown (x+\mathrm{Q})\ .\supset.\ (fy-fx)/(y-x) \leqq 0$ (5)
(4) . (5) $.\supset.\ \mathrm{D}fx \leqq 0$ (6)
(3) . (6) $.\supset.$ P]

a et b indica quantitate, et per exemplo $a<b$, et f indica quantitate functione definito in intervallo de a ad b. Si ad valore x, interno ad intervallo considerato, responde valore fx maximo inter valores de functione in toto intervallo, et si derivata de functione f, pro valore x, habe valore determinato et finito, tunc derivata vale zero.

In vero $\mathrm{D}fx$, que per definitione es limite de ratione $(fy-fx)/(y-x)$, quando y varia in toto intervallo, es etiam limite de dicto ratione, quando y varia in solo intervallo de a ad x. Tunc $fy-fx \leqq 0$, nam fx es maximo valore de functione, et $y-x<0$; ergo ratione considerato es $\geqq 0$, et suo limite es $\geqq 0$.

In modo analogo, derivata es etiam limite de ratione considerato, quando y varia in solo intervallo de x ad b. Tunc es semper $fy-fx \leqq 0$, sed $y-x>0$; ergo ratione considerato es $\leqq 0$, et suo limite es $\leqq 0$.

Ergo $\mathrm{D}fx$, que es $\geqq 0$ et $\leqq 0$, vale 0.

·2 (min | max)P·1

P·1 subsiste, si in loco de maximo valore, nos considera minimo. Plure auctore moderno voca « extremo » valore que es aut maximo aut minimo.

Exemplo. Nos vol decompone dato numero a in duo partes x et $a-x$, tale que producto de potestates m et n de duo parte fi maximo. Exponentes m,n es p. ex. numero naturale. Functione $x^m(a-x)^n|x$, es nullo pro valores 0 et a de variabile, et es positivo in interno de intervallo de 0 ad a, et es continuo. Ergo ce functio fi maximo pro valore de variabile interno ad intervallo considerato. Ce valore annulla derivata, vel satisfac aequatione :

$$mx^{m-1}(a-x)^n - nx^m(a-x)^{n-1} = 0\,;$$

si nos supprime factores x^{m-1} et $(a-x)^{n-1}$, que non es nullo in interno de intervallo considerato, aequatio fi

$$m(a-x) - nx = 0, \text{ vel } x/m = (a-x)/n\,.$$

Valore de x que satisfac ad aequatione es unico, et es valore quaesito. Ergo duo parte debe es proportionale ad exponentes, ut es noto per Algebra, III §12 P30.

Applicationes de theorema praecedente ad Geometria. (*Exercitio*).

1. Rectangulo inscripto in triangulo, et maximo in area, habe altitudine aequale ad altitudine de triangulo /2.

2. Cylindro in sphaera, maximo in volumen, habe altitudine $= 2r/\surd 3$, radio basi $= r\surd(2/3)$; r es radio de sphaera.

3. Cylindro in sphaera, maximo in superficie laterale, habe diametro de basi = altitudine = (radio sphaera)$\times\surd 2$.

4. Idem, maximo in superficie totale, habe altitudine = (radio sphaera) $\times\surd 2\surd[1-1/\surd 5]$.

5. Cono in sphaera, maximo in volumen, habe altitudine = (radio sphaera) $\times 4/3$. (Fermat a.1636).

6. Idem, maximo in superficie laterale, idem.

7. Idem, maximo in superficie totale, habe altitudine = radio $\times$ $(23-\surd 17)/16$.

8. Cylindro inscripto in cono, et maximo in volumen, habe altitudine aequale ad altitudine de cono /3.

9. Idem, maximo in superficie laterale, habe altitudine aequale ad altitudine de cono /2.

·3 $a,b\varepsilon\mathrm{q} \,.\, a \mathrel{-\!\!=} b \,.\, f,\mathrm{D}f \,\varepsilon\, \mathrm{qF}a\dashv b \;.\supset.\; f\varepsilon\,(\mathrm{qF}a\dashv b)\mathrm{cont}$

Functione que habe derivata in dato intervallo, es continuo.

[$\mathrm{Hp} \,.\, x\varepsilon\, a\dashv b \;.\supset.\; \lim[(fy-fx)|y, a\dashv b, x] =$
$\lim[(fy-fx)/(y-x) \times (y-x)\,|y, a\dashv b, x] = (\mathrm{D}fx)\times 0 = 0$]

❋ 8. Theorema de Rolle.

·1 $a,b\varepsilon\mathrm{q} \,.\, a \mathrel{-\!\!=} b \,.\, f,\mathrm{D}f \,\varepsilon\, \mathrm{qF}a\dashv b \,.\, fa = fb = 0 \;.\supset.$
$\exists\, a{}^{-}b \frown x\ni(\mathrm{D}fx = 0)$

Si functione f, habente derivata in toto intervallo de a ad b, es nullo pro valores a et b, tunc existe valore interno ad intervallo de a ad b, que redde derivata nullo.

{ Rolle a.1689 p.127:

« Les racines de chaque cascade (derivata) seront prises pour les hypotheses moyennes de la cascade suivante ». }

[$\mathrm{Hp} \,.\, \mathrm{P}7{\cdot}3 \;.\supset.\; f\varepsilon\,(\mathrm{qF}a\dashv b)\mathrm{cont}$ (1)
$\mathrm{Hp} \,.\, (1) \,.\, \S\mathrm{cont}\; 2{\cdot}3 \;.\supset.\; \max f\text{‘}a\dashv b, \min f\text{‘}a\dashv b \,\varepsilon\mathrm{q}$ (2)
$\mathrm{Hp} \,.\, \exists\, \mathrm{Q}\frown f\text{‘}a\dashv b \;.\supset.\; \max f\text{‘}a\dashv b \,\varepsilon\mathrm{Q}$
» » » $.\, x\varepsilon\, a\dashv b \,.\, fx = \max f\text{‘}a\dashv b \;.\supset.\; x\varepsilon\, a{}^{-\,-}b$ (3)
» » » » » » » » $.\, (3) \,.\, \mathrm{P}7{\cdot}1 \;.\supset.\; \mathrm{Ths}$ (4)

$\text{Hp} . \exists (-Q) \frown f`a \vdash b . \supset . \exists Q \frown (-f)`a \vdash b . (1) . \supset . \text{Ths}$ (5)
$\text{Hp} . -\exists Q \frown f`a \vdash b . -\exists (-Q) \frown f`a \vdash b . \supset . f \varepsilon \iota 0 \, \text{F} a \vdash b . \supset . \text{Ths}$ (6)
$(4) . (5) . (6) . \supset . \text{P} \,]$ Ex. Dm P9·1·2, P31·1

Functio f, que habe derivata, per theorema praecedente es continuo. Ergo existe suo maximo et suo minimo valore in toto intervallo. Si functio f sume valores positivo, tunc maximo intra illos es positivo, et responde ad valore interno ad intervallo. Ergo per theorema super maximo, derivata es nullo. In modo analogo, si functio sume valores negativo, vel si es semper nullo.

✻ 9. Theorema de valore medio.

·1 $a,b \varepsilon \text{q} . a -= b . f, \text{D}f \varepsilon \text{qF} a \vdash b . \supset . (fb - fa)/(b-a) \varepsilon \text{D}f` a \bar{\ } b$

$[\text{Hp} . g = [fx - fa - (x-a)(fb-fa)|(b-a)] | x . \supset .$
$ga = gb = 0 . \text{D}g = \text{D}f - (fb-fa)|(b-a) . \text{P8} . \supset .$
$\exists \, a \bar{\ } b \frown x \ni [\text{D}fx - (fb-fa)|(b-a) = 0] . \supset . \text{Ths} \,]$

« Si a,b es quantitate differente inter se, et functio f habe derivata in toto intervallo de a ad b, tunc ratione de incremento de functione ad incremento de variabile es uno de valores de derivata in interno de intervallo ».

Invero, nos considera functione $fx-(px+q)$, ubi varia x, et determina coefficientes p et q, in modo que ce functione es nullo pro $x=a$ et $x=b$.

Tunc functione g habe forma scripto. Per theorema de Rolle, suo derivata es nullo pro valore interno ad intervallo, unde seque P.

Theorema praecedente es multo importante in calculo differentiale, et vocare « Theorema de valore medio ».

In geometria significa « si arcu de curva, in plano, habe tangente in omni suo puncto interno ad arcu, in aliquo puncto de arcu tangente es parallelo ad chorda.

{ Cavalieri a.1635 l.VII p.15.

« Si curva linea quaecunque data tota sit in eodem plano, cui occurrat recta in duobus punctis..... poterimus aliam rectam lineam praefatae aequidistantem ducere, quae tangat portionem curvae lineae inter duos prædictos occursus continuatam ». }

Exemplo. Si nos considera radice secundo, et duo valore de variabile 100 et 101, theorema dice:

$$\sqrt{101} - \sqrt{100} \; \varepsilon \; 1/[2\sqrt{(100+\theta)}]$$
$$\text{vel} \; \sqrt{101} < 10 + 1/20 = 10{\cdot}050$$
$$\text{et} \; \sqrt{101} > 10 + 1/21 = 10{\cdot}047 \ldots$$

·2 $a,b \varepsilon \text{q} . a -= b . f, \text{D}f, g, \text{D}g \varepsilon \text{qF} a \vdash b . 0 -\varepsilon \text{D}g`a \bar{\ } b . \supset .$
$(fb-fa)/(gb-ga) \varepsilon [(\text{D}fx)/(\text{D}gx)] | x ` a \bar{\ } b$

[Hp . $h = [fx - fa - (gx - ga)(fb - fa)/(gb - ga)] \mid x$.⊃.
$ha = hb = 0$. P8 .⊃. ∃ $a^{-}b \frown x \ni [Dfx - Dgx \times (fb - fa)/(gb - ga) = 0]$
.⊃. P]

} CAUCHY *Calc. diff.* a.1829 p.37 {

Si a et b es quantitate non aequales, si f et g es functione reale definito in intervallo de a ad b, simul cum derivatas, et si derivata de g non es nullo in interno de intervallo considerato, tunc ratione de incrementos de duo functione es uno de valores que sume ratione de derivatas, respondente ad valore de variabile interno ad intervallo de a ad b.

In vero, me voca hx functione de x de forma $fx + pgx + q$, ubi p et q es quantitate que me determina in modo que $ha = hb = 0$. Functio h habe expressio scripto. Tunc per theorema de Rolle, suo derivata es nullo per uno valore inter a et b; unde seque theorema.

Ce theorema es vocato « secundo theorema de valore medio ».

✻ 10.

·1 $a,b\varepsilon$q . $a -= b$. $f\varepsilon$ qF$a^{\vdash}b$. D$f\varepsilon$ QF$a^{\vdash}b$.⊃. $f\varepsilon$ (qF$a^{\vdash}b$) cres

·2 » » (−Q) » » » decr

·3 » » (ι0) » » » const

[P9·1 ⊃ P]

Si valore de derivata de functio f es semper positivo, functio es crescente; si negativo, functio es decrescente. Si derivata habe semper valore nullo, pro omni valore de variabile inter a et b, tunc functio es constante. Seque de theorema de valore medio.

✻ 11. $a,b\varepsilon$q . $a -= b$. $f\varepsilon$ qF$a^{-}b$.⊃:

·1 $x\varepsilon a^{-}b$. D$f\varepsilon$ qF($a^{-}b - \iota x$) . lim(Df, $a^{-}b$, x)εq .⊃.
D(f, $a^{\vdash}b$, x) = lim(Df, $a^{-}b$, x)

·2 D$f\varepsilon$ qF($a^{\vdash}b$) . $x\varepsilon a^{\vdash}b$.⊃. D$fx\,\varepsilon$ Lm(Df, $a^{\vdash}b$, x)

·3 » . Dfa<0 . Dfb>0 .⊃. 0ε Df'$a^{-}b$

·4 » .⊃. Dfa^{-} Dfb ⊃ Df'$a^{-}b$

·5 » . $h\varepsilon$Q .⊃. ∃(n,x)∋}$n\varepsilon$N$_1$. $x\varepsilon$ ($a^{-}b$ f0⋯n)cres .
$x_0 = a$. $x_n = b$: $r\varepsilon$ 0⋯(n−1) .⊃$_r$. mod(Dfx_{r+1} − Dfx_r)<h{

$\cdot 6 \quad h \varepsilon \mathrm{Q} \;.\supset.\; \exists (n,x) \ni \{ n \varepsilon \mathrm{N}_1 \,.\, x \varepsilon (a \urcorner b \mathrm{f} 0 \cdots n) \mathrm{cres} \,.$
$x_0 = a \,.\, x_n = b : r \varepsilon\, 0 \cdots (n-1) \;.\supset_r.$
$\mathrm{mod}[\mathrm{D}fx_r - (fx_{r+1} - fx_r)/(x_{r+1} - x_r)] < h \}$

Derivata de uno functione, et quando non es continuo, habe plure proprietate de functiones continuo; p. ex.: si pro valore *a* derivata habe valore negativo, et pro *b* valore positivo, derivata sume valore nullo inter *a* et *b*. Vide meo scripto Ann.N. a.1884 p. 45, 153, 252; Goursat AM. a.1884, p. 49, 316.

❋ 12. Theorema de De l'Hospital.

$\cdot 1 \quad a,b \varepsilon \mathrm{q} \,.\, a \neq b \,.\, f,g \varepsilon\, \mathrm{qF} a \urcorner b \,.\, fa = ga = 0 \,.\, \mathrm{D}f, \mathrm{D}g \;\varepsilon\; \mathrm{qF} a \urcorner b \,.$
$0 \notin \mathrm{D}g \text{'} a \urcorner b \;.\supset.\; \mathrm{Lm}(fx/gx \,|\, x, a \urcorner b, a) \supset \mathrm{Lm}(\mathrm{D}fx/\mathrm{D}gx \,|\, x, a \urcorner b, a)$

$| \; x \varepsilon\, a \urcorner b \,.\, ga = 0 \,.\, gx = 0 \,.\, \mathrm{P}8 \;.\supset.\; 0 \varepsilon\, \mathrm{D}g \text{'} a \urcorner x \;.\supset.\; 0 \varepsilon\, \mathrm{D}g \text{'} a \urcorner b$ (1)
$\mathrm{Hp} \,.\, (1) \,.\, x \varepsilon\, a \urcorner b \;.\supset.\; gx \neq 0$ (2)
$(2) \;.\supset.\; fx/gx \,|\, x \;\varepsilon\; \mathrm{qf}(a \urcorner b)$ (3)
$z \varepsilon\, a \urcorner b \;.\supset.\; fz/gz = (fz - fa)/(gz - ga)$ (4)
$h = \mathrm{D}fx/\mathrm{D}gx \,|\, x \;.\supset:$
$z \varepsilon\, a \urcorner b \,.\, (4) \,.\, \mathrm{P}9{\cdot}2 \;.\supset.\; fz/gz \;\varepsilon\; h \text{'} a \urcorner z$ (5)
$x \varepsilon\, a \urcorner b \,.\, (5) \;.\supset.\; fz/gz \,|\, z \text{'} a \urcorner x \supset h \text{'} a \urcorner x$ (6)
$\quad » \quad .\, (6) \;.\supset.\; \Lambda\, fz/gz \,|\, z \text{'} a \urcorner x \supset \Lambda h \text{'} a \urcorner x$ (7)
$(7) \;.\supset.\; \bigcap[(\Lambda\, fz/gz \,|\, z \text{'} a \urcorner x) \,|\, x \text{'} a \urcorner b] \supset \bigcap[(\Lambda\, h \text{'} a \urcorner x) \,|\, x \text{'} a \urcorner b]$ (8)
$(8) \,.\, \mathrm{Df\,Lm} \;.\supset.\; \mathrm{Lm}(fz/gz \,|\, z, a \urcorner b, a) \supset \mathrm{Lm}(h, a \urcorner b, a) \;]$

Es dato duo numero *a* et *b*, distincto, et duo functione *f* et *g* reale definito in intervallo de *a* ad *b*, ambo nullo pro valore *a*, et habente derivatas, et derivata de denominatore *g* non es nullo in interno de intervallo de *a* ad *b*. Tunc omni valore limite de ratione *fz/gz*, ubi varia *z* in intervallo de *a* ad *b*, et tende ad *a*, es valore limite de ratione de duo derivata.

Demonstratione. Si functio *g* es nullo pro valore *a*, et pro aliquo valore *x* interno ad intervallo considerato, pro theorema de Rolle, suo derivata es nullo inter *a* et *x*, vel inter *a* et *b*, quod es contra hypothesi. (1)

Ergo, in hypothesi nostro, functio *g* habe semper valore non nullo in interno de intervallo de *a* ad *b*. (2)

Et ratione *fx/gx*, pro omni valore de *x* in $a \urcorner b$ habe semper valore determinato et finito. (3)

Si *z* es valore inter *a* et *b*, ratione *fz/gz* vale ratione de incrementos *fz—fa* ad *gz—ga*; nam $fa = ga = 0$. (4)

Si nos voca *h* functio D*fx*/D*gx*, ubi varia *x*, si *z* es inter *a* et *b*, de P(4), et de secundo theorema super valore medio, nos deduce que *fz/gz* es uno ex valores de *h* in intervallo de *a* ad *z*. (5)

Si nos sume valore *x* inter *a* et *b*, tunc omni valore de *fz/gz*, ubi *z* es inter *a* et *x*, es uno ex valores de *h* in intervallo de *a* ad *x*. (6)

Et classe limite generale de valores de fz/gz, ubi varia z inter a et x continere in classe limite generale de valores de h, in idem intervallo. (7)

Ergo parte commune ad omni classe praecedente, ubi x varia inter a et b continere in parte commune ad classes correspondente pro functione h. (8)

Unde, per definitione, classe limite de functione fz/gz ubi varia z, continere in classe limite de functione h. Quod era demonstrando.

·2 $a,b \varepsilon \mathrm{q} \,.\, a -= b \,.\, f, g, \mathrm{D}f, \mathrm{D}g \,\varepsilon\, \mathrm{qF}a\urcorner b \,.\, x \varepsilon\, a\urcorner b \,.\, fx = gx = 0$
$. \, 0 -\varepsilon\, \mathrm{D}g`(a\urcorner b -\iota x) \,.\, \lim(\mathrm{D}fz / \mathrm{D}gz \mid z, a\urcorner b, x) \,\varepsilon\, \mathrm{q} \cup \iota\pm\infty$
$. \supset . \lim(fz/gz \mid z, a\urcorner b, x) = \lim(\mathrm{D}fz / \mathrm{D}gz \mid z, a\urcorner b, x)$
$[\, \mathrm{P}{\cdot}1 \supset \mathrm{P} \,]$

Es dato duo numero distincto a et b, et duo functio f et g reale definito in intervallo de a ad b, simul cum derivatas Df et Dg, et pro aliquo valore x de intervallo de a ad b, interno aut extremo, ambo functione es nullo.

Si derivata de g non es nullo pro valores de variabile in intervallo considerato, differentes de valore x, et si ratione de derivatas, quando variabile tende ad x, habe limite determinato finito aut infinito, tunc limite de ratione de duo functione aequa limite de ratione de duo derivata.

Vel, si nos suppone implicito plure conditione:

« Limite de ratione de duo functione, ambo nullo pro idem valore de variabile, aequa limite de ratione de derivatas ».

Seque de theorema praecedente.

{ De l'Hospital, *Analyse des infiniment petits* a.1696 p.145:
« si l'on prend la différence du numérateur, et qu'on la divise par la différence du dénominateur, après avoir fait $x=a$, l'on aura la valeur cherchée ». }

* 13·1 $a \varepsilon \mathrm{q} \,.\, f, g, \mathrm{D}f, \mathrm{D}g \,\varepsilon\, \mathrm{qF}(a+\mathrm{Q}) \,.\, \mathrm{Lm}(f, a+\mathrm{Q}, \infty) =$
$\lim(g, a+\mathrm{Q}, \infty) = 0 \,.\, 0 -\varepsilon\, \mathrm{D}g`(a+\mathrm{Q}) \,.\supset .$
$\mathrm{Lm}(fx/gx|x, a+\mathrm{Q}, \infty) \supset \mathrm{Lm}(\mathrm{D}fx/\mathrm{D}gx|x, a+\mathrm{Q}, \infty)$
$[\, \mathrm{Lm}(fx/gx|x, a+\mathrm{Q}, \infty) = \mathrm{Lm}\{f(a+/z) \,/\, g(a+/z)\}|z, \mathrm{Q}, 0\} \,.\, \mathrm{P}12 \,.\supset .\, \mathrm{P} \,]$

·2 $a \varepsilon \mathrm{q} \,.\, f, \mathrm{D}f \,\varepsilon\, \mathrm{qF}(a+\mathrm{Q}) \,.\supset .$
$\mathrm{Lm}[(fx)/x \,|x, a+\mathrm{Q}, \infty] \supset \mathrm{Lm}(\mathrm{D}f, a+\mathrm{Q}, \infty)$

·3 $a \varepsilon \mathrm{q} \,.\, f, g, \mathrm{D}f, \mathrm{D}g \,\varepsilon\, \mathrm{qF}(a+\mathrm{Q}) \,.\, \lim(g, a+\mathrm{Q}, \infty) = \infty \,.$
$0 -\varepsilon\, \mathrm{D}g`(a+\mathrm{Q}) \,.\supset .$
$\mathrm{Lm}(fx/gx|x, a+\mathrm{Q}, \infty) \supset \mathrm{Lm}(\mathrm{D}fx/\mathrm{D}gx|x, a+\mathrm{Q}, \infty)$

✻ 14. $u \varepsilon \mathrm{Cls'q} \,.\, u \supset \delta u \,.\, m \varepsilon \mathrm{N}_1 \,.\, f \varepsilon \mathrm{qf}u \,.\, h \varepsilon \mathrm{qF}u \,.\, x \varepsilon u \,.\supset.$

·0 $\mathrm{D}^m hx = (\mathrm{D}^m h)x \quad . \quad \mathrm{D}^m(f,u,x) = \mathrm{D}^m(f,u)x$ Df

Si h es functio definito in campo u condensato, ad que pertine x, tunc $\mathrm{D}h$ es alio functio definito, et si m es numero naturale, $\mathrm{D}^m h$ habe valore determinato per III §1 P9, et es vocato « derivata de ordine m de functio h ». Nos scribe $\mathrm{D}^m hx$, lege « derivata de ordine m de h, pro valore x », in loco de $(\mathrm{D}^m h)x$, et non de $\mathrm{D}^m(hx)$, que non habe sensu.

Si f es operatio, et non es fixato suo campo de variabilitate, tunc (f,u) es functio definito, et $\mathrm{D}^m(f,u,x)$ indica « derivata de ordine m de functio f, in campo u, pro valore x ».

$g, \mathrm{D}^m g, h, \mathrm{D}^m h \,\varepsilon\, \mathrm{qF}u \,.\, a \varepsilon \mathrm{q} \,.\supset.$

·1 $\mathrm{D}^m(g+h) = \mathrm{D}^m g + \mathrm{D}^m h$

·2 $\mathrm{D}^m(a \times g) = a\mathrm{D}^m g$

·3 $\mathrm{D}^m(gx \times hx | x, u, x) = \Sigma[\, \mathrm{C}(m,r)(\mathrm{D}^{m-r} gx) \times (\mathrm{D}^r hx) \,|\, r, 0 \cdots m \,]$
{ Leibniz *MathS.* t.5 p.380 }

·4 $n \varepsilon m + \mathrm{N}_0 \,.\, x \varepsilon \mathrm{q} \,.\supset.\, \mathrm{D}^m(x^n | x, x) = \Pi[n - 0 \cdots (m-1)] \times x^{n-m}$
$n \varepsilon \mathrm{q} \,.\, x \varepsilon \mathrm{Q}$ » » » »
$n \varepsilon \mathrm{n} \,.\, x \varepsilon \mathrm{q} - \iota 0$ » » » »

·5 $x \varepsilon \mathrm{q} \,.\supset.\, \mathrm{D}^m[x^m(1-x)^m | x, x] =$
$m! \Sigma\{(-1)^r [\mathrm{C}(m,r)]^2 x^r (1-x)^{m-r} | r, 0 \cdots m\}$

·6 $\mathrm{Num}\, \theta \frown x \ni \{\mathrm{D}^m[x^m(1-x)^m \,|\, x, x] = 0\} = m$

·5·6 Polynomio de Legendre. Vide P 21.

✻ 15. Theorema de Bernoulli-Taylor.

$a, b \varepsilon \mathrm{q} \,.\, a -= b \,.\, x \varepsilon a \frown b \,.\, m \varepsilon \mathrm{N}_1 \,.\, f, \mathrm{D}^m f \,\varepsilon\, \mathrm{qF}\, a \frown b \,.\, \mathrm{D}^{m+1} fx \,\varepsilon \mathrm{q} \,.\supset.$
$\lim[\{f(x+z) - \Sigma[z^r / r!(\mathrm{D}^r fx) \,|\, r, 0 \cdots m)]\} / z^{m+1} \,|\, z, a \frown b - x, 0]$
$= (\mathrm{D}^{m+1} fx)/(m+1)!$

[P12·1 $.\supset.\, \lim\{f(x+z) - \Sigma(z^r \,/\, r!\, \mathrm{D}^r fx \,|\, r, 0 \cdots m\} / z^{m+1} \,|\, z, \ldots$
$= \lim \mathrm{D}f(x+z) - \Sigma[z^{r-1}/(r-1)!\, \mathrm{D}^r fx \,|\, r, 1 \cdots m]\}/[(m+1)z^m] | z, \ldots$
$= \lim\{(\mathrm{D}^m f(x+z) - \mathrm{D}^m fx)/[(m+1)!z]\} \,|\, z, \ldots$
DfD $.\supset.$ » $= (\mathrm{D}^{m+1} fx)/(m+1)!$]

Dato duo quantitate differente a et b, si x pertine ad intervallo de a ad b, et si f es functione reale definito in intervallo $a \frown b$, simul cum derivata de ordine m, (et cum prae-

cedentes) et si derivata de ordine $m+1$ de f pro valore x existe, tunc differentia inter $f(x+z)$ et polynomio

$$fx + zDfx + z^2/2!\,D^2fx + ... + z^m/m!\,D^mfx,$$

diviso per z^{m+1}, quando varia z, in modo que $x+z$ mane in intervallo considerato, et tende ad x, vale derivata de ordine $m+1$ de fx, diviso per $(m+1)!$.

In vero, pro $m=1$, theorema dice que $\lim[f(x+z)-fx]/z$ vale derivata de fx, quod es vero per definitione de derivata.

Pro $m = 2$, nos quaere

$$\lim\{[f(x+z)-fx-zDfx]/z^2|z,\ a^{\frown}b-x,\ 0\},$$

ubi quantitate de que nos quaere limite, se praesenta sub forma 0/0. Ergo, pro theorema de l'Hospital, nos calcula limite de ratione de duo derivata:

$$\lim[Df(x+z)-Dfx]/(2z)\mid z,\ ...$$

que, per definitione de derivata, vel pro casu $m=1$, vale $D^2fx/2$, conforme ad theorema.

Ita nos demonstra theorema pro $m=3$, ...

{ Joh. Bernoulli a.1694 t.1 p.126:

« habetur hæc series generalissima :

$$Integr.\ ndz = +nz - \frac{zzdn}{1.2.dz} + \frac{z^3ddn}{1.2.3.dz^2} - \frac{z^4dddn}{1.2.3.4.dz^3}\ \&c.$$ » }

{ Taylor a.1715 p.21 :

« Sint z et x quantitates duae variabiles, quarum z uniformiter augetur per data incrementa $\dot{z}$, et sit $n\dot{z} = v$ . . . . .

p.23 : . . . quo tempore z uniformiter fluendo fit $z+v$, fiet x,

$$x + \dot{x}\frac{v}{1\dot{z}} + \ddot{x}\frac{v^2}{1.2\dot{z}^2} + \dddot{x}\frac{v^3}{1.2.3\dot{z}^3} + \&c.$$ » }

{ MacLaurin a.1742 p.610 :

« Suppose that y is any quantity that can be expressed by a series of this form $A + Bz + Cz^2 + Dz^3 + \&c.$ where A, B, C represent invariable coefficients . . . When z wanishes, let E be the value of y, and let $\dot{E}$, $\ddot{E}$, $\dddot{E}$, &c. be then the respective values of $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. z being supposed to flow uniformly. Then

$$y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times 2\times 3\,\dot{z}^3} + \&c.$$

p.611 : This theorem was given by Dr. Taylor.

p.612 : which theorem is not materially different from Mr. Bernouilli's. » }

{ Arbogast a.1800 :

$$F(a+x) = Fa + \frac{DFa}{1}x + \frac{D^2Fa}{1.2}x^2 + \frac{D^3Fa}{1.2.3}x^3 + \text{etc.}$$ }

Nota.

Si in citatione de Bernoulli, nos pone $n = f'(a+z)$, et si nos effectua integratione indicato intra limites 0 et $b-a$, illo fi:

$$fb - fa = (b-a)f'b - (b-a)^2 f''b/2 + (b-a)^3/3!\, f'''b - \dots$$

unde formula 15 per substitutione $(x, x+z)$ in loco de b et a.

Secundo membro non es summa de uno serie; nam serie pote es divergente, re noto ad Bernoulli; vel pote habe summa differente de primo membro, ut nota Cauchy a.1823 s.2 t.4 p.230 cum exemplo e^{-/x^2}. Vide Formul. t. 4.

Theorema praecedente occurre in P18·4, planOscul P42,...

In plure libro de Calculo infinitesimale, es vocato « formula de Taylor » theorema 17·1 sequente que specta ad Lagrange.

❋ 16·1 $m \varepsilon \mathrm{N}_1 \,.\, u \varepsilon \mathrm{Cls'q} \,.\, \mathrm{Num}\, u = m+1 \,.\, f, \mathrm{D}^m f \varepsilon \mathrm{qFMedio}\, u$
$: x \varepsilon u \,.\supset_x.\, fx = 0 :\supset.\, 0 \varepsilon \mathrm{D}^m f ‘\mathrm{Medio}\, u$ [P8 ⊃ P]

·2 $m \varepsilon \mathrm{N}_1 \,.\, u \varepsilon \mathrm{Cls'q} \,.\, s \varepsilon \mathrm{N}_1 \mathrm{f} u \,.\, \Sigma(s,u) = m+1 \,.$
$f, \mathrm{D}^m f \,\varepsilon\, \mathrm{qFMedio}\, u : x \varepsilon u \,.\, r \varepsilon 0 \cdots (s_x - 1) \,.\supset_{x,r}.\, \mathrm{D}^r fx = 0$
$:\supset.\, 0 \varepsilon \mathrm{D}^m f ‘\mathrm{Medio}\, u$ [P8 ⊃ P]

Si functio f es nullo pro valore x, simul cum suo derivatas de ordine 1, 2, ... $(s-1)$, tunc nos dice que functio f es nullo « s vice » pro valore x, vel que aequatione $fx=0$ habe s radice coincidente in valore considerato.

Si functio f es nullo s_1 vice pro valore x_1, s_2 vice pro valore x_2, ..., et si summa $s_1 + s_1 + \dots$ vale $m+1$, tunc derivata de ordine m de f es nullo pro aliquo valore medio inter praecedentes. Seque de theorema de Rolle.

❋ 17. Theorema de Lagrange.

·1 $a, b \varepsilon \mathrm{q} \,.\, a -= b \,.\, n \varepsilon \mathrm{N}_1 \,.\, f, \mathrm{D}^n f \varepsilon \mathrm{qF}\, a \bar{\ } b \,.\supset.$
$fb - \Sigma[(b-a)^r / r!\, \mathrm{D}^r fa) | r, 0 \cdots (n-1)] \;\varepsilon\; (b-a)^n / n!\, \mathrm{D}^n f ‘\, a \bar{\ } b$
[$\mathrm{Hp} \,.\, k = \{fb - \Sigma[(b-a)^r\, r/!\, \mathrm{D}^r fa | r, 0 \cdots (n-1)]\} / (b-a)^n \,.$
$g = \{fx - \Sigma[(x-a)^r / r!\mathrm{D}^r fa | r, 0 \cdots (n-1)] - k(x-a)^n \} | x \,.\supset.$
$ga = \mathrm{D}ga = \mathrm{D}^2 ga = \dots = \mathrm{D}^{n-1} ga = 0 \,.\, gb = 0 \,.\supset.$
$\exists\, (a \bar{\ } b) \frown u \ni (\mathrm{D}^n gu = 0) \,.\supset.\, \exists\, (a \bar{\ } b) \frown u \ni (\mathrm{D}^n fu - n!k) = 0 \,.\supset.$
$k \varepsilon (\mathrm{D}^n f ‘ a \bar{\ } b) / n!$]

Si a et b es quantitate, a es differente de b, et si functio f habe derivatas usque ad ordine n, ubi n indica numero naturale, in toto intervallo de a ad b, tunc differentia inter fb et summa de primos n termine de serie de Taylor vale $(b-a)^n/n!$ multiplicato per valore medio de derivata de ordine n.

In facto, si nos voca k quantitate in primo membro diviso per $(b-a)\upharpoonleft n$, et si nos pone

$gx = fx - fa - (x-a)\mathrm{D}fa - \ldots - (x-a)^{n-1}/(n-1)!\,\mathrm{D}^{n-1}fa - k(x-a)^n$,

id es, si nos voca g functione in secundo membro, ubi varia x, tunc functio g es nullo, simul cum suo derivatas de ordine 1, 2, ... $n-1$, pro $x = a$, et es nullo pro $x = b$. Ergo per theorema de Rolle, vel per P16·2, suo derivata de ordine n, $\mathrm{D}^n fx - n!k$, es nullo pro aliquo valore inter a et b. Unde seque theorema.

Exemplo. Si nos substitue $a = 0$, $b = x$, $f = (1+x)^m|x$, seque :

$$(1+x)^m - \Sigma[\mathrm{C}(m,r)x^r\,|r, 0\cdots(n-1)]\ \varepsilon\ \mathrm{C}(m,n)x^n\,(1+\theta x)^{m-n}$$

unde deriva §lim P31·6.

} LAGRANGE a.1797, *Th. des Fonctions analytiques* p.49 :
« D'où résulte enfin ce théorème nouveau et remarquable par sa simplicité et généralité, qu'en désignant par u une quantité inconnue, mais renfermée entre les limites 0 et x, on peut développer successivement toute fonction de x et d'autres quantités quelconques suivant les puissances de x, de cette manière :

$$\begin{aligned} \mathrm{f}x &= \mathrm{f}. + x\mathrm{f}'u, \\ &= \mathrm{f}. + x\mathrm{f}'. + \frac{x^2}{2}\mathrm{f}''u, \\ &= \mathrm{f}. + x\mathrm{f}'. + \frac{x^2}{2}\mathrm{f}''. + \frac{x^3}{2.3}\mathrm{f}'''u, \end{aligned}$$

les quantités f., f'., f''., etc. étant les valeurs de la fonction fx et de ses dérivées f'x, f''x, etc., lorsqu'on y fait $x = 0$ ». {

·2 $\mathrm{Hp}{\cdot}1 \,.\, p\varepsilon\mathrm{N}_1 \,.\supset.\, fb - \Sigma[(b-a)^r/r!\,\mathrm{D}^r fa|r, 0\cdots(n-1)]\ \varepsilon$
$\{(b-a)^p(b-x)^{n-p}/[p\times(n-1)!]\,\mathrm{D}^n fx\}\,|x\,\text{‘}\,a\overline{\ }b$

} CAUCHY, *Exercices* t.1 a.1826 p.26, pro $p=1$;
SCHLÖMILCH a.1847 p.177 {

[$k=$ (vide DmP·1) . $h = \{fb - \Sigma[(b-x)^r/r!\,\mathrm{D}^r fx\,|r, 0\cdots(n-1)] - k(b-x)^p(b-a)^{n-p}\}\,|x \,.\supset.\, ha=hb=0 \,.\supset.\, \exists\, a\overline{\ }b \frown x\ni(\mathrm{D}hx=0) \,.\supset.$
$\exists\, a\overline{\ }b \frown x\ni[-(b-x)^{n-1}\mathrm{D}^n fx + kp(n-1)!(b-x)^{p-1}(b-a)^{n-p} = 0] \,.\supset.\, \mathrm{P}$]

·21 $\mathrm{P}{\cdot}2 \,.\, p=n \,.\supset.\, \mathrm{P}{\cdot}1$

P·2 da expressione de resto in serie de Taylor, invento per Cauchy contine P·1 ut casu particulare.

·3 $a,b\varepsilon\mathrm{q} \,.\, a -\!\!= b \,.\, n\varepsilon\mathrm{N}_1 \,.\, f\varepsilon\,\mathrm{qF}a\mapsto b \,.\, \mathrm{D}^n f\,\varepsilon\,(\mathrm{qF}a\mapsto b)(\mathrm{cres}_0\cup\mathrm{decr}_0)$
$.\supset.\, fb - \Sigma[(b-a)^r/r!\,\mathrm{D}^r fa\,|r, 0\cdots n]\ \varepsilon\ \Theta(b-a)^n[\mathrm{D}^n fb - \mathrm{D}^n fa]/n!$

} MORERA RdM. a.1892 p.36 {

❋ 18·1 $a,b\varepsilon\mathrm{q} \,.\, a -\!\!= b \,.\, f, \mathrm{D}^2 f\,\varepsilon\,\mathrm{qF}a\mapsto b \,.\, x\varepsilon\, a\overline{\ }b \,.\supset.$
$fx - fa - (x-a)(fb-fa)/(b-a)\ \varepsilon\ (x-a)(x-b)\mathrm{D}^2 f\,\text{‘}\,a\overline{\ }b\ /2$

$[\ h = |fz - fa - (z-a)(fb-fa)/(b-a) - (z-a)(z-b)[fx - fa - (x-a)$
$(fb-fa)/(b-a)]/[(x-a)(x-b)]|\ |z\ .\supset.\ ha = hb = hx = 0\ .\supset.$
$0 \varepsilon\ Dh`a^{-}x\ .\ 0 \varepsilon\ Dh`x^{-}b\ .\supset.\ 0 \varepsilon\ D^2h`a^{-}b\ .\supset.\ P\ ]$

Exprime resto in interpolatione de primo gradu, vel in regula de « partes proportionale », adoptato in tabulas de log, sin, etc.

·2 $\mathrm{Hp}\cdot 1\ .\ m,n \varepsilon Q\ .\supset.\ f[(ma+nb)/(m+n)] - (mfa+nfb)/(m+n)$
$\varepsilon\ -(b-a)^2 mn(m+n)^{-2}/2\ D^2f`(a^{-}b)$ [P·1 $\supset$P]

$(ma+nb)/(m+n)$ es valore « medio arithmetico inter a et b, cum pondo m et n ». Si D^2f in intervallo $a^{-}b$ es semper positivo, tunc valore de functione respondente ad medio arithmetico inter a et b es minore de medio arithmetico de valores de functione respondentes ad a et b.

Si $m = n = 1$, et $f = x^p\ |x$, seque Arithmetica §5 P14·2 (pag. 42).

·3 Hp P·1 . $D^2f\ \varepsilon\ QF\ a^{-}b\ .\ r \varepsilon N_1 + 1\ .\ z \varepsilon\ (a^{-}b\ F\ 1\cdots r)\mathrm{cres}\ .$
$m \varepsilon\ QF\ 1\cdots r\ .\supset.\ f[(\Sigma mz)/(\Sigma m)] < (\Sigma mfz)/(\Sigma m)$ [P·2 $\supset$P]

Generalizatione de praecedente. Pro pondo $m = 1$, et $f = x^p\ |x$, seque §ΣP8·3 (pag. 123).

·4 $a,b \varepsilon q\ .\ a<b\ .\ f \varepsilon\ qF a \dashv b\ .\ x \varepsilon\ a^{-}b\ .\ Dfx = 0\ .\ D^2fx > 0\ .\supset.$
$\exists\ (c,d) \ni [c \varepsilon\ a^{-}x\ .\ d \varepsilon\ x^{-}b\ .\ fx = \min f`(c \dashv d)]$

$[\ \mathrm{P}15\ .\supset.\ \lim[(f(x+h) - fx)/h^2\ |h,\ a^{-}b - x,\ 0] = D^2fx/2\ .\supset.$
$\exists\ (c,d) \ni [c \varepsilon\ a^{-}x\ .\ d \varepsilon\ x^{-}b : h \varepsilon\ c^{-}d - x\ .\supset_h .\ f(x+h) > fx\ .\supset.\ \mathrm{Ths}\ ]$

Si $a^{-}b$ es intervallo, ubi es definito functio f, et pro aliquo valore x interno ad intervallo, derivata de ordine 1 es nullo, et derivata de ordine 2 es positivo, tunc existe intervallo $c^{-}d$, parte de $a^{-}b$, continente in suo interno valore x, tale que fx es minimo de valores de f in intervallo $c^{-}d$.

✻ 19·1 $a,b \varepsilon q\ .\ a -\!\!= b\ .\ f \varepsilon\ qF a^{-}b\ .\ m \varepsilon N_1\ .\ x \varepsilon\ (a^{-}b\ f\ 0\cdots m)\mathrm{sim}\ .$
$g = \Sigma|fx_r \Pi[(z-x_s)/(x_r-x_s)\ |s,\ (1\cdots m) - \iota r]\ |r,\ 1\cdots m|\ |z\ .$
$D^m f\ \varepsilon\ qF\ a^{-}b\ .\ y \varepsilon\ a^{-}b\ .\supset.$
$fy - gy\ \varepsilon\ \Pi[(y-x_r)|r,\ 1\cdots m]\ D^m f`(a^{-}b)/m!$

$[\ \mathrm{Hp}\ .\ k = (fy-gy)/\Pi[(y-x_r)|r,1\cdots m]\ .\ h = |fz - gz - k\ \Pi[(z-x_r)|r,1\cdots m]|\ |z$
$.\supset.\ hx_1 = 0\ .\ hx_2 = 0\ .\ \ldots\ .\ hx_m = 0\ .\ \mathrm{P}16\cdot 1\ .\supset.$
$\exists\ a^{-}b \cap z \ni (D^m hz = 0)\ .\supset.\ \exists\ a^{-}b \cap z \ni (D^m fz - m!k = 0)\ .\supset.\ \mathrm{Ths}\ ]$

| AMPÈRE, GergonneA. a.1826 t.16 p.329 ;
Dm. H. A. SCHWARZ, TorinoA. a.1882 ;
STIELTJES, a.1882 Amsterdam Ak. s.2 t.17 p.239-254 |

Functio g in P·1 es illo functio integro de gradu $m-1$, que pro valores $x_1 x_2 \ldots x_m$ coincide cum f, et es dicto « functio interpolare ».

P·1 exprime $fy-gy$, que vocare « resto aut errore in interpolatione ».

Newton a.1686 t.3 prop. XL lemma 5, da g sub alio forma; Waring a.1776, et Lagrange a.1795 (Œuvres t.7 p.285), sub forma hic adoptato.

✻ 20.

·1 $a,h\varepsilon\mathrm{q} \,.\, f\varepsilon\, \mathrm{qF}(a+\Theta h) \,.\, \mathrm{l}'\, [(\mathrm{mod}\, \mathrm{D}^n fx)|(n,x)`(\mathrm{N}_1 \,;\, a+\Theta h)]\, \varepsilon \mathrm{Q} \,.$
$\supset .\; f(a+h) = \Sigma\{h^r/r!\, \mathrm{D}^r fa \,|r,\, \mathrm{N}_0\}$

[$k = \mathrm{l}'[(\mathrm{mod}\, \mathrm{D}^n fx)|(n,x)`(\mathrm{N}_1 ; a+\theta h)] \,.\, \mathrm{P}17{\cdot}1 \,.\supset.$
$\mathrm{mod}[f(a+h)-\Sigma(h^r/r!\mathrm{D}^r fa|r,0\cdots n)] < kh^{n+1}/(n+1)! \,.\, \S\lim \mathrm{P}13{\cdot}1 \,.\supset.\, \mathrm{P}$]

Si limite supero de derivata de ordine n de fx, ubi n sume omni valore integro, et x omni valore in intervallo considerato, es finito, tunc serie de Taylor converge ad primo membro.

·2 $k\varepsilon\, \mathrm{Cls}'\mathrm{q} \,.\, k \supset \delta k \,.\, f\varepsilon\, \mathrm{q\,f}(k;\mathrm{N}_0) : n\varepsilon \mathrm{N}_0 \,.\, x\varepsilon k \,.\supset_{k,x} .\, \mathrm{D}[f(z,n)|z,$
$k\,,x]\,\varepsilon \mathrm{q} \;:\; \Sigma\{[\mathrm{l}'\, \mathrm{mod}\, \mathrm{D}[f(z,n)|z,k,z]\,|z\,`k]|n,\, \mathrm{N}_0\}\; \varepsilon\mathrm{Q} : x\varepsilon k \;:\supset.$
$\mathrm{D}\{\Sigma[f(z,n)|n,\mathrm{N}_0]|z,k,x\} = \Sigma\{\mathrm{D}[f(z,n)|z,k,x]|n,\, \mathrm{N}_0\}$

Comm(D, Σ)

✻ 21·1

$x,a\varepsilon\mathrm{q} \,.\, \mathrm{mod}x,\, \mathrm{mod}a\; \varepsilon\theta \,.\supset.\; /\sqrt{(1-2ax+a^2)} =$
$\Sigma\{a^n/(n!2^n)\, \mathrm{D}^n[(x^2-1)^n|x,x]|n,\, \mathrm{N}_0\}$

Coefficiente de $a\uparrow n$, es dicto « polynomio de Legendre ». Plure Auctore indica illo per $\mathrm{X}_n\, x$. Es reductibile ad casu particulare praecedente :

$\mathrm{D}^n\, (x-a)^n\, (x-b)^n\, |x.$

✻ 22·1

$n\varepsilon\mathrm{N}_1 \,.\, a\varepsilon\, \mathrm{qF}\, 0\cdots n \,.\supset : f\varepsilon\, \mathrm{qFq} \,.\, \mathrm{D}f = [\Sigma(a_r x^r|r,0\cdots n)|x,$
$\mathrm{q}] \,.=.\, f = \{f0 + \Sigma[a_r x^{r+1}/(r+1)|r,0\cdots n]|x,\mathrm{q}\}$

[$\mathrm{D}\{fx - \Sigma[a^r x^{r+1}/(r+1)\,|\,r,0\cdots n]\,|\,x,\mathrm{q}\} = (\iota 0 ; \mathrm{q}) \,.\, \mathrm{P}10{\cdot}3,\; \supset \mathrm{P}$]

Functio que habe pro derivata polynomio de gradu n es polynomio de gradu $n+1$. Termine que non depende de x es $f0$.

·2 $m\varepsilon\mathrm{q} \,.\, n\varepsilon\, \mathrm{q}-\iota 1 \,.\, k\varepsilon\, \mathrm{Intrv} \,.\, a\varepsilon k \,.\supset:$
$f\varepsilon\, \mathrm{QF}k \,.\, \mathrm{D}f = [m(fx)^n|x,k] \,.=.$
$[(fx)^{-n}\mathrm{D}fx - m\,|x,k] = (\iota 0 ; k) \,.=.$
$\mathrm{D}[(fx)^{-n+1}/(-n+1) - mx\,|x,k] = (\iota 0 ; k) \,.=.$
$[(fx)^{-n+1}/(-n+1) - mx\,|x,k]\; \varepsilon\; (\mathrm{qF}k)\mathrm{const} \,.=.$
$[(fx)^{-n+1}/(-n+1) - (fa)^{-n+1}/(-n+1) - m(x-a)\,|x,k] = (\iota 0 ; k) \,.=.$
$f = \{[(fa)^{-n+1} + (-n+1)m(x-a)]\uparrow/(-n+1)|x,k\} \,.$
$k \supset \mathrm{q}\frown x\ni[(fa)^{-n+1} + (-n+1)m(x-a) > 0]$

Nos quaere functio f positivo definito in aliquo intervallo k, que pro omni valore de x in k, satisfac aequatione $\mathrm{D}fx = m(fx)^n$, id es, que habe derivata proportionale ad aliquo potestate, diverso de 1, de functio. Aequatione vale $(fx)^{-n}\mathrm{D}fx - m = 0$; primo membro es derivata de functio

scripto in linea 4, que resulta constante, vel aequale ad suo valore pro $x = 0$, unde resulta valore de functio f. Intervallo k contine solo valores de x, que redde basi positivo. Nam nos habe definito potestate cum exponente q, pro basi positivo.

Casu $n = 1$ fore tractato in P52·1.

Aequatione de typo praecedente, que contine functio et suo derivata, vocare « aequatione differentiale ».

Cx D

* 30. $u \varepsilon$ Cls'q . $n \varepsilon N_1$. $f \varepsilon$ Cxn fu . $x \varepsilon u \frown \delta u$.⊃. P1

·1 $a,b \varepsilon$ q . $a -= b$. $n \varepsilon N_1$. f, Df ε Cxn F $a \overline{\vdash} b$.⊃.
$(fb - fa)/(b - a)$ ε Med Df'$(a \overline{\ } b)$

·2 $a,b \varepsilon$ q . $a -= b$. $m,n \varepsilon N_1$. $f \varepsilon$ CxnF$a \overline{\ } b$. $x \varepsilon a \overline{\ } b$. $D^m fx \varepsilon$ Cxn
.⊃. P15

·3 Hp ·1 . $m \varepsilon N_1$. $D^m f$ ε Cxn F $a \overline{\ } b$. $h \varepsilon a \overline{\ } b - x$.⊃. $f(x+h) -$
$\Sigma[(h^r / r!\ D^r fx) \mid r, 0 \cdots (m-1)]$ ε $h^m / m!$ Med $D^m f$'$(x + \theta h)$

Si f es numero complexo, aut vectore, aut puncto functione de variabile reale, tunc semper nos defini derivata ut limite de ratione de incremento de functione ad incremento de variabile, quando incremento de variabile tende ad 0.

Aliquo theorema super derivatas de functione reale accipe modificatione pro numeros complexo. Ita theorema de valore medio (P9), fi P·1 :

« Ratione de incremento de functio complexo ad incremento de variabile reale es valore medio inter valores de derivata ». Illo non es semper uno ex valores de derivata.

Theorema de Taylor subsiste sine modificatione, P·2. Resto in formula de Taylor exige consideratione de valore medio.

Dtrm D

* 31·1 $a,b \varepsilon$ q . $a < b$. f,g,h,Df,Dg,Dh ε qF$a \overline{\vdash} b$.⊃.
∃ $a \overline{\ } b \frown x \ni \{$Dtrm[(D$fx$,D$gx$,D$hx$), (fa,ga,ha), (fb,gb,hb)] $= 0\}$

[$k =$ Dtrm[(fx, gx, hx), (fa, ga, ha), (fb, gb, hb)] $|x$.⊃.
$ka = kb = 0$. P8 .⊃. P]

Nos considera tres functio, f, g, h, definito, cum derivatas, in aliquo intervallo $a \overline{\ } b$; tunc determinante

$$\begin{bmatrix} fx, & gx, & hx \\ fa, & ga, & ha \\ fb, & gb, & hb \end{bmatrix}$$

es nullo pro $x = a$, et pro $x = b$. Ergo suo derivata, que resulta, si in primo linea nos scribe derivatas, es nullo, pro aliquo valore inter a et b.

Si nos pone $h = (\iota 1 ; a \frown b)$, id es nos suppone que h habe valore constante 1, in toto intervallo de a ad b, seque secundo theorema de valore medio, P9·2.

Subst q′ D

❋ 32. (Subst n | Cxn) P30

❋ 33. (q′ | q) §D P1-5, 6·1

·1 $u \varepsilon \mathrm{q}' \mathrm{f} \mathrm{N}_0 . a \varepsilon \mathrm{q}' . \infty \text{-} \varepsilon \mathrm{Lm}(\mathrm{mod} u_n a^n) | n . x \varepsilon \mathrm{q}' . \mathrm{mod} x < \mathrm{mod} a . \supset .$
$\mathrm{D} \{ [\Sigma(u_n z^n | n, \mathrm{N}_0)] | z, \mathrm{q}' \frown z \ni (\mathrm{mod} z < \mathrm{mod} a), x \} = \Sigma[n u_n x^{n-1} | n, \mathrm{N}_1]$

vct D

❋ 40. (vct | Cx) P30
(pnt »

Puncto px, que depende de variabile reale x, es vocato « puncto mobile ». Variabile reale x es vocato « tempore », et pote coincide cum tempore physico.

Incremento de positione de puncto, vel differentia de duo positione de puncto mobile, es vectore.

Derivata Dpx de puncto es vectore, vocato « velocitate ». Derivata de ordine duo $\mathrm{D}^2 px$ es dicto « acceleratione ». Si puncto es materiale, producto $m\mathrm{D}^2 px$ de suo massa per acceleratione es « fortia » vel « vi ». Plure Auctore de Mechanica sume ce proprietate ut definitione de fortia.

$m(\mathrm{D}px)^2/2$ vocare « energia » vel « vi viva ».

$k \varepsilon \mathrm{Cls'q} . k \supset \delta k . x \varepsilon k . u, v, \mathrm{D}u, \mathrm{D}v \; \varepsilon \; \mathrm{vF}k . \supset .$

·1 $\mathrm{D}[ux \times vx | x, k, x] = (ux) \times (\mathrm{D}vx) + (vx) \times \mathrm{D}ux$

·2 $\mathrm{D}[(ux) \, \alpha \, (vx) | x, k, x] = (\mathrm{D}ux) \, \alpha \, vx + (ux) \, \alpha \, (\mathrm{D}vx)$

Derivata de producto interno et alterno de duo vectore variabile es analogo ad derivata de producto de duo functio numerico. Nota que in producto alterno non lice commuta factores.

·3 $ux \text{-}= 0 . \supset . \mathrm{DU}ux = (\mathrm{cmp} \perp ux) \mathrm{D}ux / (\mathrm{mod} ux)$

·4 » $\mathrm{Dmod} ux = \mathrm{D}ux \times \mathrm{U}ux$

469. ***tempore,*** tempus, F temps, H tiempo, I tempo, DR tempo (in musica). ⊃ tempor-ario A, tempor-iza A. ⊂ temp- + -ore (56).

470. temp- ⊃ temp-ore, temp-estate A, temp-eratura DR.
⊂ tepe + -n- (341) (secundo Bopp, Bréal et alios).
Secundo Fick, temp- de «tempore» ‖ A thing, D ding.

471. tepe ⊃ tep-ore A, tep-ido A. ‖ S tap, R tep-lo = L tep-ore.

472. ***mobile*** FI, H moble, D mobil. ⊃ mobilitate A.
⊂ mo- (374) + -bile (208).

473. ***velocitate,*** A velocity, F velocité (non in Mathematica), H velocidad, I velocità. ⊂ veloce — -e + -itate (8).

474. ***veloce*** FI, H veloz. ⊂ (secunde Bréal) velo + -ce (397).
Van. considera primo elemento ut ‖ L vola.

475. velo (de nave), vela I, F voile.
⊂ vehe (343) + -lo (224), post contractione.

476. ***acceleratione,*** AF acceleration, H aceleracion, I accelerazione.
⊂ accelera + -tione (12).

477. ***accelera*** I, A accelerate, F accélère, H acelera.
⊂ ad (41) + celere — -e + -a (4).

473. ***celere*** I. ⊃ celer-itate A. ⊂ cele- + -re.

479. cele- ⊃ cele-re, cele-bre = frequentato. ‖ L colle, prae-celle, cole, cul-mine, col-umna. ‖ G bu-col-o, pol-o. ‖ A hill, AD holm, R cholm'. = es celere, es alto.

480. -re ⊃ cele-re ⌒ ac-re. = -ente.

481. ***massa*** IR, A mass, DF masse, H masa.
⊂ G maza = pasta, substantia.

482. ***vi,*** vis ⊃ vi-olento A. ‖ G i-s, S g'i.

483. ***fort-ia*** (non L), AF force, H fuerza, I forza. ⊂ forte — -e + -ia (143).

484. ***forte*** AFI, H fuerte, DR (militare) fort. ⊂ fer + -te ?.

485. -te ⊃ men-te, gen-te, par-te, supersti-te, interpre-te, equ-i-te.

486. ***energia*** G ἐνέργεια, A energy, D energie, F énergie, HIR energia.
⊂ en- + erg- + -ia (43).

487. en G =‖ L in. ⊃ en-ergia, en-cyclopaedia DR.

488. erg- ergo, ergon, G ἔργο-ν. =‖ A work, D werk = L labore.
⊃ erg = unitate de labore in Physica.

✻ 41. rectaT planN

$k \varepsilon$ Cls'q . $k \supset \delta k$. $p \varepsilon$ pFk . $x \varepsilon k$. $px - \varepsilon\, p'(k - \iota x)$.⊃.

·0 rectaTpx = lim[recta(px, py) | y, k, x] Df

Si k es classe de quantitates, condensato, et p es puncto functione definito des k, et x es valore in classe k, et puncto px es differente de omni alio puncto de trajectoria de p, per valores de variabile differentes de x, id es, si px non es puncto multiplo de curva, tunc rectaTpx, lege « recta tangente ad trajectoria de p, respondente ad valore x de variabile », es limite de recta que transi per px, et per altero puncto py, quando y varia in classe k, et tende ad x.

Classe k es aut classe q, aut Q, aut es intervallo, etc., ut in Df de derivata. Symbolo rectaTpx vale (rectaTp)x, et non rectaT(px), id es, nos determina recta tangente ad traiectoria de p, pro valore x de variabile, et non tangente ad puncto px, que non habe sensu.

·1 Dpx ε v $-\iota 0$.⊃. rectaTpx = recta(px, Dpx)

[Df rectaT .⊃. rectaT px = lim [recta(px, py) | y, k, x]
§vct 31·4 .⊃. » = » » (px, $py - px$) | y, k, x]
» ·11 .⊃. » = » » » ($py - px$)/($y - x$) y, k, x]
§lim P54·2 .⊃. » = recta [px, lim (»
Df D .⊃. » = recta(px, Dpx)]

Si derivata de puncto mobile p, pro valore x de variabile, es vectore determinato, non nullo, tunc recta tangente in px es recta que transi per puncto px, et es parallelo ad derivata Dpx.

In facto, recta tangente es limite de recta per px et py; vel de recta per px, et parallelo ad vectore $py - px$, ad que me substitue vectore parallelo ($py - px$)/($y - x$). Limite de ce vectore es derivata Dpx; unde seque theorema.

·2 $n \varepsilon N_1$. Dpx = D^2px = ... = D$^n px$ = 0 . D^{n+1}px ε v $-\iota 0$.⊃.
rectaTpx = recta(px, D^{n+1}px)

[Df rectaT . §D P15 .⊃. rectaTpx = lim[recta(px,py)|y,k,x]
= lim recta(px, $py - px - \Sigma[(y-x)^r / r!\, D^r px \,|\, r, 1 \cdots n]$ / $(y-x)^{n+1}$)|y,k,x]
= recta[px, D^{n+1}px/($n+1$)!] = recta(px, D^{n+1}px)]

Si pro valore x, derivata de puncto es nullo, tunc recta tangente habe directione de primo inter derivatas sequente, que non es nullo.

·3 $\text{planN}\,px = \text{p}\frown y\ni[\text{proj}(\text{rectaT}\,px)y = px]$ Df
= « plano normale ad trajectoria de p »

·4 $\text{Hp P·1} \,.\supset.\, \text{planN}\,px = \text{p}\frown y\ni[(y-px)\times\text{D}px = 0]$
$= \text{p}\frown y\ni\{\text{D}[\text{mod}(y-pz)|z, k, x] = 0\}$
$= \text{p}\frown y\ni[\text{real}(y-px)/\text{D}px = 0] = \text{plan}(px, \text{ID}px)$

Euclide l.3 Df2, dice que recta es tangente « ἐφάπτεσθαι » ad circulo (l.1 Df15) si habe uno solo puncto commune cum circulo.

Nos pote applica idem Df ad ellipsi, etc.; sed non ad omni curva.

Descartes, *La Geometrie* Œuvres, t. 6, p. 418 dice que tangente es recta que seca curva in duo puncto « ioins en vn »; id es, si æquatione que determina ce punctos de intersectione habe duo « racines entierement ésgales ».

Df considerato se transforma in P·0, si nos considera recta per duo puncto « juncto in uno », ut limite de recta per duo puncto distincto.

489. ***tangente*** DFHI, A tangent, R tangens' (in trigonometria).
$\subset$ tange + -nte (142).

490. ***tange*** HI. $\supset$ (489), tang-ibile A. $\subset$ tage (242) + -n- (341).

✻ 42. planO

$\text{Hp P41} \,.\, p\text{‘}(k-\iota x) \frown \text{rectaT}\,px = \Lambda \,.\supset:$

·0 $\text{planO}\,px = \lim[\text{plan}(\text{rectaT}\,px, py)\,|y, k, x]$ Df

Si p es puncto functione definito in classe condensato k, et si recta tangente in px non habe alio puncto commune cum trajectoria de p, in classe considerato, tunc planOpx, lege « plano osculatore ad trajectoria de p pro valore x », es limite de plano que contine recta tangente in puncto px, et puncto py, quando y varia in k, et tende ad x.

·1 $\text{D}px\,\varepsilon\,\text{v}-\iota 0 \,.\, \text{D}^2px\,\varepsilon\,\text{v}-\text{qD}px \,.\supset.$
$\text{planO}\,px = \text{plan}(px, \text{D}px, \text{D}^2px)$

[$\text{planO}\,px = \lim\{\text{plan}[\text{rectaT}\,px, py]\,|y, k, x\}$
P1·1 $.\supset.$ » $\{\text{recta}(px, \text{D}px), py]\,|y, k, x\}$
» $\{px, \text{D}px, [py-px-(y-x)\text{D}px]/(y-x)^2\,|y, k, x\}$
§D P15 $.\supset.$ » $= \text{plan}(px, \text{D}px, \text{D}^2px)$]

Si, pro valore x de variabile, derivata de puncto p es vectore non nullo, et si derivata de ordine duo de p es vectore non parallelo ad derivata de ordine uno, tunc plano osculatore es plano per px, et parallelo ad derivata primo et secundo

Vel: plano osculatore ad trajectoria de puncto p, pro tempore x, es plano de puncto, de suo velocitate, et de suo acceleratione, si ce elementos determina plano.

Vel: plano osculatore contine vi, que move puncto.

In facto, per definitione de plano osculatore, et per theorema super recta tangente, plano osculatore es limite de plano per px, Dpx, et py. Vectore $py - px$ jace in ce plano; ergo plano osculatore es limite de plano per px et Dpx, et parallelo ad vectore $py - px - (y-x)\mathrm{D}px$, et si diviso per $(y-x)^2$. Per theorema de Taylor, limite de isto vectore es $\mathrm{D}^2px/2$, unde seque theorema.

·2 $m,n \varepsilon \mathrm{N}_1 . \mathrm{D}px = \mathrm{D}^2px = \ldots = \mathrm{D}^{m-1}px = 0 . \mathrm{D}^m px \,\varepsilon\, \mathrm{v}-\iota 0 .$
$\mathrm{D}^{m+1}px, \ldots \mathrm{D}^{m+n-1}px \,\varepsilon\, \mathrm{qD}^m px . \mathrm{D}^{m+n}px \,\varepsilon\, \mathrm{v}-\mathrm{qD}^m px .\supset.$
$\mathrm{planO}px = \mathrm{plan}(px, \mathrm{D}^m px, \mathrm{D}^{m+n}px)$

Si m—1 derivata successivo de punto p, pro valore x considerato, es nullo et derivata de ordine m non es nullo, et si n—1 derivata de ordine post m es parallelo ad derivata de ordine m, et derivata de ordine $m+n$ non es parallelo ad derivata de ordine m, tunc plano osculatore es plano determinato per puncto px, et per suo derivatas de ordine m et $m+n$.

·3 $\mathrm{Hp}\cdot 1 .\supset. \mathrm{T}px = \mathrm{UD}px \quad . \quad \mathrm{N}px = \mathrm{UDT}px \quad .$
$\mathrm{B}px = [\mathrm{I}(\mathrm{T}px)\alpha(\mathrm{N}px)]$ Df

Tpx, Npx, Bpx es vectores unitario parallelo ad « tangente », « normale principale », « binormale », de linea p.

·4 $\mathrm{rectaN}px = \mathrm{planN}px \frown \mathrm{planO}px = \mathrm{recta}(px, \mathrm{N}px)$ Df
= « normale principale ».

·5 $\mathrm{rectaB}px = \mathrm{recta}(px, \mathrm{B}px)$ Df
= « binormale » (Saint Venant a.1845).

Notione de plano osculatore occurre in Tinseau, « Solutions de quelques problèmes relatifs à la théorie des surfaces courbes », Mém. Sav. Étrang. t. 9, a.1781.

491. ***osculatore*** I, A osculatory, F osculateur. $\subset$ oscula- + -tore.

492. oscula- = basia, combasia. $\subset$ osculo — -o + -a (92).

493. osculo = basio, parvo bucca. $\subset$ os + -culo.

494. os, ore = bucca. $\supset$ (493), or-atore,...
=‖ S as, ‖ (secundo Van.) L es (1), in sensu « respira ».

495. -culo $\supset$ se-culo, arti-culo, mole-cul-a, os-culo.
$\subset$ -co (201) — -o + -ulo (224).

✻ 43. Ax Cc Rc

$k \varepsilon \mathrm{Cls'q} \,.\, k \supset \delta k \,.\, p \varepsilon \mathrm{pF}k \,.\, x \varepsilon k \,.\supset.$

·0 $\mathrm{Ax}\,px = \lim[(\mathrm{planN}\,px \cap \mathrm{planN}\,py) | y, k, x]$ Df

Axpx vocare « axi de plano osculatore ad curva », vel « axi de curva » (Monge, *Géométrie descriptive*, a.VII, p.106). Es intersectione de duo plano normale consecutivo.

·1 $\mathrm{Cc}\,px = \iota[\mathrm{Ax}\,px \cap \mathrm{planO}\,px]$ Df

» = « centro de curvatura ». Es puncto de intersectione de axi cum plano osculatore.

·2 $\mathrm{Rc}\,px = \mathrm{d}(px\,,\, \mathrm{Cc}px)$ Df

= « radio de curvatura ».

$\mathrm{D}px \,\varepsilon\, \mathrm{v}{-}\iota 0 \,.\, \mathrm{D}^2px \,\varepsilon\, \mathrm{v}{-}\mathrm{qD}px \,.\supset.$

·3 $\mathrm{Ax}\,px = \mathrm{planN}\,px \cap z\ni[(z-px)\times(\mathrm{D}^2px) = (\mathrm{D}px)^2]$

$[\mathrm{Hp} \,.\, f = [(z-py)\times\mathrm{D}py \,|y, k] \,.\supset.\, \mathrm{Ax}px = \lim\{z\ni[fx = 0 \,.\, fy = 0]|y,k,x\}$
$= \lim\{z\ni[fx = 0 \,.\, (fy-fx)/(y-x) = 0]\,|y, k, x\}$
$= z\ni\{fx = 0 \,.\, \lim[(fy-fx)/(y-x) \,|\, y, k, x] = 0\} = z\ni[fx = 0 \,.\, \mathrm{D}fx = 0]$
$= \mathrm{planN}px \cap z\ni[-(\mathrm{D}px)^2 + (z-px)\times\mathrm{D}^2px = 0] \,]$

·4 $\mathrm{Cc}\,px = px - \mathrm{D}px \,/\, \mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px)$

$[\, z = \mathrm{Cc}px \,.\supset.\, \mathrm{real}(z-px)/\mathrm{D}px = 0 \,.\, \mathrm{D}[\mathrm{real}(z-px)/\mathrm{D}px] \,|\, x, k, x] = 0$ (1)
$\mathrm{real}\,\mathrm{D}[(z-px)/\mathrm{D}px|x, k, x] = \mathrm{real}\{-1-[(z-px)/\mathrm{D}px][\mathrm{D}^2px/\mathrm{D}px]\} =$
$= -1 - \mathrm{real}[(z-px)/\mathrm{D}px]\mathrm{real}[\mathrm{D}^2px/\mathrm{D}px]$
$-\mathrm{Imag}[(z-px)/\mathrm{D}px]\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px)$ (2)
$(1) \,.\, (2) \,.\supset.\, 1+\mathrm{Imag}[(z-px)/\mathrm{D}px]\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px) = 0 \,.\supset.$
$\mathrm{Imag}[(z-px)/\mathrm{D}px] = -/\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px)$ (3)
$(z-px)/\mathrm{D}px = \mathrm{real}[(z-px)/\mathrm{D}px] + \mathrm{Imag}[(z-px)/\mathrm{D}px] \,.\, (1) \,.\, (3) \,.\supset.$
$(z-px)/\mathrm{D}px = -/\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px) \,.\supset.$
$z-px = -\mathrm{D}px/\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px) \,.\supset.\, \mathrm{P} \,]$

·5 $\mathrm{Rc}\,px = (\mathrm{D}px)^2/\mathrm{mod}[(\mathrm{cmp}\perp\mathrm{D}px)\mathrm{D}^2px]$ Bg

$[\, \mathrm{Df\,Cc} \,.\, \mathrm{P}{\cdot}3 \,.\supset.\, (\mathrm{Cc}px-px)\times(\mathrm{D}^2px) = (\mathrm{D}px)^2 \,.\supset.\, \mathrm{P} \,]$

Radio de curvatura vale quadrato de velocitate diviso per componente normale de acceleratione, considerato in valore absoluto. Componente normale de acceleratione vocare saepe « acceleratione normale ».

Constructio graphico de centro de curvatura, dato px, $\mathrm{D}px$, D^2px:

Construe punctos $px+\mathrm{D}px$, $px+\mathrm{D}px+(\mathrm{cmp}\perp\mathrm{D}px)\mathrm{D}^2px$; per puncto $px+\mathrm{D}px$ duce normale ad recta$[px, px+\mathrm{D}px+(\mathrm{cmp}\perp\mathrm{D}px)\mathrm{D}^2px]$, in plano osculatore, que seca normale ad curva in Ccpx.

·6 $\mathrm{Rc}px = (\mathrm{mod}\,\mathrm{D}px)^3 \,/\, \mathrm{mod}(\mathrm{D}px \,\alpha\, \mathrm{D}^2px)$

·7 $\mathrm{Ax}px = \mathrm{recta}(\mathrm{Cc}px, \mathrm{B}px)$

496. ***axi,*** AL axis, DF axe, I asse, H eje, F essieu.
|| D achse, G axon, R osį, S acs'a.

497. ***curvatura*** HI, A curvature, F courbure.
$\subset$ curva (verbo) + -to — -o + -ura (153).

498. curva (verbo) = fac curvo. $\subset$ curvo — -o + -a (4).

499. ***curvo*** HI, ***curva*** (nomen) HI. AD curve, F courbe.
|| R crivi-tį, criva-ja (linija). $\subset$ cur- + -vo (214).
curva = curvo — -o + -a (126, linea).

500. cur- $\supset$ cur-vo. || L cor-ona, cir-co, ci-n-ge,...
G cyr-to = cur-vo, cyl-indro, cy-clo,... (V a n.).

❋ 44. curvatura torsio

$k \varepsilon \mathrm{Cls'q} \,.\, k \supset \delta k \,.\, p \varepsilon \mathrm{pF}k \,.\, x \varepsilon k \,.\, \mathrm{D}px \,\varepsilon\, \mathrm{v}\text{-}\iota 0 \,.\, \mathrm{D}^2 px \,\varepsilon\, \mathrm{v}\text{-}\mathrm{qD}px \,.\supset.$

·1 $\text{curvatura}\, px = \mathrm{D\,T}px \,/\, \mathrm{modD}px = \mathrm{N}px \,/\, \mathrm{Rc}\,px$ Df

·2 $\text{torsio}\, px = \mathrm{D\,B}\,px \times \mathrm{N}px \,/\, \mathrm{mod\,D}\,px$ Df

·3 $\text{torsio}px = -(\mathrm{D}px\; \alpha\; \mathrm{D}^2px\; \alpha\; \mathrm{D}^3px \,/\Psi] \,/\, (\mathrm{D}px\; \alpha\; \mathrm{D}^2px)^2$

Nos considera « curvatura » ut vectore, et « torsione » ut quantitate cum signo. Torsio es positivo in luppolo et in cucurbita, es negativo in viti vinifera, et in viti commune de Mechanica.

·4 $\mathrm{D\,B}\,px \,/\, \mathrm{mod\,D}px = (\text{torsio}px)\,\mathrm{N}px$

·5 $\mathrm{DN}\,px \,/\mathrm{mod\,D}\,px = -(/\,\mathrm{Rc}\,px)\,\mathrm{T}px - (\text{torsio}\,px)\,\mathrm{B}px$

} FRENET, *Sur les courbes à double courbure*, Thèse 10 juillet 1847, JdM. t.17 a.1851.

$/\,\text{torsio}\; px$ = radio de torsione.

$\mathrm{Cc}\,px - (/\,\text{torsio}\,px)(\mathrm{D\,Rc}\,px)\mathrm{B}\,px$ = « centro de sphaera osculatrice » = « puncto de inviluppo de planN px ».

recta$\{px, (\mathrm{Rc}\,px)\,\mathrm{T}px - (/\,\text{torsio}\,px)\,\mathrm{B}px\}$ = « generatrice de superficie rectificante inviluppo de plan$(px, \mathrm{I\,N}\,px)$ ».

$(o+\mathrm{T}px)|x$ describe indicatrice de tangentes.
$(o+\mathrm{N}px)|x$ » » normales principale.
$(o+\mathrm{B}px)|x$ » » binormales.

501. ***torsione*** (L. a. +100), AF torsion, H tortijon, I torsione.
$\subset$ tortione (L. classico) $\subset$ torto — -o + -ione.

502. torto I, ADF tort, H tuerto. $\supset$ tortu-oso A, tort-ura AD.
$\subset$ *torcto (vocabulo supposito) $\subset$ torque — -ue + -to.

503. torque, HI torce, F tordre. || (secundo B r é a l) L trepe (raro), trep-ido, || G trepe, tropo, trop-ico, R trepet', S trap.

✻ 45·1 $k \varepsilon \text{Interv} . p \varepsilon \text{pF}k . \text{D}p \varepsilon \text{vF}k . z_1, z_2 \varepsilon k . z_1 < z_2 . \supset .$
$pz_1 \, \alpha \, pz_2 \; \varepsilon \; (z_2 - z_1) \, pz_1 \, \alpha \, \text{Med}\, \text{D}p \text{‘} z_1 \frown z_2$

[$b \varepsilon \varphi^2 . f = \{[pz_1 \, \alpha \, pz \, \alpha \, b - (z-z_1)/(z_2-z_1) \, pz_1 \, \alpha \, pz_2 \, \alpha \, b] / \Psi | z, z_1 \frown z_2 \} . \supset .$
$f \varepsilon \text{qF} z_1 \frown z_2 . fz_1 = 0 . fz_2 = 0 . \text{D}f = \{[pz_1 \, \alpha \, \text{D}pz \, \alpha \, b - |(z_2-z_1) \, pz_1 \, \alpha \, pz_2 \, \alpha$
$b] / \Psi | z, z_1 \frown z_2 \}$. Theorema de Rolle $. \supset . \; pz_1 \, \alpha \, pz_2 \, \alpha \, b / \Psi \, \varepsilon \, (z_2 - z_1) \, pz_1 \, \alpha$
$\text{D}p \text{‘} z_1 \frown z_2 \, \alpha \, b / \Psi$ (1)

(1) . Df Med $. \supset .$ P]

·2 $\text{Hp}\cdot 1 . \text{D}p \; \varepsilon \; (\text{vF}k)\text{cont} . x \varepsilon k . \supset .$
$\lim[(pz_1 \, \alpha \, pz_2)/(z_2 - z_1) \, | z, (k\text{f}\, 1 \cdots 2)\text{cres}, (x,x)] = px \, \alpha \, \text{D}px$

·3 $\text{Hp}\cdot 2 . \supset . \lim[\text{d}(pz_1, pz_2)/(z_2 - z_1) \, | z, \ldots] = \text{modD}px$

·4 $\text{Hp}\cdot 2 . \text{D}px -\!\!= 0 . \supset .$
$\lim[\text{recta}(pz_1, pz_2) \, | z, (k\text{f}\, 1 \cdots 2)\text{cres}, (x,x)] = \text{recta}(px, \text{D}px)$

Recta tangente, in uno puncto de curva, per definitione, es positione limite de recta que uni ce puncto ad alio puncto de curva, dum secundo puncto tende ad primo. Theorema dice, que, si puncto habe derivata de ordine 1 continuo et non nullo, tunc recta tangente es positione limite de recta que uni duo puncto differente de curva, dum ambo puncto varia et tende ad puncto dato.

Si hypothesi non es vero, thesi pote es in defectu. P. ex. recta tangente ad curva descripto per puncto $px = o + x^2 a + x^3 b$, ubi $o \varepsilon \text{p} . a, b \varepsilon \text{v} . a \alpha b -\!\!= 0$, que vocare « parabola de ordine 3/2 », habe $\text{rectaT}\, p0 = \text{recta}(0, a)$. Positione limite de $\text{recta}(px, p-x)$, que uni duo puncto de curva, que tende ad $p0$ dum x tende ad 0, es $\text{recta}(0, b)$, differente de praecedente.

✻ 46·1 $\text{Hp P}45\cdot 1 . \text{D}^2 p \; \varepsilon \; \text{vF}k . z_1, z_2, z_3 \varepsilon k . z_1 < z_2 < z_3 . \supset .$
$pz_1 \, \alpha \, pz_2 \, \alpha \, pz_3 \; \varepsilon \; (z_2 - z_1)(z_3 - z_1)(z_3 - z_2) \, pz_1 \, \alpha \, \text{MedD}p \text{‘} z_1 \frown z_2 \, \alpha$
$\text{MedD}^2 p \text{‘} z_1 \frown z_3 / 2$

·2 $\text{Hp}\cdot 1 . \text{D}^2 p \; \varepsilon \; (\text{vF}k)\text{cont} . \supset . \lim\{(pz_1 \, \alpha \, pz_2 \, \alpha \, pz_3)/[(z_2 - z_1)$
$(z_3 - z_1)(z_3 - z_2)] | z, (k\text{f}\, 1 \cdots 3)\text{cres}, (x,x,x)\} = px \, \alpha \, \text{D}px \, \alpha \, \text{D}^2 px / 2$

·3 $\text{Hp}\cdot 2 . \text{D}px -\!\!= 0 . \supset . \lim \text{d}[pz_1, \text{rectaT}(p, z_2)]/(z_2 - z_1)^2 \, | z, \ldots]$
$= \text{mod}(\text{D}px \, \alpha \, \text{D}^2 px)/(2 \text{modD}px)$

·4 $\text{Hp}\cdot 2 . \supset . \lim[\text{ang}(\text{D}pz_1, \text{D}pz_2)/(z_2 - z_1) \, | z, \ldots] =$
$\text{mod}(\text{D}px \, \alpha \, \text{D}^2 px)/(\text{modD}px)^2$

·5 $\text{D}^2 p \; \varepsilon \; (\text{vF}k)\text{cont} . \text{D}px \, \alpha \, \text{D}^2 px -\!\!= 0 . \supset .$
$\lim[\text{plan}(pz_1, pz_2, pz_3) \, | z, (k\text{f}\, 1 \cdots 3)\text{cres}, (x,x,x)] =$
$\text{plan}(px, \text{D}px, \text{D}^2 px)$

✻ 47·1 Hp P46·1 . $D^3p \,\varepsilon\, \mathrm{vF}k$. $z_1, z_2, z_3, z_4 \varepsilon k$. $z_1 < z_2 < z_3 < z_4$.⊃.
$pz_1 \,\alpha\, pz_2 \,\alpha\, pz_3 \,\alpha\, pz_4 \,\varepsilon\, (z_2-z_1)(z_3-z_1)(z_3-z_2)(z_4-z_1)(z_4-z_2)$
$(z_4-z_3)\, pz_1 \,\alpha\, \mathrm{MedD}p\text{‘}z_1\frown z_2 \,\alpha\, \mathrm{MedD}^2pz_1\frown z_3 \,\alpha\, \mathrm{MedD}^3pz_1\frown z_4/12$

[$f = \{pz_1 \,\alpha\, pz_2 \,\alpha\, pz_3 \,\alpha\, pz|\Psi - (z-z_1)(z-z_2)(z-z_3)/[(z_4-z_1)(z_4-z_2)(z_4-z_3)]$
$pz_1 \,\alpha\, pz_2 \,\alpha\, pz_3 \,\alpha\, pz_4|z, z_1\frown z_4\}$.⊃. $f\varepsilon \mathrm{qF}k$. $fz_1 = fz_2 = fz_3 = fz_4 = 0$. Theorema de Rolle .⊃. $pz_1 \,\alpha\, pz_2 \,\alpha\, pz_3 \,\alpha\, pz_4 \,\varepsilon\, (z_4-z_2)(z_4-z_3)\, pz_1 \,\alpha\, pz_2 \,\alpha\, pz_3 \,\alpha$
$\mathrm{MedD}^3p\text{‘}z_1\frown z_4/3!$ (1)

(1) . P45·1 . P46·1 .⊃. P]

Hp·1 . $D^3p \,\varepsilon\, (\mathrm{vF}k)\mathrm{cont}$.⊃.

·2 $\lim\{(pz_1 \,\alpha\, pz_2 \,\alpha\, pz_3 \,\alpha\, pz_4)/[(z_2-z_1)(z_3-z_1)(z_3-z_2)(z_4-z_1)(z_4-z_2)$
$(z_4-z_3)] \,|\, z, (k\mathrm{f}1\cdots4)\mathrm{cres}, (x,x,x,x)\} = px \,\alpha\, \mathrm{D}px \,\alpha\, \mathrm{D}^2px \,\alpha\, \mathrm{D}^3px/12$

·3 $\mathrm{D}px \,\alpha\, \mathrm{D}^2px -\!\!= 0$.⊃. $\lim\{\mathrm{d}[pz_1, \mathrm{planO}(p,z_2)]/(z_2-z_1)^3 \,|\, z, \ldots\} =$
$\mathrm{mod}(\mathrm{D}px \,\alpha\, \mathrm{D}^2px \,\alpha\, \mathrm{D}^3px)/[6\, \mathrm{mod}(\mathrm{D}px \,\alpha\, \mathrm{D}^2px)]$

·4 $\mathrm{D}px -\!\!= 0$.⊃. $\lim\{\sin(\mathrm{D}pz_1, \mathrm{D}pz_2, \mathrm{D}pz_3)/[(z_2-z_1)(z_3-z_1)$
$(z_3-z_2)] \,|\, z, \ldots\} = [(\mathrm{D}px \,\alpha\, \mathrm{D}^2px \,\alpha\, \mathrm{D}^3px)/\psi]/[2(\mathrm{mod}\,\mathrm{D}px)^3]$

·5 $\mathrm{D}px \,\alpha\, \mathrm{D}^2px -\!\!= 0$.⊃.
$\lim\{\mathrm{d}[\mathrm{rectaT}(p,z_1), \mathrm{rectaT}(p,z_2)]/(z_2-z_1)^3 \,|\, z, \ldots\} =$
$\mathrm{mod}(\mathrm{D}px \,\alpha\, \mathrm{D}^2px \,\alpha\, \mathrm{D}^3px)/[12\, \mathrm{mod}(\mathrm{D}px \,\alpha\, \mathrm{D}^2px)]$

·6 $\mathrm{D}px \,\alpha\, \mathrm{D}^2px -\!\!= 0$.⊃. $\lim \mathrm{ang}[\mathrm{rectaT}(p,z_1), \mathrm{planO}(p,z_2)]/$
$(z_2-z_1)^2 = \mathrm{mod}(\mathrm{D}px \,\alpha\, \mathrm{D}^2px \,\alpha\, \mathrm{D}^3px)/[2\, \mathrm{mod}(\mathrm{D}px \,\alpha\, \mathrm{D}^2px)]$

·7 $\mathrm{D}px \,\alpha\, \mathrm{D}^2px -\!\!= 0$.⊃.
$\lim\{\mathrm{ang}[\mathrm{planO}(p,z_1), \mathrm{planO}(p,z_2)]/(z_2-z_1) \,|\, z, \ldots\} =$
$\mathrm{mod}\, \mathrm{D}px\; \mathrm{mod}(\mathrm{D}px \,\alpha\, \mathrm{D}^2px \,\alpha\, \mathrm{D}^3px)/[\mathrm{D}px \,\alpha\, \mathrm{D}^2px]^2$

Vide meo libro *Applicazioni geometriche* a.1887 p.110.

✻ 49·1 $g\varepsilon \mathrm{v}$.⊃:
$p\varepsilon\, \mathrm{pFq}$. $\mathrm{D}^2p = (\iota g;\mathrm{q})$.=. $p = [(p0 + t\mathrm{D}p0 + t^2g/2)|t, \mathrm{q}]$

[… .=. $\mathrm{D}p = [(\mathrm{D}p0 + tg)|t, \mathrm{q}]$.=. …]

Si puncto mobile p habe acceleratione constante g, tunc suo lege de motu habe expressione scripto. Suo trajectoria es parabola.

Puncto materiale pesante, in vacuo, prope superficie de Terra, habe acceleratione constante, dicto « gravitate ».

·2 $p\varepsilon\, \mathrm{pFq}$. $o\varepsilon \mathrm{p}$: $x\varepsilon \mathrm{q}$.⊃$_x$. $\mathrm{D}^2px \,\varepsilon\, \mathrm{q}(px - o)$:⊃.
$o\alpha p\alpha \mathrm{D}p \,\varepsilon\, (\mathrm{p}^3\mathrm{fq})\mathrm{const}$

[$x\varepsilon \mathrm{q}$.⊃. $o \,\alpha\, px \,\alpha\, \mathrm{D}^2px = 0$.⊃. $\mathrm{D}[(o \,\alpha\, px \,\alpha\, \mathrm{D}px)|x, \mathrm{q}, x] = 0$.⊃. P]

Si p es puncto mobile, et fortia que move illo es semper directo verso puncto fixo o, tunc tripuncto $o \,\alpha\, px \,\alpha\, \mathrm{D}px$, id es triangulo considerato in magnitudine, in plano suo, et in sensu, es constante dum varia x.

e D

❋ 50·1 $x \varepsilon \mathrm{q} . \supset . \mathrm{D}(\mathrm{e}^x | x, x) = \mathrm{e}^x$

Dem 1. [Df D $. \supset . \mathrm{D}(\mathrm{e}^x | x, x) = \lim[(\mathrm{e}^{x+h} - \mathrm{e}^x)/h \,|h, \mathrm{q}, 0]$
$= $ » $\mathrm{e}^x(\mathrm{e}^h - 1)/h$ »
$= \mathrm{e}^x \lim(\mathrm{e}^h - 1)/h,$ »
§e P4·7 $. \supset .$ » $= \mathrm{e}^x$]

Dem 2. [$\mathrm{D}(\mathrm{e}^x \,|\, x, x) = \mathrm{D}[\Sigma(x^r / r! \,|\, r, \mathrm{N}_0) \,|\, x, x]$
$= \Sigma[\mathrm{D}(x^r \,|\, x, x)/r! \,|\, r, \mathrm{N}_0] = \Sigma[x^{r-1}/(r-1)! \,|\, r, \mathrm{N}_1]$
$= \Sigma[x^r / r! \,|\, r, \mathrm{N}_0] = \mathrm{e}^x$]

Si x es quantitate, tunc derivata de e^x, quando x varia, pro valore x, vale e^x.

Si nos pone $x = 0$, nos habe: $\mathrm{D}(\mathrm{e}^x \,|\, x, 0) = 1$, derivata de e^x quando x varia, pro valore 0, vale 1. Curva diagramma de functio $(\mathrm{e}^x \,|\, x, \mathrm{q})$ es dicto « logarithmica ». Ergo « Tangente ad logarithmica $y = (\mathrm{e}^x \,|\, x, \mathrm{q})$ in puncto de abscissa 0, fac cum axi de x angulo de 45^0 ».

·2 $x \varepsilon \mathrm{Q} . \supset . \mathrm{D}(\log, x) = /x$

Dem 1. [Df D $. \supset . \mathrm{D}(\log, x) = \lim\{[\log(x+h) - \log x]/h \,|h, \mathrm{Q}-x, 0\}$
§Log·6·7 $. \supset .$ » $= \lim\{[\log(1+h/x)]/h$ » »
» ·8 $. \supset .$ » $= /x \times \lim\{\log(1+h/x) \curlywedge (x/h)$ » »
Comm(lim, log) $. \supset .$ » $= /x \times \log \lim\{$ » » »
Df e $. \supset .$ » $= /x \times \log \mathrm{e} = /x$]

Dem 2. [§e P2·01 . §D P4·2 $. \supset . \mathrm{D}(\log, \mathrm{Q}, x) = /\mathrm{D}(\mathrm{e} \curlywedge y | y, \mathrm{q}, \log x)$
$= /(\mathrm{e} \curlywedge \log x) = /x$]

·3 $a \varepsilon \mathrm{Q} - \iota 1 . x \varepsilon \mathrm{Q} . \supset . \mathrm{D}({}^a\mathrm{Log}\, x \,| x, x) = ({}^a\mathrm{Log}\, \mathrm{e})/x$
[§e P2·1 . P·2 $. \supset$ P]

·4 $a \varepsilon \mathrm{Q} . x \varepsilon \mathrm{q} . \supset . \mathrm{D}(a^x | x, x) = a^x \log a$
[Df D $. \supset .$ $\mathrm{D}(a^x | x, \mathrm{q}, x) = \lim[(a^{x+h} - a^x)/h \,|h, \mathrm{q}, 0]$
§q 5·2 $. \supset .$ » » $[a^x(a^h - 1)/h$ »
Comm(lim, ×) $. \supset .$ » $= a^x \lim$ » »
§e P5·2 $\supset$ P]

❋ 51·1 $x \varepsilon \mathrm{q} . \supset . \mathrm{D}[\mathrm{e}^x(x-1) \,|\, x, x] = x\mathrm{e}^x$.
$x \varepsilon \mathrm{q} . \supset . \mathrm{D}[\mathrm{e}^x(x^2 - 2x + 2) \,|\, x, x] = x^2 \mathrm{e}^x$

·2 $x \varepsilon \mathrm{Q}+1 . \supset . \mathrm{D}(\log\log, x) = /(x \log x)$.
$a, x \varepsilon \mathrm{q} . a^2 > x^2 . \supset . \mathrm{D}\{\log[(a-x)/(a+x)] | x, x\} = 2a/(x^2 - a^2)$

·3 $a \varepsilon \mathrm{Q} . x \varepsilon \mathrm{q} . \supset . \mathrm{D}\{\log[x + \sqrt{(a+x^2)}] | x, x\} = /\sqrt{(a+x^2)}$.

$a \varepsilon \mathrm{q} \,.\, x \varepsilon \mathrm{q} - \iota 0 \;.\supset.\; \mathrm{D} \{ \log[\sqrt{(a^2+x^2)} - a]/x \,|\, x, .x \}$
$= a/[x\sqrt{(a^2+x^2)}]$

·4 $a,b,x \varepsilon \mathrm{q} \,.\, b > a^2 \;.\supset.$
$\mathrm{D}\{\log[x+a+\sqrt{(x^2+2ax+b)}]x, x\} = /\sqrt{(x^2+2ax+b)}$

·5 $a,b,x \varepsilon \mathrm{q} \,.\, a -= b \,.\, (x+a)(x+b) > 0 \;.\supset.$
$\mathrm{D}\{\lfloor \log[\sqrt{(x+a)}+\sqrt{(x+b)}]/[\sqrt{(x+a)}-\sqrt{(x+b)}]\rfloor | x, x\} =$
$/\sqrt{[(x+a)(x+b)]}$

·6 $a \varepsilon \mathrm{Q} \,.\, x \varepsilon \mathrm{q} \,.\, n \varepsilon \mathrm{N}_1 \;.\supset.\; \mathrm{D}^n(a^x \,|x, x) = a^x(\log a)^n$

·7 $x \varepsilon \mathrm{Q} \,.\, n \varepsilon \mathrm{N}_1 \;.\supset.\; \mathrm{D}^n(\log, x) = (-1)^{n-1}(n-1)!\, x^{-n}$

·8 » $\mathrm{D}^{n+1}(x^n \log x \,|x, x) = n! \,/x$

·9 $x \varepsilon \mathrm{Q} \,.\, n \varepsilon \mathrm{N}_1 \;.\supset.$
$\mathrm{D}^n(\log x\,/x \,|x, x) = (-1)^n n!\, x^{-n-1}[\log x - \Sigma/(1 \cdots n)]$

* 52·1 $a \varepsilon \mathrm{q} \;.\supset:\; f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}f = af \;.=.\; fx = [(f0)\, \mathrm{e} \urcorner (ax) | x, \mathrm{q}]$

[$\mathrm{D}f = af \;.=.\; [(\mathrm{D}fx - afx) \,| x, \mathrm{q}] = (\iota 0 \mathbin{:} \mathrm{q})$
§D 2·5 $.\supset:$ $.=.\; \mathrm{D}(\mathrm{e}^{-ax} fx \,|x, \mathrm{q}) =$ »
§D 10·3 $.\supset:$ $.=.\; (\mathrm{e}^{-ax} fx \,|x, \mathrm{q}) \;\varepsilon\; (\mathrm{qFq})\, \mathrm{const}$
Df const $.\supset.\cdot.$ $.=:\; x \varepsilon \mathrm{q} \;.\supset_x.\; \mathrm{e}^{-ax} fx = [(\mathrm{e}^{-ax} fx) \,|x|0 = f0$]

·2 $a \varepsilon \mathrm{q} - \iota 0 \,.\, b \varepsilon \mathrm{q} \;.\supset:\; f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}f = [(afx+b)|x, \mathrm{q}] \;.=.$
$\mathrm{D}[(fx+b/a) \,|\, x, \mathrm{q}] = [a(fx+b/a) \,|\, x, \mathrm{q}] \;.=.$
$[(fx+b/a) \,|\, x, \mathrm{q}] = [(f0+b/a)\mathrm{e}^{ax} \,|\, x, \mathrm{q}] \;.=.$
$f = [(f0)\mathrm{e}^{ax} + b/a(\mathrm{e}^{ax}-1)|x, \mathrm{q}]$

Calculo de f, functio reale de variabile reale, que pro omni valore de x, satisfac aequatione $\mathrm{D}fx = afx + b$, ubi a et b quantitate constante.

·3 $a,b,m \varepsilon \mathrm{q} \,.\, m -= a \;.\supset:\; f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}f = [(afx+b\mathrm{e}^{mx})|x, \mathrm{q}] \;.=.$
$\mathrm{D}\{[\mathrm{e}^{-ax}fx - b\mathrm{e}^{(m-a)x}/(m-a)] \,|\, x, \mathrm{q}\} = (\iota 0 \mathbin{:} \mathrm{q}) \;.=.$
$\{[\mathrm{e}^{-ax}fx - b\mathrm{e}^{(m-a)x}/(m-a)] \,|\, x, \mathrm{q}\} \;\varepsilon\; (\mathrm{qFq})\, \mathrm{const} \;.=:$
$x \varepsilon \mathrm{q} \;.\supset_x.\; \mathrm{e}^{-ax}fx - b\mathrm{e}^{(m-a)x}/(m-a) = f0 - b/(m-a) \;:=.$
$f = \{[(f0)\mathrm{e}^{ax} + b\mathrm{e}^{mx}/(m-a) - b\mathrm{e}^{ax}/(m-a)]|x, \mathrm{q}\}$

·4 $a \varepsilon \mathrm{q} - \iota 0 \;.\supset:\; f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}^2 f = a^2 f \;.=.$
$f = \{[(f0)(\mathrm{e}^{ax}+\mathrm{e}^{-ax})/2 + (\mathrm{D}f0)(\mathrm{e}^{ax}-\mathrm{e}^{-ax})/(2a)]|x, \mathrm{q}\}$

[$f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}^2 f = a^2 f \;.=.\; [\mathrm{e}^{ax}(\mathrm{D}^2fx - a^2fx) \,|\, x, \mathrm{q}] = (\iota 0 \mathbin{:} \mathrm{q})$
$.=.\; \mathrm{D}[(\mathrm{e}^{ax}\mathrm{D}fx - a\mathrm{e}^{ax}fx) \,|\, x, \mathrm{q}] = (\iota 0 \mathbin{:} \mathrm{q})$
$.=.\; [(\mathrm{e}^{ax}\mathrm{D}fx - a\mathrm{e}^{ax}fx) \,|\, x, \mathrm{q}] \;\varepsilon\; (\mathrm{qFq})\mathrm{const}$
$.=:\; x \varepsilon \mathrm{q} \;.\supset_x.\; \mathrm{e}^{ax}\mathrm{D}fx - a\mathrm{e}^{ax}fx = \mathrm{D}f0 - af0$
$.=.\; \mathrm{D}f = [afx + (\mathrm{D}f0 - af0)\mathrm{e}^{-ax} \,|\, x, \mathrm{q}]$
$.=. \ldots$]

VI

CALCULO DIFFERENTIALE

VI. CALCULO DIFFERENTIALE.

D (derivata)

* 1. $u \varepsilon$ Cls'q . $f \varepsilon$ qFu .⊃:

·0 $x,y \varepsilon u$.⊃. $\Delta(f; x,y) = fy - fx$ Def

·1 $x,y \varepsilon u$. $x -= y$.⊃. $D(f; x,y) = (fy - fx)/(y - x)$ Def

·2 $x \varepsilon u \frown \delta u$.⊃. $D(f; x,x) = \lim[D(f; x,y) \,|y, u, x]$ Def

·3 » $Dfx = D(f; x,x)$ Def

u indica classe de quantitates; f es quantitate functione definito in campo u.

Tunc, si x et y es individuo pertinente ad classe u, differentia $fy-fx$ es dicto « incremento de functione », et indicato per $\Delta(f; x,y)$.

Si y es differente de x, tunc $D(f; x,y)$, dicto « ratione incrementale », indica ratione de incremento de functione ad incremento $y-x$ de variabile.

$D(f; x,x)$ indica limite de ratione præcedente, si y varia in campo u, et verge ad x. Per definitione de limite (pag. 230, Prop. 40·0), x, que pertine ad campo u, es proximo ad alios u, vel pertine ad classe derivata de u: $x \varepsilon u \frown \delta u$.

In loco de $D(f; x,x)$, nos scribe Dfx, que vocare « derivata de functione f pro valore x ».

Secundo membro in Prop·2 contine litera y que es apparente, nam seque signo de inversione |. Litera u pote es eliminato, per æqualitate

$$u = \text{Variab}\, f$$

« u es campo de variabilitate de functione definito f ».

Ergo secundo membro contine variabiles reale independente f et x; et pote es indicato per Dfx.

Scriptura $\mathrm{D}fx$, que contine tres signo D, f, x, sine parenthesi, debe es decomposito in

$$(\mathrm{D}f)\,x$$

« derivata de functione f, pro valore x », vel in

$$\mathrm{D}\,(f, x)$$

« derivata dependente de functione f et de valore x ».

$\mathrm{D}fx$ non pote es decomposito in $\mathrm{D}(fx)$, « derivata de numero fx », expressione sine sensu; dato f et x, resulta determinato fx; sed non viceversa. Dato numero fx, nos non pote determina f et x, unde depende derivata. Derivata non depende de solo numero fx. P. ex:

$$\mathrm{D}\sin 0 = 1$$

significa $(\mathrm{D}\sin)0 = 1$, « derivata de functione sinu, pro valore 0 vale 1 ».

Non lege $\mathrm{D}\,(\sin 0) = 1$, vel $\mathrm{D}\,0 = 1$, que non habe sensu.

* * *

Derivata habe numeroso applicatione in Geometria, in Mechanica, in Physica, in Œconomia etc., que nos considera infra. Nunc me expone suo interpretatione in Geometria.

Nos repræsenta functione f per curva loco de punctos

$$px = o + xi + (fx)j,$$

ubi o es puncto, i, j es vectore unitario et orthogono:

$$o\,\varepsilon\,\mathrm{p}\,.\,i, j\,\varepsilon\,\mathrm{v}\,.\,i^2 = j^2 = 1\,.\,i\times j = 0.$$

Variabile x varia in campo de variabilitate de f. Tunc curva descripto per puncto px vocare « diagramma de functione f ».

Differentia de duo puncto es vectore:

$$\Delta(p; x,y) = py - px = (y-x)i + (fy-fx)j,$$

parallelo ad vectore:

$$\mathrm{D}(p; x,y) = (py-px)/(y-x) = i+(fy-fx)/(y-x)j = i+\mathrm{D}(f; x,y)j$$

Numero z vocare « inclinatione » de vectore $i + zj$ in dato systema coordinato.

Ergo inclinatione de conjungente duo puncto de curva es $(fy-fx)/(y-x)$. Suo limite vocare « inclinatione de curva in puncto px ». Ergo:

« $\mathrm{D}fx$ vale inclinatione de diagramma de functione f, in puncto de abscissa x », vel « inclinatione de recta tangente ad curva ».

Functione f, que nos deriva, es « definito », vel es considerato simul cum suo campo de variabilitate: $f\,\varepsilon\,\mathrm{q}\,\mathrm{F}\,u$, ubi symbolo F « functione definito » es introducto in pag. 79.

Functiones que occurre in Analysi habe campo de variabilitate non semper definito. Tunc nos adde isto campo pro obtine functiones definito.

Exemplo 1.

$$\mathrm{D}\,(x^2|x,\ \mathrm{q})0 = 0.$$

Lege: « Derivata de elevatione ad potestate 2, in campo de numeros reale, pro valore 0, vale 0 ». Vel « inclinatione de parabola diagramma de functione $y = x^2$ ubi x sume omni valore reale, in origine o, vale 0; vel vectore i es tangente ad parabola ». Campo de variabilitate es toto campo de numeros reale.

Exemplo 2.

$$\mathrm{D}\,(/,\ \mathrm{q}{-}\iota 0)\,1 = -1.$$

« Derivata de functione reciproco, in campo de valores reale differente de 0, pro valore 1, vale -1 ». Vel « si nos considera hyperbola diagramma de functione $y = 1/x$, ubi x sume omni valore reale, valore 0 excepto, suo inclinatione in puncto de abscissa 1 vale -1; vel tangente fac angulo -45^0 cum axi oi.

Campo de variabilitate de functione que nos deriva pote coincide cum toto campo de numeros reale, $u = \mathrm{q}$, ut in exemplo 1, vel es intervallo, vel habe forma plus complexo. P. ex., nos pote deriva functione rationale de variabile rationale rFr. Suffice in generale que x pertine ad classe u et ad classe derivata.

Si nos muta campo de variabilitate, derivata pote sume valores differente. Per exemplo:

$$\mathrm{D}\,(\mathrm{mod},\ \mathrm{Q}_0)\,0 = 1$$
$$\mathrm{D}\,(\mathrm{mod},\ -\mathrm{Q}_0)\,0 = -1.$$

« Derivata de functione modulo, in campo de numeros $\gtreqless 0$, pro valore 0 vale 1, et in campo de numeros $\leqq 0$ vale -1 ». Vel « diagramma de functione modulo consta de duo semirecta bisectrice de angulos $(-i, j)$ et (i, j). Ce linea non habe inclinatione determinato in origine, sed inclinationes $+1$ et -1 ad dextera et ad sinistra de origine ».

Nota.

Leibniz indica derivata de y relativo ad x, per signo $\dfrac{dy}{dx}$ ubi « recta aliqua pro arbitrio assumpta vocetur dx » (MathS. t.5 p.220) et « ipsas dx, dy, ut ipsarum x,y differentiis sive incrementis, vel decrementis momentaneis proportionales haberi posse » (p.169).

Quantitates dx et dy vocare « differentiale ». dx es quantitate arbitrario, non nullo; dy vale derivata $\times dx$.

In aliquo casu Leibniz pone $d\overline{x}=1$; et scribe (*Briefwechsel* t.1 p.226):

$$d\overline{x} = 1\ ,\ d\overline{x}^2 = 2x\ ,\ d\overline{x}^3 = 3x^2 \text{ etc.}\quad d\overline{\sqrt{x}} = \frac{1}{2\sqrt{x}} \text{ etc.}$$

Tunc $dy = \mathrm{D}y$, et differentiale es identico ad derivata.

! Newton indica derivata per uno puncto supra functione; Lagrange per uno accentu, Arbogast per $Df{\cdot}x$. Vide Theorema de Taylor.

Cauchy (*Œuvres* s.1 t.4 p.255) indica derivatas per D_x, D_y,... ubi indice designa variabile.

Jacobi distingue derivatas de functione de plure variabile, per signo ∂; ce derivatas vocare « partiale ». Sed ille nota que ce notatione non suffice, et debe es completato per lingua commune. Werke, t.3 p.396 a.1841:

« quando sine graviori incommodo licet, quanquam maxime affectanda sunt signa, quibus et omnis ambiguitas tollatur, et formulae sine omni interpretatione verbali adjecta, per se clarae et intelligibiles fiant, in hoc tamen casu... ».

P. ex. si nos habe functione de 3 variabile $f\varepsilon$ qf(q!q!q), et si $u\varepsilon$ qfq, $w\varepsilon$ qf(q!q), es necesse plure specie de d pro indica 4 derivata:

$D\,f(x,y,z)|x$, $D\,f(x,ux,z)|x$, $D\,f[x,y,w(x,y)]|x$, $D\,f[x,ux,w(x,ux)]|x$,

que es derivata de $f(x,y,z)$ pro x, ubi:

1. y et z es constante;
2. y es functione de x; z es constante;
3. z es functione de x et de y, et y es constante;
4. z es functione de x et de y, et y es functione de x.

derivata. Per etymologia, vide pag. 159, N. 280. Es nomen de symbolo δ super classes, et de symbolo D super functiones.

differentiale, AD differentiale, F différentielle, H differencial, I differenziale, R differentsial'. $\subset$ differentia (166, pag. 70) + -le (6, pag. 19).

✻ 2. $u\varepsilon$ Cls'q . $f\varepsilon$ qFu .$\supset$:

·1 $x,y\varepsilon u$.$\supset$. $\Delta(f;y,x) = -\Delta(f;x,y)$

·2 $x,y,z\varepsilon u$.$\supset$. $\Delta(f;x,y) + \Delta(f;y,z) + \Delta(f;z,x) = 0$

·3 $x,y\varepsilon u$.$\supset$. $D(f;y,x) = D(f;x,y)$

·4 $x,y\varepsilon u$. $y-=x$.$\supset$. $fy = fx + (y-x)D(f;x,y)$

✻ 3. $u\varepsilon$ Cls'q . $f\varepsilon$ qFu .$\supset$:

·1 $x\varepsilon u\frown\delta u$. $Dfx\ \varepsilon$ q .$\supset$. $\lim(f,u,x) = fx$

Dem. $\lim(f,u,x) = \lim[fx+(y-x)D(f;x,y)\,|y,u,x] = fx$

Si Dfx es quantitate (determinato et finito), tunc limite de functione, si variabile, in campo de variabilitate de f, verge ad x, es valore fx.

Tunc, in definitione de derivata, Prop. 1·2, lice substitue ad fx valore dato per Prop. 3·1. Parte de hypothesi non es necessario, et definitione sume forma plus generale:

·2 $x\varepsilon\,\delta u$.$\supset$. $Dfx = \lim\{[fy-\lim(f,u,x)]/(y-x)\ |\ y,u,x\}$ Df

·3 $u \supset \delta u \,.\, \mathrm{D}f \,\varepsilon\, \mathrm{qF}u \,.\supset.\, f\varepsilon\, (\mathrm{qF}u)\mathrm{cont}$

Dem. Prop. 3·1. Def cont (pag. 238). $\supset$. P

Si classe u pertine ad suo classe derivata, vel si campo u es condensato, secundo nomenclatura de pag. 180, et si functio f habe derivata in toto campo u, tunc isto functio es continuo. Resulta de Prop. ·1, et de definitione de continuitate.

Non omni functio continuo habe derivata. Uno exemplo es functio « modulo » ante considerato.

Novo exemplo:

$\mathrm{D}\,[x \sin /x \,|x,\ /(\mathrm{N}_1 \pi)]\, 0 = 0$

$\mathrm{D}\,[\quad » \quad ,\ /(2\mathrm{N}_1 \pi + \pi/2)]\, 0 = 1$

$\mathrm{D}\,[\quad » \quad ,\ /(2\mathrm{N}_1 \pi - \pi/2)]\, 0 = -1.$

Functio $x \sin/x$ non es definito pro $x = 0$. Si nos tribue ad illo valore 0, tunc functio es continuo, sed pro $x = 0$ non habe derivata in campo « q » de numeros reale. Nam limite de ratione de incremento de functione ad incremento de variabile depende de campo de variabilitate.

* 4. $a,b,x \varepsilon\, \mathrm{q} \,.\supset.\, \mathrm{D}[(ax+b)\,|x, \mathrm{q}]x = a$

Dem. $y\varepsilon\, \mathrm{q}\text{-}\iota x \,.\supset.\, \mathrm{D}[(ax+b)\,|x; x,y] = a$

Si a, b, x es quantitate, tunc derivata de functione lineare $ax + b$, ubi varia x, in campo de quantitates, pro valore dato x, vale a.

Nota que in scriptura $(ax+b)|x$, litera x es apparente, et non es idem litera x que occurre in hypothesi; sed nullo confusione ori in isto casu, et in casu simile in calculo integrale.

In vero, ratione de incremento de functione ad incremento de variabile habe valore constante a.

* 5. Derivata de summa.

·1 $u\varepsilon\, \mathrm{Cls'q} \,.\, f,g \varepsilon\, \mathrm{qF}u \,.\, x\varepsilon\, u \frown \delta u \,.\, \mathrm{D}fx, \mathrm{D}gx\, \varepsilon \mathrm{q} \,.\supset.$

$\mathrm{D}[(fx+gx)|x, u]x = \mathrm{D}fx + \mathrm{D}gx$ Distrib(D, +)

« Derivata de summa de duo functione vale summa de derivatas de functiones ».

[Df D $.\supset.$ $\mathrm{D}[(fx+gx)|x,u]x = \lim\{[(fy+gy)-(fx+gx)]/(y-x)\,|y, u, x]$

Distrib(/,+). $\supset$. $\quad » = \lim\{(fy-fx)/(y-x) + (gy-gx)/(y-x) \quad »$

Distrib(lim,+) $.\supset.$ P]

·2 Hyp·1 . $a\varepsilon \mathrm{q} \,.\supset.\, \mathrm{D}(a\times fx\,|x, u)x = a\times \mathrm{D}fx$

« Derivata de producto de quantitate constante per functione vale quantitate constante per derivata de functione ».

✻ 6. DERIVATA DE PRODUCTO.

Hyp 5·1 .⊃. $D(fx \times gx \mid x, u)x = fx \times Dgx + gx \times Dfx$

Dem. $y \varepsilon u - \iota x$.⊃. $D(fx \times gx \mid x; x,y) =$
$fx \times D(g;x,y) + gx \times D(f;x,y) + D(f;x,y) \times D(g;x,y) \times (y-x)$

« Derivata de producto de duo functione vale primo factore per derivata de secundo, plus secundo factore per derivata de primo ».

✻ 7. DERIVATA DE QUOTIENTE.

·1 $x \varepsilon$ q$-\iota 0$.⊃. $D(/, q-\iota 0)x = -/x^2$
Dem. $y \varepsilon$ q$-\iota 0-\iota x$.⊃. $D(/; x,y) = -/(xy)$

·2 $u \varepsilon$ Cls'q . $f,g \varepsilon$ qFu . $0 -\varepsilon f`u$. $x \varepsilon u \frown \delta u$. Dfx, $Dgx \varepsilon$ q .⊃.
$D(gx/fx \mid x,u)x = (fx \times Dgx - gx \times Dfx)/(fx)^2$

Dem. $y \varepsilon u - \iota x$.⊃. $D(gx/fx \mid x; x,y) = (gy/fy - gx/fx)/(y-x) =$
$(fx\, gy - gx\, fy)/[(y-x)\, fx\, fy] = [fx\, D(g;x,y) - gx\, D(f;x,y)] / (fx\, fy)$

« Derivata de quotiente de duo functione vale denominatore per derivata de numeratore, minus numeratore per derivata de denominatore, toto diviso per quadrato de denominatore ». Nos suppone que numeratore et denominatore es functione definito in campo u, ubi denominatore non sume valore zero, et que, per valore x in u et prope alios u, ambo functione habe derivata.

✻ 8. DERIVATA DE POTESTATE.

·1 $m \varepsilon N_1$. $x \varepsilon$q .⊃. $D(x^m \mid x, q)x = mx^{m-1}$
Dem. 1 $y \varepsilon$ q$-\iota x$.⊃. $D(x^m \mid x; x,y) = \Sigma(y^{m-r}x^{r-1} \mid r, 1 \cdots m)$.⊃. P
Dem. 2 $m = 1$.⊃ P (1)
$m \varepsilon N_1$. $D(x \wedge m \mid x, q)x = mx \wedge (m-1)$. P6 .⊃.
$D(x^{m+1} \mid x, q)x = D[(x^m \times x) \mid x, q]x = mx^{m-1} \times x + x^m = (m+1)x \wedge m$ (2)
(1) . (2) . Induct .⊃. P

Si m es numero naturale, et x es quantitate, tunc derivata de x^m, ubi varia x, in campo de numeros reale, pro valore x de variabile, vale mx^{m-1}.

Nota que in formula $x^m \mid x$, litera x es apparente; formula vale $z^m \mid z$,

et indica « potestate m » ; litera x non es idem litera x, que occurre in Hp. Si nos pone $m = 2$, formula fi :

$$x \varepsilon \mathrm{q} \;.\supset.\; \mathrm{D}(x^2 \mid x, \mathrm{q})x = 2x,$$

et si nos pone $x = 1$, formula fi :

$$\mathrm{D}(x^2 \mid x, \mathrm{q})1 = 2.$$

« derivata de quadrato, in campo de valores reale, pro valore 1, vale 2 ».

In formula incompleto $\mathrm{D}x^2 = 2x$, que occurre in plure libro, non lice substitutione materiale de valore numerico ad x.

Demonstratione 1. In vero, ratione de incremento de functione ad incremento de variabile es summa de m termine, que omni tende ad x^{m-1}.

Demonstratione 2. Potestate es casu particulare de producto, et regula deriva ex regula de derivatione de producto.

·2 $n \varepsilon \mathrm{N}_1 \,.\, x \varepsilon \mathrm{q}-\iota 0 \;.\supset.\; \mathrm{D}(x^{-n} | x, \mathrm{q}-\iota 0)x = -nx^{-n-1}$

Dem. $\mathrm{D}(x^{-n}|x, \mathrm{q}-\iota 0)x = \mathrm{D}(/x^n | x, \mathrm{q}-\iota 0)x = -nx^{n-1}/x^{2n} = -nx^{-n-1}$

Regula de potestate mane, si exponente es negativo, pro variabile non nullo.

·3 $x \varepsilon \mathrm{Q} \;.\supset.\; \mathrm{D}(\surd, \mathrm{Q})x = /(2\surd x)$

Dem. $\mathrm{D}(\surd; x, y) = /(\surd x + \surd y)$

✻ 9. Derivata de functione de functione.

$u, v \varepsilon \mathrm{Cls'q} \,.\, f \varepsilon \mathrm{qF}u \,.\, g \varepsilon u\mathrm{F}v \,.\, x \varepsilon v \frown \delta v \,.\, gx \,\varepsilon\, \delta u \,.\, \mathrm{D}f(gx), \mathrm{D}gx \;\varepsilon \mathrm{q}$
$\;.\supset.\; \mathrm{D}(fg)x = \mathrm{D}f(gx) \times \mathrm{D}gx$

Dem. $y \varepsilon v-\iota x \,.\, \mathrm{P}2{\cdot}4 \;.\supset.\; fgy - fgx = \mathrm{D}(f; gx, gy) \times (gy - gx) \;.\supset.$
$\mathrm{D}(fg; x, y) = \mathrm{D}(f; gx, gy) \times \mathrm{D}(g; x, y) \;.\supset.\; \mathrm{P}$

Nos suppone : u et v es classe de quantitates ; f es quantitate functione definito in campo u ; g es u functione definito in campo v. Nos sume individuo x in v et prope alios v ; gx es prope alios u. Functione f pro valore gx, et functione g pro valore x habe derivatas determinato et finito.

Tunc derivata de functione fg, functione f de functione g, vale derivata de f pro valore gx, multiplicato per derivata de g pro valore x.

Vel derivata de fgx pro x vale derivata de fgx pro gx, multiplicato per derivata de g pro x.

In vero, ratione incrementale de functione fg, pro valores distincto x et y, vale producto de ratione incrementale de f pro valores gx et gy, distincto aut non, per ratione incrementale de g pro valores x et y. Unde ad limite seque theorema.

❋ 10. FUNCTIONE INVERSO.

·0 $u \varepsilon \mathrm{Intv} \,.\, f \varepsilon (\mathrm{qF}u)\mathrm{cres\ cont} \,.\supset.\, f^{-1} \varepsilon (\mathrm{qF}f\text{'}u)\ \mathrm{cres\ cont}$

Si in aliquo intervallo u, functione f es crescente et continuo, tunc suo functione inverso es definito in campo de valores sumpto per f, et es ibi crescente et continuo.

·1 (decr | cres) P·0

Idem pro functione decrescente.

·2 $u \varepsilon \mathrm{Intv} \,.\, f \varepsilon (\mathrm{qF}u)(\mathrm{cres} \cup \mathrm{decr})\mathrm{cont} \,.\, y \varepsilon f\text{'}u \,.\, \mathrm{D}f(f^{-1}y) \varepsilon \mathrm{q}-\iota 0 \,.$
$\supset.\, \mathrm{D}f^{-1}y = /\mathrm{D}f(f^{-1}y)$

Dem. $y' \varepsilon f\text{'}u - \iota y \,.\supset.\, \mathrm{D}(f^{-1}; y, y') = (f^{-1}y' - f^{-1}y)/(y' - y) = /\mathrm{D}(f; f^{-1}y, f^{-1}y')$

Si functione f es crescente aut decrescente, et continuo, et si y es uno ex valores de functione, tunc derivata de functione inverso de f pro valore y, es reciproco de derivata de functione directo pro valore respondente ad y. Nos suppone que derivata de functione directo existe et non es nullo.

❋ 11. DERIVATA DE RADICE.

·1 $m \varepsilon \mathrm{N}_1 \,.\, x \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{D}(\sqrt[m]{x} \,|\, x, \mathrm{Q})x = /[m \sqrt[m]{x}^{m-1}]$

Dem. $\mathrm{D}(\sqrt[m]{\ }, \mathrm{Q})x = /\mathrm{D}(z^m \,|\, z, \mathrm{Q})\sqrt[m]{x} = /[(mz^{m-1} \,|\, z, \mathrm{Q})\text{'}\sqrt[m]{x}] = /[m(\sqrt[m]{x})^{m-1}]$

·2 $m \varepsilon \mathrm{r} \,.\, x \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{D}(x^m \,|\, x, \mathrm{Q})x = mx^{m-1}$

❋ 12. DERIVATA DE EXPONENTIALE.

·1 $x \varepsilon \mathrm{q} \,.\supset.\, \mathrm{D}(\mathrm{e}^x \,|\, x, \mathrm{q})x = \mathrm{e}^x$

Dem. $\mathrm{Def\,D} \,.\supset.\, \mathrm{D}(\mathrm{e}^x \,|\, x, \mathrm{q})x = \lim[(\mathrm{e}^{x+h} - \mathrm{e}^x)/h \,|\, h, \mathrm{q}, 0]$
$= \text{»}\ \mathrm{e}^x(\mathrm{e}^h - 1)/h\ \text{»}$
$= \mathrm{e}^x \lim(\mathrm{e}^h - 1)/h\ \text{»}$
§e P5·1 (pag. 245) $.\supset.$ » $= \mathrm{e}^x$

Derivata de functione exponentiale $(\mathrm{e}^x | x, \mathrm{q})$ vale se ipso.

Si nos pone $x = 0$, nos habe: $\mathrm{D}(\mathrm{e}^x \,|\, x, \mathrm{q})0 = 1$, derivata de e^x quando x varia, pro valore 0, vale 1. Curva diagramma de functione $(\mathrm{e}^x \,|\, x, \mathrm{q})$ es « logarithmica ». Ergo « Tangente ad logarithmica in puncto de abscissa 0, fac cum axi de x angulo de 45^0 ».

·2 $a\varepsilon Q \,.\, x\varepsilon q \,.\supset.\, D(a^x|x, q)x = a^x \log a$

Dem. $h\varepsilon q \,.\supset.\, D(a^x|x; x, x+h) = (a^{x+h}-a^x)/h = a^x(a^h-1)/h$

§e P5·2 $.\supset.\, \lim[(a^h-1)/h|h, q, 0] = \log a \,.\supset.\,$ P

✻ 13. Derivata de logarithmo.

·1 $x\varepsilon Q \,.\supset.\, D(\log, Q)x = /x$

Dem 1. Df D $.\supset.\, D(\log, Q)x = \lim\{[\log(x+h)-\log x]/h\,|h, Q-x, 0\}$

§Log·6·7 $.\supset.$ » $= \lim\{[\log(1+h/x)]/h$ » »

» ·8 $.\supset.$ » $= /x \times \lim\{\log(1+h/x)\curvearrowright(x/h)$ » »

Comm(lim, log) $.\supset.$ » $= /x \times \log\lim\{$ » » »

Df e $.\supset.$ » $= /x \times \log e = /x\,\}$

Dem 2. §e P2·01 . §D P12·1 $.\supset.\, D(\log, Q)x = /D(e\curvearrowright y|y, q)\log x$

$= /(e\curvearrowright\log x) = /x$

·2 $a\varepsilon Q-\iota 1 \,.\, x\varepsilon Q \,.\supset.\, D({}^a\mathrm{Log}\,x\,|x, x) = ({}^a\mathrm{Log}\,e)/x$

Dem. ${}^a\mathrm{Log}\,x = {}^a\mathrm{Log}\,e \times \log x \,.\supset.\,$ P

·3 $m\varepsilon q \,.\, x\varepsilon Q \,.\supset.\, D(x^m|x, Q)x = mx^{m-1}$

Dem. $x\curvearrowright m = e\curvearrowright(m\log x)$

·4 $u\varepsilon \mathrm{Cls'q} \,.\, f\varepsilon QFu \,.\, g\varepsilon qFu \,.\, x\varepsilon u\frown\delta u \,.\, Dfx, Dgx\,\varepsilon\, q \,.\supset.$

$D(fx\curvearrowright gx\,|x, u)x = (fx\curvearrowright gx)\times\log fx\times Dgx + gx\times fx\curvearrowright(gx-1)\times Dfx$

Dem. $fx\curvearrowright gx = e\curvearrowright(gx\log fx)$

✻ 14. Exercitio.

$x\varepsilon q-\iota 0 \,.\supset.\, D(\mathrm{mod}, q)x = \mathrm{sgn}\,x$.

$D(\mathrm{mod}, Q_0)0 = 1$. $D(\mathrm{mod}, -Q_0)0 = -1$

Derivata de functione « mod » depende de campo de variabilitate.

$a,b,c,d\varepsilon q \,.\, x\varepsilon q \,.\, a+bx -= 0 \,.\supset.$

$D[(c+dx)/(a+bx)\,|\,x, q-\iota-a/b]x = (ad-bc)/(a+bx)^2$

$x\varepsilon q \,.\supset.\, D[x/\sqrt{(1+x^2)}\,|\,x, q]x = (1+x^2)\curvearrowright(-3/2)$

$x\varepsilon q \,.\supset.\, D[e^x(x-1)\,|\,x, q]x = xe^x$

$x\varepsilon q \,.\supset.\, D[e^x(x^2-2x+2)\,|\,x, q]x = x^2e^x$

$x\varepsilon q \,.\supset.\, D[(e^x+e^{-x})/2\,|\,x. q]x = (e^x-e^{-x})/2$

» » $-$ » » » $+$ »

Functiones in secundo membro es « sinu » et « cosinu hyperbolico ».

$x\varepsilon\,\mathrm{Q}+1 \;.\supset.\; \mathrm{D}(\log\log, \mathrm{Q}+1)x = /(x\log x)$

$a,x\varepsilon\mathrm{q} \;.\; a^2 > x^2 \;.\supset.$

$\mathrm{D}\{\log[(a-x)/(a+x)] \mid x, (-a)^{\frown}a\}\, x = 2a/(x^2-a^2)$

$a\varepsilon\mathrm{Q} \;.\; x\varepsilon\mathrm{q} \;.\supset.\; \mathrm{D}\{\log[x+\surd(a+x^2)] \mid x, \mathrm{q}\}x = /\surd(a+x^2)$

$a\varepsilon\mathrm{q} \;.\; x\varepsilon\,\mathrm{q}\text{-}\iota 0 \;.\supset.\; \mathrm{D}[\log\{[\surd(a^2+x^2)+a]/x\} \mid x, \mathrm{q}\text{-}\iota 0]x$
$= a/[x\surd(a^2+x^2)]$

$a,b,x\varepsilon\mathrm{q} \;.\; b>a^2 \;.\supset.$

$\mathrm{D}\{\log[x+a+\surd(x^2+2ax+b)] \mid x, \mathrm{q}\}x = /\surd(x^2+2ax+b)$

$a,b,x\varepsilon\mathrm{q} \;.\; a<b<x \;.\supset.$

$\mathrm{D}\{[\log[\surd(x-a)+\surd(x-b)]/[\surd(x-a)-\surd(x-b)]] \mid x, b+\mathrm{Q}\}x =$
$/\surd[(x-a)(x-b)]$

✻ 15. Derivata de numero complexo, de vectore et de puncto, functione de variabile reale.

·1 $u\varepsilon\,\mathrm{Cls'q} \;.\; n\varepsilon\mathrm{N}_1 \;.\; f\varepsilon\,\mathrm{Cx}n\,\mathrm{F}u \;.\; x\varepsilon\, u{\frown}\delta u \;.\supset.$ P1, 2, 3, 5

·2 (vct | Cx) P·1

(pnt »

Definitione de derivata, dato in Prop. 1, et principale regulas de derivatione, subsiste pro numero complexo functione dato in campo reale.

Et si nos considera puncto et vectore in loco de complexo.

Derivata de punto mobile, si variabile es « tempore », es vectore dicto « velocitate ».

tempore, vocabulo de Latino internationale.
$\supset$ = F. temps = H. tiempo = I. tempo.
F. temps $\supset$ A. tense (in Grammatica).
I. tempo $\supset$ A.D.R. tempo (in Musica).
$\supset$ tempor-ale A.F.H.I., tempor-ario A.F.H.I.
Vocabularios etymologico non es concorde in origine de ce vocabulo.

velocitate $\supset$ = A. velocity = H. velocidad = I. velocità.
F. vélocité (non in Mathematica).
$\subset$ veloc-e (= F. véloce = H. veloz = I. veloce) — -e + -itate (p.19).

❋ 16. $u \varepsilon \text{Cls'q} \,.\, x \varepsilon u \frown \delta u \,.\, f,g \varepsilon \text{vF}u \,.\, \text{D}fx, \text{D}gx \, \varepsilon \, \text{v} \;.\supset.$

·1 $\text{D}(fx \times gx \mid x, u)x = fx \times \text{D}gx + gx \times \text{D}fx$

·2 $\text{D}(fx \,\alpha\, gx|x, u)x = \text{D}fx \,\alpha\, gx + fx \,\alpha\, \text{D}gx$

Derivata de producto interno et alterno de duo vectore variabile es analogo ad derivata de producto de duo functio numerico. Nota que in producto alterno non lice commuta factores.

·3 $fx -= 0 \;.\supset.\; \text{D}\, \text{mod}fx = \text{D}fx \times \text{U}fx$

·4 $\quad$ » $\quad \text{D}\, \text{U}fx = (\text{cmp} \lfloor fx)\text{D}fx/(\text{mod}fx)$

❋ 17. Functione imaginario de variabile imaginario.

·0 $u \varepsilon \text{Cls'q'} \,.\, f \varepsilon \text{q'F}u \;.\supset.$ P1, 2, ...

·1 $x \varepsilon \text{q'} \;.\supset.\; \text{D}(\text{e}^x|x, \text{q'})x = \text{e}^x$
[(q'|q) Dem P12·1]

Si x es numero imaginario, tunc derivata de e^x, ubi varia x, in campo de numeros imaginario, pro valore x, vale e^x, ut pro campo reale.

·2 $x \varepsilon \text{q} \;.\supset.\; \text{D}(\text{e}^{\text{i}x}|x, \text{q})x = \text{ie}^{\text{i}x}$

❋ 18. Derivata de functiones trigonometrico.

·1 $x \varepsilon \text{q} \;.\supset.\; \text{D}(\text{s}, \text{q})x = \text{c}x \;\;.\;\; \text{D}(\text{c}, \text{q})x = -\text{s}x$
[$\text{D}[\text{e}\urcorner(\text{i}x)\,|x, \text{q}]x = \text{i}\,\text{e}\urcorner(\text{i}x)$. Oper imag . Oper real $.\supset.$ P]

Derivata de sin vale cos, et derivata de cos vale −sin. Resulta de derivata de exponentiale.

·2 $x \varepsilon \text{q}-(\text{n}+/2)\pi \;.\supset.$
$\text{D}[\text{t}, \text{q}-(2\text{n}+1)\pi/2]x = /(\text{c}\,x)^2 = 1+(\text{t}\,x)^2$
[» $= \text{D}(\text{s}x/\text{c}x\,|x)x = (\text{c}x\,\text{Ds}x - \text{s}x\,\text{Dc}x)/(\text{c}x)^2 = /(\text{c}x)^2$]

·3 $y \varepsilon (-1)\frown 1 \;.\supset.\; \text{Ds}^{-1}y = /\sqrt{(1-y^2)} \;\;.\;\; \text{Dc}^{-1}y = -/\sqrt{(1-y^2)}$
[$\text{Ds}^{-1}y = /\text{Ds}(\text{s}^{-1}y) = /(\text{c}\,\text{s}^{-1}y) = /\sqrt{(1-y^2)}$]

·4 $y \varepsilon \text{q} \;.\supset.\; \text{Dt}^{-1}y = /(1+y^2)$
[$\text{Dt}^{-1}y = /\text{Dt}(\text{t}^{-1}y) = /[1+(\text{t}\,\text{t}^{-1}y)^2] = /(1+y^2)$]

* 19·1 $x \varepsilon\, \mathrm{q}-\mathrm{n}\pi \;.\supset.\; \mathrm{D}/\mathrm{t}x = -/(\mathrm{s}x)^2$

·2 $x \varepsilon\, \theta\pi/2 \;.\supset.\; \mathrm{D}\log \mathrm{s}x = /\mathrm{t}x$

·3 $x \varepsilon\, \theta\pi/2 \;.\supset.\; \mathrm{D}\log \mathrm{c}x = -\mathrm{t}x$

·4 $x \varepsilon\, \theta\pi \;.\supset.\; \mathrm{D}\log \mathrm{tang}(x/2) = /\sin x$

·5 $x \varepsilon -\pi/2\,\bar{}\,\pi/2 \;.\supset.$
$\mathrm{D}\log \mathrm{tang}(\pi/4+x/2) = /\cos x$

·6 $a\varepsilon \mathrm{Q} \,.\, b\varepsilon \mathrm{q} \,.\, \mathrm{mod}\, b < a \,.\, x\varepsilon \mathrm{q} \;.\supset.$
$\mathrm{D}\, \mathrm{c}^{-1}[(b+a\cos x)/(a+b\cos x)] = \sqrt{}(a^2-b^2)/(a+b\cos x)$

·7 $x\varepsilon \mathrm{q} \;.\supset.\; \mathrm{D}\{[/(4\sqrt{2})]\log[(1+x\sqrt{2}+x^2)/(1-x\sqrt{2}+x^2)] +$
$[/(2\sqrt{2})]\mathrm{tang}^{-1}[x\sqrt{2}\,/(1-x^2)]\} = /(1+x^4)$

* 20·1 Theorema de maximo et minimo.

$u\varepsilon \mathrm{Cls'q} \,.\, f\varepsilon\, \mathrm{qF}u \,.\, x\varepsilon\, \mathrm{in}u \,.\, fx = \max f\text{‘}u \,.\, \mathrm{D}fx\, \varepsilon \mathrm{q} \;.\supset.\; \mathrm{D}fx = 0$

Dem. $\mathrm{Hp} \,.\, y\varepsilon\, u \;.\supset.\; fy - fx \leqq 0$ (1)
$(1) \,.\, y\varepsilon\, u \frown (x-\mathrm{Q}) \;.\supset.\; \mathrm{D}(f;x,y) \geqq 0$ (2)
$(1) \,.\, y\varepsilon\, u \frown (x+\mathrm{Q}) \;.\supset.\; \mathrm{D}(f;x.y) \leqq 0$ (3)
$(2) \,.\, (3) \;.\supset.\; \mathrm{D}fx \geqq 0 \,.\, \mathrm{D}fx \leqq 0 \;.\supset.\; \mathrm{D}fx = 0$

u es classe, ubi es definito functione f. Si ad valore x, interno ad campo considerato, responde valore fx, maximo ex valores de functione in toto campo, et si derivata de f, pro valore x, es determinato et finito, tunc ce derivata vale zero.

In vero, fx es maximo valore de functione. Ergo, si y es in campo u, seque $fy-fx \leqq 0$. Tunc, si $y<x$, ratione incrementale es $\geqq 0$. Si $y>x$, ratione considerato es $\leqq 0$. Dfx, limite de ambo ratione, es $\geqq 0$ et $\leqq 0$, ergo D$fx = 0$.

·2 (min | max)P·1

P·1 subsiste, si in loco de maximo valore, nos considera minimo. Plure auctore moderno voca « extremo » valore que es aut maximo aut minimo.

Exemplo. Nos vol decompone dato numero a in duo partes x et $a-x$, tale que producto de potestates m et n de duo parte fi maximo. Exponentes m,n es p. ex. numero naturale. Functione $x^m(a-x)^n|x$, es nullo pro valores 0 et a de variabile, et es positivo in interno de intervallo de 0 ad a, et es continuo. Ergo ce functione fi maximo pro valore de variabile interno ad intervallo considerato. Ce valore annulla derivata, vel satisfac æquatione :

$$mx^{m-1}(a-x)^n - nx^m(a-x)^{n-1} = 0;$$

si nos supprime factores x^{m-1} et $(a-x)^{n-1}$, que non es nullo in interno de intervallo considerato, æquatione fi

$$m(a-x) - nx = 0, \text{ vel } x/m = (a-x)/n.$$

Valore de x que satisfac æquatione es unico, et es valore quæsito. Ergo duo parte debe es proportionale ad exponentes, ut es noto per Algebra, III §12 P30, pag. 110.

Applicationes de theorema praecedente ad Geometria. (*Exercitio*).

1. Rectangulo inscripto in triangulo, et maximo in area, habe altitudine æquale ad altitudine de triangulo /2.
2. Cylindro in sphæra, maximo in volumen, habe altitudine $= 2r/\surd 3$, radio basi $= r\surd(2/3)$; r es radio de sphæra.
3. Cylindro in sphæra, maximo in superficie laterale, habe diametro de basi = altitudine = (radio sphæra)$\times\surd 2$.
4. Idem, maximo in superficie totale, habe altitudine = (radio sphæra) $\times\surd 2\surd[1-1/\surd 5]$.
5. Cono in sphæra, maximo in volumen, habe altitudine = (radio sphæra) $\times 4/3$. (Fermat a.1636).
6. Idem, maximo in superficie laterale, idem.
7. Idem, maximo in superficie totale, habe altitudine = radio $\times$ $(23-\surd 17)/16$.
8. Cylindro inscripto in cono, et maximo in volumen, habe altitudine æquale ad altitudine de cono /3.
9. Idem, maximo in superficie laterale, habe altitudine æquale ad altitudine de cono /2.

Si functione es maximo, non in interno, sed p. ex. pro limite supero de campo, tunc derivata, si existe, es $\geqq 0$.

Functione « mod » es minimo pro $x=0$, et non habe derivata.

$x\uparrow(2/3)$ es minimo pro $x=0$, et derivata es ∞.

❋ 21. Theorema de Rolle.

$$a,b\varepsilon\mathrm{q} \,.\, a-\!\!=b \,.\, f,\mathrm{D}f\,\varepsilon\,\mathrm{qF}a\!\!-\!\!b \,.\, fa=fb=0 \,.\supset.\, 0\,\varepsilon\,\mathrm{D}f\text{`}a\!\!-\!\!b$$

Si functione reale f, habente derivata in toto intervallo de a ad b, es nullo pro valores a et b, tunc existe valore interno ad intervallo, que redde derivata nullo.

{ Rolle a.1689 p.127 :

« Les racines de chaque cascade (derivata) seront prises pour les hypotheses moyennes de la cascade suivante ». }

Versione : inter duo valores que annulla functione, existe valore que annulla derivata.

Dem.

Functione f, que habe derivata, per theorema præcedente es continuo:

Hp . P2·3 $.\supset.$ $f \varepsilon (\mathrm{qF}a \dashv b)\mathrm{cont}$ (1)

Ergo existe suo maximo et suo minimo valore in toto intervallo:

Hp . (1) . § cont 2·3 $.\supset.$ $\max f\text{‘}a \dashv b, \min f\text{‘}a \dashv b\, \varepsilon \mathrm{q}$ (2)

Si functio f sume valores positivo, tunc maximo intra illos es positivo, et responde ad valore interno ad intervallo:

Hp . $\exists\, \mathrm{Q} \frown f\text{‘}a \dashv b \,.\, x \varepsilon a \dashv b \,.\, fx = \max f\text{‘}a \dashv b$ $.\supset.$ $x \varepsilon a^{-}b$ (3)

Ergo per theorema super maximo, derivata es nullo:

Hp (3) . P20 $.\supset.$ Ths (4)

Si functione f sume valores negativo, $-f$ sume valores positivo, et ex ratiocinio præcedente seque thesi:

Hp . $\exists\, (-\mathrm{Q}) \frown f\text{‘}a \dashv b$ $.\supset.$ $\exists\, \mathrm{Q} \frown (-f)\text{‘}a \dashv b$. (4) $.\supset.$ Ths (5)

Si functione sume valores non positivo, et non negativo, es semper nullo, et derivata $= 0$:

Hp . $-\exists\, \mathrm{Q} \frown f\text{‘}a \dashv b \,.\, -\exists\, (-\mathrm{Q}) \frown f\text{‘}a \dashv b$ $.\supset.$ $f \varepsilon \iota 0\, \mathrm{F} a \dashv b$ $.\supset.$ Ths (6)

In omni casu thesi es vero:

(4) . (5) . (6) $.\supset.$ P

✻ 22. Theorema de valore medio.

$$a,b \varepsilon \mathrm{q} \,.\, a -= b \,.\, f, \mathrm{D}f \,\varepsilon\, \mathrm{qF}a \dashv b \;.\supset.\; \mathrm{D}(f;a,b) \,\varepsilon\, \mathrm{D}f\text{‘}\,a^{-}b$$

Dem. Hp . $g = \{[fx - fa - (x-a)\mathrm{D}(f;a,b)] \mid x, a \dashv b\}$ $.\supset.$ $ga = gb = 0$.
P21 $.\supset.$ $\exists\, a^{-}b \frown x \ni [\mathrm{D}fx - \mathrm{D}(f;a,b) = 0]$ $.\supset.$ Ths

Si a, b es quantitate differente inter se, et functione reale f habe derivata in toto intervallo de a ad b, tunc ratione de incremento de functione ad incremento de variabile es uno ex valores de derivata in interno de intervallo.

In vero, si nos pone $hx = fa + (x-a)\mathrm{D}(f;a,b)$, seque $ha = fa$, $hb = fb$. Tunc functione $gx = fx - hx$, ubi varia x, in intervallo dato, es nullo pro $x = a$ et $x = b$. Per theorema de Rolle, suo derivata es nullo pro valore interno ad intervallo, unde seque P.

Theorema præcedente es multo importante in calculo differentiale.

In Geometria, si nos considera puncto $o + xi + (fx)j$, tunc puncto $o + xi + (hx)j$ describe recta que i trans punctos de curva, de abscissa a et b. Theorema dice: chorda es parallelo ad tangente in aliquo puncto de arcu.

{ Cavalieri a.1635 l.vii p.15.

« Si curva linea quæcunque data tota sit in eodem plano, cui occurrat recta in duobus punctis ..... poterimus aliam rectam lineam præfatæ æquidistantem ducere, quæ tangat portionem curvæ lineæ inter duos prædictos occursus continuatam ». }

Puncto $o+(fx)i$ describe recta oi. Theorema dice que $\mathrm{D}(f;\,a,b)$, id es spatio descripto per puncto diviso per tempore, es uno ex valores de $\mathrm{D}fx$, numero que mete velocitate de puncto.

Sub ambo forma, de Geometria et de Mechanica, propositione es quasi evidente. Suo enunciatione, cum hypothesi necessario et sufficiente es in:

Grassmann a.1862, Werke t.1 p.323,

Weierstrass, vide G. Cantor JfM. a.1870 t.72 p.141,

Ossian-Bonnet, vide Serret Calc. diff. a.1868 p.19.

❋ 23.

·1 $a,b\varepsilon \mathrm{q}\,.\,a -\!\!= b\,.\,f\varepsilon\,\mathrm{qF}a\text{-}b\,.\,\mathrm{D}f\varepsilon\,\mathrm{QF}a\text{-}b\ .\supset.\ f\varepsilon$ cres

·2 » » $(-\mathrm{Q})$ » $.\supset.\ f\varepsilon$ decr

·3 » » $(\iota 0)$ » $.\supset.\ f\varepsilon$ const

[P22 $\supset$ P]

De theorema de valore medio seque:

Functione reale, que in aliquo intervallo habe derivata semper positivo, es crescente.

Si derivata es negativo, functione es decrescente.

Si derivata es semper nullo, functione habe valore constante.

Si duo functione f et g habe derivata semper æquale, $\mathrm{D}f = \mathrm{D}g$, tunc $f-g$ habe derivata semper nullo. Ergo differentia de duo functione es constante.

Si functione h es derivata de f, $h = \mathrm{D}f$, tunc f vocare functione primitivo, vel « integrale » de h.

Exemplo.

$n\varepsilon\mathrm{Q}\ .\supset.\ (x^n \mathrm{e}^{-x}|x,\ 0\,\overline{\ \ }\,n)\ \varepsilon$ cres . $(x^n \mathrm{e}^{-x}|x,\ n+\mathrm{Q})\ \varepsilon$ decr

Functione considerato es crescente de 0 ad n, et post decresce.

$(x^x|x,\ \theta/\mathrm{e})\ \varepsilon$ decr . $(x^x|x,\ /\mathrm{e}+\mathrm{Q})\ \varepsilon$ cres

Nos habe: $x\varepsilon\mathrm{q}\ .\supset.\ 1-\cos x \geqq 0.$

Tunc functione $x-\sin x$, que habe pro derivata functione $1-\cos x$, es crescente; et es nullo pro $x=0$; ergo:

$$x\varepsilon\mathrm{Q}\ .\supset.\ x-\sin x > 0,$$

vel $$x\varepsilon\mathrm{Q}\ .\supset.\ x < \sin x.$$

Tunc functione $x^2/2-1+\cos x$, que habe pro derivata functione $x-\sin x$, es crescente; illo es nullo pro $x=0$; ergo:

$$x\varepsilon\mathrm{Q}\ .\supset.\ x^2/2-1+\cos x > 0,$$

vel $x \varepsilon Q . \supset . \cos x > 1 - x^2/2$.

Tunc functione $x^3/3! - x + \sin x$ es crescente, et nullo pro $x = 0$;

ergo: $x \varepsilon Q . \supset . \sin x > x - x^3/3!$,

et ita porro, resulta formula 16·3 de pag. 252.

❋ 24. Approximationes.

Quantitates que occurre in praxi es in generale determinato per approximatione; id es, es dato intervallo que contine illo. Isto intervallo pote es dato per extremos:

Densitate de ferro = 7700⌐7900 Kg/m³,
» nive = 50⌐500 Kg/m³.

Pote es dato per uno extremo, et suo amplitudo (vel longore, Long de pag. 142). Per exemplo, scriptura

$$e = 2{\cdot}71828...$$

significa, apud plure Auctore, que cifras scripto es primo cifras de evolutione de numero e in fractione decimale, id es:

$$e \; \varepsilon \; 2{\cdot}71828 + \theta \times 10^{-5}$$

et $$\text{Long } (2{\cdot}71828...) = 10^{-5}.$$

Aliquo Auctore da intervallo per puncto medio, et demi amplitudo; et scribe Log3 = 0·4771213 in loco de

$$\text{Log}3 \; \varepsilon \; 0{\cdot}4771213 \pm (\theta \times 10^{-7})/2.$$

In omni casu, amplitudo de intervallo es uno unitate de ultimo ordine decimale.

Si x es valore noto per approximatione, id es, si x es intervallo, et super illo nos fac operatione f, tunc fx (id es, $f'x$) es determinato solo per approximatione, id es fx es intervallo (nos suppone continuitate de f). Si dx es differentia de duo valore de x, et dfx differentia de valores correspondente de fx, theorema de valore medio dice que

$$\text{d}fx = (\text{D}fx)\text{d}x,$$

ubi in Dfx, x es aliquo valore in intervallo dato.

Nos calcula dfx per excessu, et resulta:

$$x \varepsilon \text{Intv} . f, \text{D}f \; \varepsilon \; \text{qF}x . \supset . \text{Long}fx \leqq (\text{l}'\text{modD}fx) \times \text{Long}x.$$

« Si x es intervallo, et f es functione reale definito, cum suo derivata, in intervallo x, tunc longore de intervallo fx, vel errore in calculo de fx, non supera longore de x, vel errore

in variabile x, multiplicato per maximo valore absoluto de derivata in intervallo ».

Ex regulas de derivatione seque:

$$x \varepsilon \theta \,.\, h \varepsilon \theta x \,.\, m \varepsilon \mathrm{N}_1 \,.\supset.\, \mathrm{Long}(x-\theta h)^m < h$$

« Errore in calculo de potestate m de numero approximato $x-\theta h$, dato per limite supero x, et amplitudo de approximatione h, si x es < 1, es minore de $m \times$ errore de numero ».

$$x \varepsilon 1+\mathrm{Q} \,.\, h \varepsilon \mathrm{Q} \,.\, m \varepsilon \mathrm{N}_1 \,.\supset.\, \mathrm{Long}\ \sqrt[m]{(x+\theta h)} < h$$

« Errore in calculo de radice m de numero majore de 1, es minore de errore in radicando ».

$$x \varepsilon 1+\mathrm{Q}.\ h \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{Long\ Log}(x+\theta h) < h$$

« Errore in logarithmo decimale (aut naturale) de numero es minore que errore de numero ».

$$x \varepsilon 1+\mathrm{Q} \,.\, h \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{Long}\ \sin(x+\theta h) < h$$

« Errore in sinu (vel cosinu) es minore de errore de arcu ».

Exemplo. Nos quære $(9{\cdot}25...)^{10}$. Vale $10^{10}\times(0{\cdot}925...)^{10}$. Me calcula $(0{\cdot}925)^{10}$, per exemplo cum tabulas de logarithmos, $= 0{\cdot}45858....$ Numero 0·925... habe errore $< 10^{-3}$; ergo suo potestate habe errore $< 10^{-2}$. Me supprime cifras decimale post secundo, et scribe: $(0{\cdot}925...)^{10} = 0{\cdot}45..$,; et post multiplicatione per 10^{10}:

$$(9{\cdot}25...)^{10} = 10^9\times 4{\cdot}5...$$

ubi secundo cifra pote es 5 (ut es scripto) aut 6; causa accumulatione de duo errore: in suppressione de cifras post 0·45, et in determinatione de basi 9·25....

Aliquo Auctore (PERRY) voca cifras « immorale » cifras que non resulta ex hypothesi, sed ex prolongatione arbitrario de calculo trans limites legitimo. Qui scribe per exemplo:

$$(9{\cdot}25...)^{10} = 10^9\times 4{\cdot}5858...$$

scribe 5 cifras, de que 3 es immorale. Ille affirma que omni numero inter 9·25 et 9·26 ad potestate 10 es comprehenso inter $10^9\times 4{\cdot}5858$ et $10^9\times 4{\cdot}5859$, quod non es vero; nam illos es comprehenso solo inter $10^9\times 4{\cdot}5$ et $10^9\times 4{\cdot}7$.

❋ 25. INTEGRALE DE POLYNOMIO.

$$n \varepsilon \mathrm{N}_1 \,.\, a \varepsilon \mathrm{qF}\,0\cdots n \,.\supset\colon f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}f = [\Sigma(a_r x^r|r, 0\cdots n)|x, \mathrm{q}]$$
$$.=.\, f = \{f0+\Sigma[a_r x^{r+1}/(r+1)|r, 0\cdots n]|x, \mathrm{q}\}$$

$$[\ \mathrm{D}\{fx-\Sigma[a_r x^{r+1}/(r+1)\ |\ r,0\cdots n]\ |\ x, \mathrm{q}\} = (\iota 0\colon \mathrm{q})\ .\ \mathrm{P}{\cdot}233.\ \supset \mathrm{P}\ ]$$

Si n es numero naturale, et si a es successione de quantitates cum indices 0, 1, ... n, tunc functione f, que habe pro derivata polynomio :

$$Dfx = a_0+a_1x+a_2x^2+...+a_nx^n$$

pro omni valore reale de x, vale :

$$fx = f0+a_0x+a_1x^2/2+a_2x^3/3+...+a_nx^{n+1}/(n+1)$$

ubi varia x, in toto campo de numeros.

* 26. Altero theorema de valore medio.

·1 $a,b\varepsilon q \,.\, a{-}{=}b \,.\, f,g\varepsilon\, qF\, a\frown b \,.\, m\varepsilon N_0 \,.\, Df =$
$[(x-a)^m gx | x, a\frown b] \,.\supset.\, fb-fa\ \varepsilon\ (b-a)^{m+1}/(m+1) \times g`a\frown b$

Dem. $k = (fb-fa)/(b-a)^{m+1} \,.\, h = [fx-fa-(x-a)^{m+1}k \,|x, a\frown b] \,.\supset.$
$ha{=}hb{=}0$. Theorema de Rolle $.\supset.\ \exists\ a\frown b \cap x\ni(Dhx{=}0) \,.\supset.$
$\exists\ a\frown b \cap x\ni[Dfx-(m+1)k(x-a)^m{=}0] \,.\supset.\ \exists\ a\frown b \cap x\ni[gx-(m+1)k{=}0]$
$.\supset.\ k\varepsilon\ (g`a\frown b)/(m+1)$

Si derivata de functione f habe forma $(x-a)^m gx$, dum x varia de a ad b, tunc incremento de functione vale integrale de primo factore $(x-a)^m$, per valore medio de secundo.

·2 $a,b\varepsilon q \,.\, a{-}{=}b \,.\, f,g\varepsilon\, qFa\frown b \,.\, n\varepsilon N_1 \,.\, c\varepsilon\, qF1\cdots n \,.$
$Df = \{\Sigma[c_r(x-a)^{r-1} \,|r, 1\cdots n] + (x-a)^n gx \,|\, x, a\frown b\} \,.\supset.$
$fb\ \varepsilon\ fa\ +\ \Sigma[c_r(b-a)^r/r \,|r, 1\cdots n] + (b-a)^{n+1}/(n+1) \times g`a\frown b$

Dem. Hp $.\supset.$ $D\{fx-fa-\Sigma[c_r(x-a)^r \,|\, r, 1\cdots n] \,|\, x, a\frown b\} =$
$[(x-a)^n gx \,|\, x, a\frown b]$. P·1 $.\supset.$ P

Si derivata de functione es polynomio ordinato secundo potestates de $(x-a)$, plus termine de forma $(x-a)^n gx$, tunc incremento de functione vale integrale de polynomio, plus termine que nos calcula cum regula præcedente.

Exemplo.

Nos habe (Prop. 13) : $x\varepsilon -1+Q \,.\supset.$
$D[\log(1+x) \,|x, -1+Q]x = 1/(1+x)$

Ex Algebra (pag. 121 P4) : $x\varepsilon -1+Q \,.\, n\varepsilon N_1 \,.\supset.$
$1/(1+x) = \Sigma[(-1)^r x^r \,|r, 0\cdots(n-1)] + (-1)^n x^n/(1+x)$

Nos integra cum regula præcedente: $\quad x\varepsilon -1+\mathrm{Q} \;.\supset.$

$$\log(1+x)\ \varepsilon\ \Sigma[(-1)^r x^{r+1}/(r+1)\ |r,\ 0\cdots(n-1)]\ +$$
$$(-1)^n x^{n+1}/(n+1)\times 1/(1+\theta x)$$

Si $-1<x\leqq 1$, limite supero et infero de $1/(1+\theta x)$ es finito; $\lim x^{n+1}/(n+1)\ |n = 0$. Seque serie de MERCATOR p.246 P6·1.

Alio exemplo.

Nos habe: $\quad x\varepsilon\mathrm{q} \;.\supset.\ \mathrm{D}\,\mathrm{tang}^{-1}x = 1/(1+x^2)$

et: $\quad x\varepsilon\mathrm{q}\,.\,n\varepsilon\mathrm{N}_1 \;.\supset.$

$$1/(1+x^2) = \Sigma[(-1)^r x^{2r}\ |r,\ 0\cdots(n-1)] + (-1)^n x^{2n}/(1+x^2)$$

Nos integra et habe P7·2 de pag. 264, unde serie 7·1

·3 $\quad a,b\varepsilon\mathrm{q}\,.\,a\,{-}{=}\,b\,.\,f,\mathrm{D}f,g,\mathrm{D}g\,\varepsilon\,\mathrm{qF}a\,\overline{\ }\,b\,.\,0\,{-}\varepsilon\,\mathrm{D}g\text{‘}a\,\overline{\ }\,b \;.\supset.$
$\quad (fb-fa)/(gb-ga)\ \varepsilon\ [\,(\mathrm{D}fx)/(\mathrm{D}gx)\,]\,|x\text{‘}\,a\,\overline{\ }\,b$

Dem. $\quad \mathrm{Hp}\,.\,h = \{[fx-fa-(gx-ga)(fb-fa)/(gb-ga)]\ |\ x,\ a\,\overline{\ }\,b\} \;.\supset.$
$ha = hb = 0\,.\,\mathrm{P}21 \;.\supset.\ \exists\, a\,\overline{\ }\,b\cap x\ni[\mathrm{D}fx-\mathrm{D}gx\times(fb-fa)/(gb-ga)=0]$
$.\supset.\ \mathrm{P}$

{ CAUCHY *Calc. diff.* a.1829 p.37 }

Ratione de incrementos de duo functione es uno ex valores sumpto per ratione de derivatas.

In vero, me voca hx functione de x de forma $fx+pgx+q$, ubi p et q es quantitate que me determina in modo que $ha=hb=0$. Functione h habe expressione scripto. Tunc per theorema de Rolle, suo derivata es nullo per uno valore inter a et b; unde seque theorema.

·4 $\quad a,b\varepsilon\mathrm{q}\,.\,a<b\,.\,f,g,h,\mathrm{D}f,\mathrm{D}g,\mathrm{D}h\ \varepsilon\ \mathrm{qF}a\,\overline{\ }\,b \;.\supset.$
$\quad \exists\, a\,\overline{\ }\,b\cap x\ni\{\mathrm{Dtrm}[(\mathrm{D}fx,\mathrm{D}gx,\mathrm{D}hx),(fa,ga,ha),(fb,gb,hb)]=0\}$

[$k = \mathrm{Dtrm}[(fx,gx,hx),(fa,ga,ha),(fb,gb,hb)]\ |x \;.\supset.$
$\quad ka = kb = 0\,.\,\mathrm{P}21 \;.\supset.\ \mathrm{P}$]

Nos considera tres functione, f, g, h, definito, cum derivatas, in aliquo intervallo $a\,\overline{\ }\,b$; tunc determinante

$$\begin{bmatrix} fx, & gx, & hx \\ fa, & ga, & ha \\ fb, & gb, & hb \end{bmatrix}$$

es nullo pro $x=a$, et pro $x=b$. Ergo suo derivata, que resulta, si in primo linea nos scribe derivatas, es nullo, pro aliquo valore inter a et b.

Si nos pone $h = (\iota 1\text{‘}a\,\overline{\ }\,b)$, id es nos suppone que h habe valore constante 1, in toto intervallo de a ad b, seque P·3. Si $\quad gx = (x-a)\uparrow m$, seque P·1; et pro $m=0$, resulta P22.

❋ 27. Theorema de De l'Hospital.

Limite de ratione de duo functione æqua ratione de limites, si numeratore et denominatore habe limite determinato et finito, et limite de denominatore non es nullo (pag. 217 Prop. 8·3 et 9·2).

Si denominatore verge ad 0, et numeratore habe limite non nullo, limite de ratione es infinito (ibi Prop. 9·3).

Si numeratore et denominatore verge ad 0, tunc es utile regula sequente:

·1 $a,b\varepsilon\mathrm{q} \,.\, a \mathrel{-\!=} b \,.\, f, g, \mathrm{D}f, \mathrm{D}g \;\varepsilon\; \mathrm{qF}a\urcorner b \,.\, x\varepsilon\, a\urcorner b \,.\, fx = gx = 0$
$.\; 0 \mathrel{-\varepsilon} \mathrm{D}g‘(a\urcorner b - \iota x) \,.\, \lim(\mathrm{D}fz \,/\, \mathrm{D}gz \mid z, a\urcorner b, x) \;\varepsilon\; \mathrm{q} \cup \iota\underline{\pm}\infty$
$.\supset.\; \lim(fz/gz \mid z, a\urcorner b, x) = \lim(\mathrm{D}fz \,/\, \mathrm{D}gz \mid z, a\urcorner b, x)$

Dem. $fz/gz = (fz - fx)/(gz - gx) \;\varepsilon\; (\mathrm{D}fu/\mathrm{D}gu) \mid u‘z\urcorner x \;.\supset.\; \mathrm{P}$

Es dato duo numero a et b, distincto, et duo functione f et g reale definito in intervallo de a ad b, habente derivatas. Ambo functione es nullo pro valore x de intervallo. Derivata de denominatore g non es nullo pro valores differente de x. Ratione de duo derivata habe limite finito aut infinito. Tunc limite de ratione fz/gz, ubi varia z in intervallo dato et verge ad x, æqua limite de ratione de duo derivata.

{ De l'Hospital, *Analyse des infiniment petits* a.1696 p.145: « si l'on prend la différence du numérateur, et qu'on la divise par la différence du dénominateur, après avoir fait $x=a$, l'on aura la valeur cherchée ». }

·2 $a,b\varepsilon\mathrm{q} \,.\, a \mathrel{-\!=} b \,.\, f,g\varepsilon\, \mathrm{qF}a\urcorner b \,.\, fa = ga = 0 \,.\, \mathrm{D}f, \mathrm{D}g \;\varepsilon\; \mathrm{qF}a\urcorner b \,.$
$0 \mathrel{-\varepsilon} \mathrm{D}g‘a\urcorner b \;.\supset.\; \mathrm{Lm}(fx/gx|x, a\urcorner b, a) \supset \mathrm{Lm}(\mathrm{D}fx/\mathrm{D}gx|x, a\urcorner b, a)$

·3 $a\varepsilon\mathrm{q} \,.\, f,g,\mathrm{D}f,\mathrm{D}g \;\varepsilon\; \mathrm{qF}(a+\mathrm{Q}) \,.\, \lim(f, a+\mathrm{Q}, \infty) =$
$\lim(g, a+\mathrm{Q}, \infty) = 0 \,.\, 0 \mathrel{-\varepsilon} \mathrm{D}g‘(a+\mathrm{Q}) \;.\supset.$
$\mathrm{Lm}(fx/gx|x, a+\mathrm{Q}, \infty) \supset \mathrm{Lm}(\mathrm{D}fx/\mathrm{D}gx|x, a+\mathrm{Q}, \infty)$

[$\mathrm{Lm}(fx/gx|x, a+\mathrm{Q}, \infty) = \mathrm{Lm}\{f(a+/z) \,/\, g(a+/z)]|z, \mathrm{Q}, 0\} \,.\, \mathrm{P}{\cdot}1 \;.\supset.\; \mathrm{P}$]

·4 $a\varepsilon\mathrm{q} \,.\, f, \mathrm{D}f \,\varepsilon\, \mathrm{qF}(a+\mathrm{Q}) \;.\supset.$
$\mathrm{Lm}[(fx)/x \,|x, a+\mathrm{Q}, \infty] \supset \mathrm{Lm}(\mathrm{D}f, a+\mathrm{Q}, \infty)$

·5 $a\varepsilon\mathrm{q} \,.\, f,g,\mathrm{D}f,\mathrm{D}g \;\varepsilon\; \mathrm{qF}(a+\mathrm{Q}) \,.\, \lim(g, a+\mathrm{Q},\infty) = \infty \;.$
$0 \mathrel{-\varepsilon} \mathrm{D}g‘(a+\mathrm{Q}) \;.\supset.$
$\mathrm{Lm}(fx/gx|x, a+\mathrm{Q}, \infty) \supset \mathrm{Lm}(\mathrm{D}fx/\mathrm{D}gx|x, a+\mathrm{Q}, \infty)$

✻ 28. Derivata de serie.

$$k \varepsilon \mathrm{Intv} \,.\, f \varepsilon \mathrm{q\,f}(k{:}\mathrm{N}_0) : x \varepsilon k \,.\, n \varepsilon \mathrm{N}_0 \,.\supset_{x,n}.\, \mathrm{D}[f(z,n) | z,k] x \,\varepsilon \mathrm{q} :$$
$$\Sigma\{\mathrm{l}' [\mathrm{mod}\, \mathrm{D}f(z,n) | z\text{‘}k] | n, \mathrm{N}_0\} \,\varepsilon \mathrm{Q} \,.\, x \varepsilon k :\supset.$$
$$\mathrm{D}\{\Sigma[f(z,n) | n, \mathrm{N}_0] | z, k\} x = \Sigma\{\mathrm{D}[f(z,n) | z, k] x \,|\, n, \mathrm{N}_0\}$$

k es intervallo ; *f* es quantitate functione de duo variabile, uno in campo *k*, altero in campo de numeros N_0. [Tunc $f(x,n)$, si varia *n*, repræsenta serie, de que omni termine depende de *x*]. Nos suppone que pro omni valore de *x* in *k*, et pro omni indice *n*, termine $f(x,n)$ habe derivata pro *x* ; et que serie de limites supero de valores absoluto de derivatas de $f(z,n)$, pro *z* variante in *k*, es convergente. Tunc derivata de serie vale serie de derivatas.

Dem.

Per definitione, derivata vale limite de ratione incrementale :

$$\mathrm{D}\{\Sigma[f(z,n)|n,\mathrm{N}_0] \,|z,k\} x = \lim(\mathrm{D}\{\Sigma[f(z,n) \,|n, \mathrm{N}_0] \,|\, z;\ x,y\} \,|\, y,k,x) \qquad (1)$$

Sed ratione incrementale de serie es serie de rationes incrementale :

§ lim P21·5 P22·1 (p.222) $.\supset.$

$$\mathrm{D}\{\Sigma[f(z,n) \,|n, \mathrm{N}_0] \,|z;\ x,y\} = \Sigma\{\mathrm{D}[f(z,n) \,|z;\ x,y] \,|\, n, \mathrm{N}_0\} \qquad (2)$$

Nunc ratione incrementale de omni termine verge ad suo derivata :

$$n \varepsilon \mathrm{N}_0 \,.\supset.\, \lim\{[\mathrm{D}f(z,n) \,|z;\ x,y] \,|y, k, x\} = \mathrm{D}[f(z,n) \,|z, k] x \qquad (3)$$

Et per theorema de valore medio, ratione incrementale es uno ex valores de derivata :

$$n \varepsilon \mathrm{N}_0 \,.\, y \varepsilon k \,.\supset.\, \mathrm{D}[f(z,n) \,|z;\ x,y] \;\varepsilon\; \mathrm{D}[f(z,n) \,|z, k]\text{‘}\, x\overline{\ }y \qquad (4)$$

Ergo serie de limites supero de valores absoluto de rationes incrementale de termines de serie dato converge ad limite $\leqq$ ad serie de limites supero de derivatas :

$$\Sigma(\mathrm{l}'\{\mathrm{mod}\mathrm{D}[f(z,n)|z;\ x,y] \,|y\text{‘}k\} \,|n, \mathrm{N}_0) \leqq \Sigma\{\mathrm{l}'\mathrm{mod}[\mathrm{D}f(z,n)|z,k] \,|n, \mathrm{N}_0\} \qquad (5)$$

Nos deduce, per theorema præcedente, que limite de serie vale serie de limites, unde seque theorema :

Comm(lim, Σ) (p.233 P43·1) $.\supset.$ P

Theorema præcedente deriva de theorema de limite de serie. Ergo vale si serie de derivatas es de « convergentia uniforme simplice » (Dini).

Serie $\Sigma[f(x,n) \,|n, \mathrm{N}_0]$ es de convergentia uniforme simplice si :

$$0 \;\varepsilon\; \mathrm{Lm}\{\mathrm{l}'\mathrm{mod}\Sigma[f(x,n) \,|\, n, m+\mathrm{N}_0] \,|\, x\text{‘}k\} \,|\, m.$$

et es æquiconvergente (vide pag. 234), si $0 = \lim$.....

Hypothesi de continuitate de derivata (que occurre in aliquo libro) non es necessario.

✻ 29. $a,b \varepsilon \mathrm{q} \,.\, a -= b \,.\, f \varepsilon \mathrm{qF} a \frown b \;.\supset:$

·1 $x \varepsilon a \frown b \,.\, \mathrm{D}f \varepsilon \mathrm{qF}(a \frown b - \iota x) \,.\, \lim(\mathrm{D}f, a \frown b, x) \varepsilon \mathrm{q} \;.\supset.$
$\mathrm{D}fx = \lim(\mathrm{D}f, a \frown b, x)$

·2 $\mathrm{D}f \varepsilon \mathrm{qF}(a \frown b) \,.\, x \varepsilon a \frown b \;.\supset.\; \mathrm{D}fx \,\varepsilon\, \mathrm{Lm}(\mathrm{D}f, a \frown b, x)$
[P22 $\supset$ P·1·2]

·3 » $.\, \mathrm{D}fa < 0 \,.\, \mathrm{D}fb > 0 \;.\supset.\; 0 \varepsilon \mathrm{D}f \text{‘} a \frown b$
[Hyp $.\supset.\; \exists\, a \frown b \frown x \ni (fx = \max f \text{‘} a \frown b) \;.\supset.$ P]

·4 » $.\supset.\; \mathrm{D}fa \frown \mathrm{D}fb \supset \mathrm{D}f \text{‘} a \frown b$
[P·3 $\supset$ P·4]

·5 » $.\, h \varepsilon \mathrm{Q} \;.\supset.\; \exists (n,x) \ni \{ n \varepsilon \mathrm{N}_1 \,.\, x \varepsilon (a \frown b\, \mathrm{f} 0 \cdots n) \mathrm{cres} \,.$
$x_0 = a \,.\, x_n = b : r \varepsilon 0 \cdots (n-1) \;.\supset_r.\; \mathrm{mod}(\mathrm{D}fx_{r+1} - \mathrm{D}fx_r) < h \}$

·6 $h \varepsilon \mathrm{Q} \;.\supset.\; \exists (n,x) \ni \{ n \varepsilon \mathrm{N}_1 \,.\, x \varepsilon (a \frown b\, \mathrm{f} 0 \cdots n) \mathrm{cres} \,.$
$x_0 = a \,.\, x_n = b : r \varepsilon 0 \cdots (n-1) \;.\supset_r.\; \mathrm{mod}[\mathrm{D}fx_r - \mathrm{D}(f; x_r, x_{r+1})] < h \}$

Derivata de uno functione, et quando non es continuo, habe plure proprietate de functiones continuo; p. ex.:

·1 Si derivata Df existe pro omni valore in intervallo differente de x, et si quando variabile verge ad x, derivata verge ad limite, tunc existe derivata de f pro valore x, æquale ad ce limite.

·2 Si derivata Df existe in toto intervallo, tunc Dfx es uno ex limes de derivata Df, ubi variabile verge ad x.

·3 Si pro valore a derivata habe valore negativo, et pro b valore positivo, derivata sume valore nullo inter a et b (Darboux a.1873).

·4 Omni valore inter duo valore de derivata es valore de derivata.

·5 Dato quantitate positivo h, nos pote divide intervallo de a ad b in partes cum valores $x_0 = a$, x_1, x_2..., $x_n = b$ in modo que differentia de duo valore consecutivo de derivata es in valore absoluto minore de h.

·6 Et in modo que differentia inter derivata et ratione incrementale pro duo valores consecutivo fi minore de h.

Vide meo scripto Ann.N. a.1884 p. 45, 153, 252;
Goursat AM. a.1884, p. 49, 316.

✻ 30. DERIVATA DE ORDINE SUPERIORE.

$u \varepsilon \mathrm{Cls}\text{‘q} \,.\, f \varepsilon \mathrm{qF}u \,.\, m \varepsilon \mathrm{N}_1 \,.\, x \varepsilon \delta^m u \;.\supset:\; \mathrm{D}^m fx = (\mathrm{D}^m f)x$ Df

$\mathrm{D}^m fx$ « derivata de ordine m de functione f pro valore x » debe es decomposito ut es scripto [et non in $\mathrm{D}^m(fx)$]. Nos suppone que functione f es definito in campo u, et que x pertine ad classe derivata de ordine m de u.

Exercitio.

$f, D^m f, g, D^m g \; \varepsilon \; qFu \; .\supset.$

$D^m [(fx+gx)|x,u]x = D^m fx + D^m gx$

$D^m (a \times fx|x,u)x = a \times D^m fx$

$D^m (fx \times gx|x, u)x = \Sigma[\, C(m,r)(D^{m-r} fx) \times (D^r gx) \,|r, 0 \cdots m \,]$

{ Leibniz *MathS.* t.5 p.380 }

$n \varepsilon\, m+N_0 \,.\, x \varepsilon q \;.\supset.\; D^m(x^n|x, q)x = \Pi[n-0\cdots(m-1)] \times x^{n-m}$

$a \varepsilon Q \,.\, x \varepsilon q \,.\, n \varepsilon N_1 \;.\supset.\; D^n(a^x \,|x, q)x = a^x(\log a)^n$

$x \varepsilon Q \,.\, n \varepsilon N_1 \;.\supset.\; D^n(\log, Q)x = (-1)^{n-1}(n-1)!\, x^{-n}$

» $\quad D^{n+1}(x^n \log x \,|x, Q)x = n!\,/x$

$x \varepsilon Q \,.\, n \varepsilon N_1 \;.\supset.$

$D^n(\log x\,/x \,|x, Q)x = (-1)^n n!\, x^{-n-1}[\log x - \Sigma/(1\cdots n)]$

$n \varepsilon N_1 \,.\, x \varepsilon q \;.\supset.\; D^n(e^{ix} \,|x, q)x = i^n e^{ix} \quad .$

$D^n sx = s(x+n\pi/2) \quad . \quad D^n cx = c(x+n\pi/2)$

$a,b \varepsilon q \;.\supset.\; D^n \,[s(ax+b) \,|x, q]x = a^n \, s(ax+b+n\pi/2)$

$y \varepsilon q \,.\, n \varepsilon N_1 \;.\supset.\; D^n t^{-1} y = (n-1)!\, [c(t^{-1}y)]^n s[n(\pi/2 + t^{-1}y)]$

✻ 31. Interpolatione de primo gradu.

·1 $a,b \varepsilon q \,.\, a -\!= b \,.\, f, D^2 f \,\varepsilon\, qF a\overline{\ }b \,.\, x \varepsilon\, a^{-}b \;.\supset.$

$fx - fa - (x-a)D(f;a,b) \;\varepsilon\; (x-a)(x-b)(D^2 f `a^{-}b)/2$

Dem. $\quad h = \{fz - fa - (z-a)D(f;a,b) - (z-a)(z-b)[fx - fa - (x-a)D(f;a,b)]/[(x-a)(x-b)] \,|z, a\overline{\ }b\} \;.\supset.\; ha = hb = hx = 0 \;.\supset.$
$0 \varepsilon\, Dh`a\overline{\ }x \,.\, 0 \varepsilon\, Dh`x\overline{\ }b \;.\supset.\; 0 \varepsilon\, D^2 h`a\overline{\ }b \;.\supset.\;$ P

Si a et b es quantitates differente, et f es functione definito in intervallo de a ad b, tunc functione « interpolante » de primo gradu in x:

$$fa + (x-a)D(f;\, a,b)$$

coincide cum fx pro $x = a$ et $x = b$.

In praxi, valore de isto functione es considerato ut valore approximato de fx. Errore in ce approximatione, id es, differentia inter valore vero fx et valore de functione interpolare, es expresso per formula scripto, ubi figura derivata de ordine 2.

Interpolatione in tabula de logarithmos

Commune tabula de logarithmos decimale contine logarithmos de numeros integro inter 10^n et 10^{n+1}. Si x es numero integro inter ce limites, $10^n \leqq x < 10^{n+1}$, tabula da Log$x$ et Log$(x+1)$.

Si t es fractione proprio, regula de interpolatione dice que per approximatione es

$$\text{Log}(x+t) = \text{Log}x + t[\text{Log}(x+1) - \text{Log}x]$$

Errore, vel differentia d de duo membro resulta de theorema præcedente, ubi nos pone

$$fx = \text{Log}x, \qquad \text{D}fx = \text{M}/x, \qquad \text{D}^2fx = -1/x^2.$$

M, modulo de logarithmos, $=$ Log e $= 0{\cdot}434\ldots$ (pag. 242). Tunc

$$d \,\varepsilon\, t(1-t)\text{M}/[2(x+\theta)^2]$$

Ergo $d > 0$; $t(1-t) < 1/4$, $x > 10^n$, ergo $d < 0{\cdot}054\ldots \times 10^{-2n} < 10^{-(2n+1)}$.

Errore in interpolatione in tabula de logarithmos es minore de uno unitate decimale de ordine $2n+1$.

Per exemplo, in tabula de pag. 119, que da logarithmos de numeros de 10 ad 100, errore in interpolatione es minore de uno unitate de ordine 3, vel da tres cifra decimale pro logarithmo di omni numero.

In interpolatione de tabula de Log sin x, si h es differentia de duo valore consecutivo de x, scripto in tabula, errore vale $\text{M}h^2/(8 \sin x^2)$. Si $x > 12^0 22'$, et $h = 1'$, errore resulta $< 10 \wedge (-7)$.

·2 $\text{Hp}{\cdot}1 \,.\, m,n \varepsilon \text{Q} \,.\supset.\, f[(ma+nb)/(m+n)] - (mfa+nfb)/(m+n)$
$\varepsilon\, -(b-a)^2 mn(m+n)^{-2}/2\; \text{D}^2 f`(a \frown b)$

Dem. $[(ma+nb)/(m+n) \mid x]\, \text{P}{\cdot}1 \,.\supset.\, \text{P}$

$(ma+nb)/(m+n)$ es valore « medio arithmetico inter a et b, cum pondo m et n ». Formula exprime differentia inter functione de valore medio, et valore medio de functione. Resulta de Prop. præcedente.

Exemplo. Si $fx = x^2$, seque $\text{D}^2fx = 2 > 0$; ergo:

$$[(ma+nb)/(m+n)]^2 < (ma^2+nb^2)/(m+n).$$

Si $fx = \log x$, seque « medio arithmetico supera medio geometrico », ut es scripto in pag. 110 § Q P26.

❋ 32. SERIE ASYMPTOTICO DE POTESTATES.

$u \varepsilon \text{Intv} \,.\, x \varepsilon u \,.\, n \varepsilon \text{N}_1 \,.\, f \,\varepsilon\, \text{q}\,\text{F}u \,.\, \text{D}^n fx \,\varepsilon \text{q} \,.\supset.\, \lim[\{fy-$
$\Sigma[(y-x)^r/r!(\text{D}^r fx) \mid r, 0\cdots(n-1)]\}/(y-x)^n \mid y, u, x] = (\text{D}^n fx)/n!$

[$\text{P}27{\cdot}1 \,.\supset.\, \lim\{fy - \Sigma[(y-x)^r / r!\; \text{D}^r fx \mid r,\, 0\cdots(n-1)]\, /(y-x)^n \mid y, u, x\}$
$= \lim\{\text{D}fy - \Sigma[(y-x)^{r-1}/(r-1)!\; \text{D}^r fx \mid r,\, 1\cdots(n-1)]\}/[n(y-x)^{n-1}] \mid y, u, x\}$
$= \lim\{(\text{D}^{n-1}fy - \text{D}^{n-1}fx)/[n!(y-x)]\} \mid y, u, x\}$
$\text{Def D} \,.\supset.$ » $= (\text{D}^n fx)/n!$]

Si functione f dato in intervallo u, habe derivata de ordine n (et præcedentes) pro valore x, tunc limite de differentia inter fy et polynomio

$$fx + (y-x)\mathrm{D}fx + (y-x)^2/2!\,\mathrm{D}^2fx + \ldots + (y-x)^{n-1}/(n-1)!\,\mathrm{D}^{n-1}fx,$$

diviso per $(y-x)^n$, quando varia y, in intervallo considerato, et tende ad x, vale derivata de ordine n diviso per $n!$.

Functione de y, que diviso $(y-x)^n$, verge ad limite finito, si y verge ad x, vocare « infinitesimo de ordine n ».

Ergo, theorema dice que fy contine fx, quantitate finito, plus $(y-x)\mathrm{D}fx$, infinitesimo de primo ordine, plus $(y-x)^2/2\,\mathrm{D}^2fx$, infinitesimo de secundo ordine, plus etc.

In vero, pro $n=1$, theorema dice que $\lim(fy-fx)/(y-x)$ vale derivata de fx, quod es vero per definitione de derivata.

Pro $n=2$, nos quære

$$\lim\{[fy-fx-(y-x)\mathrm{D}fx]/(y-x)^2 \mid y, u, x\},$$

ubi quantitate de que nos quære limite, se præsenta sub forma 0/0. Ergo, pro theorema de l'Hospital, nos calcula limite de ratione de duo derivata:

$$\lim(\mathrm{D}fy-\mathrm{D}fx)/[2(y-x)] \mid y, \ldots$$

que, per definitione de derivata, vel pro casu $n=1$, vale $\mathrm{D}^2fx/2$, conforme ad theorema.

Ita nos demonstra theorema pro $n=3$, ...

Isto theorema es uno ex formas de « formula de Taylor ». Me habe dato ce propositione, sub conditiones scripto, in notas ad Genocchi, *Calcolo differenziale* a. 1884 p. XIX; versione in Germanico pag. 321. Vide Mathesis a. 1889 p. 110, Torino A. a. 1891; Cesàro, *Calcolo infinitesimale* a. 1905 p. 94.

Expressione « série asymptotique » es de H. Poincaré Acta M. t. 8 a. 1886 p. 295, que da alio theorema interessante series de isto natura.

Plure Auctore deduce P32 de P34. Tunc occurre existentia de derivata de ordine n prope x, et suo continuitate, quod non es necessario.

Theorema præcedente occurre in P33, rectaT, planOscul ...

✻ 33. Maximo et minimo de functione.

·1 $a,b\,\varepsilon\,\mathrm{q} \,.\, a<b \,.\, f\,\varepsilon\,\mathrm{qF}a\text{-}b \,.\, x\,\varepsilon\,a\text{-}b \,.\, \mathrm{D}fx=0 \,.\, \mathrm{D}^2fx>0 \,.\supset.$
$\exists\,(c,d)\ni[c\,\varepsilon\,a\text{-}x \,.\, d\,\varepsilon\,x\text{-}b \,.\, fx = \min f`(c\text{-}d)]$

Dem. P32 $.\supset.\ \lim[(fy-fx)/(y-x)^2 \mid y, a\text{-}b, x] = \mathrm{D}^2fx/2 > 0 \,.\supset.$
$\exists\,(c,d)\ni[c\,\varepsilon\,a\text{-}x \,.\, d\,\varepsilon\,x\text{-}b : y\,\varepsilon\,c\text{-}d \,.\supset_y.\, fy > fx \,.\supset.$ Ths

Si $a\text{-}b$ es intervallo, ubi es definito functione f, et pro aliquo

valore x interno ad intervallo, derivata de ordine 1 es nullo, et derivata de ordine 2 es positivo, tunc existe intervallo $c\frown d$, parte de $a\frown b$, continente in suo interno valore x, tale que fx es minimo ex valores de f in intervallo $c\frown d$.

·2 $(<, \max) \mid (>, \min)$ P·1

Et si derivata secundo es negativo, functione es maximo.

·3 $a,b \varepsilon q \,.\, a<b \,.\, f \varepsilon qFa\frown b \,.\, x \varepsilon a\frown b \,.\, n \varepsilon 2N_1 : r \varepsilon 1\cdots(n-1) \,.\supset_r.$ $D^r fx = 0 : D^n fx > 0 :\supset. \exists(c,d) \ni [c \varepsilon a\frown x \,.\, d \varepsilon x\frown b \,.\, fx = \min f`c\frown d]$

·4 $(<, \max) \mid (>, \min)$ P·3

·5 $a,b \varepsilon q \,.\, a<b \,.\, f \varepsilon qFa\frown b \,.\, x \varepsilon a\frown b \,.\, n \varepsilon 2N_1+1 : r \varepsilon 1\cdots(n-1) \,.\supset_r.$ $D^r fx = 0 : D^n fx > 0 :\supset.$
$\exists(c,d) \ni [c \varepsilon a\frown x \,.\, d \varepsilon x\frown b : y \varepsilon c\frown d \,.\supset_y .\, D(f;x,y) > 0]$

·6 $(< \mid >)$ P·5

✻ 34. Theorema de Lagrange.

$a,b \varepsilon q \,.\, a -= b \,.\, n \varepsilon N_1 \,.\, f, D^n f \varepsilon qF a\frown b \,.\supset.$
$fb - \Sigma[(b-a)^r / r!\, D^r fa | r, 0\cdots(n-1)] \;\varepsilon\; (b-a)^n / n!\, D^n f`\, a\frown b$

Dato duo quantitate a et b, differentes, et functione f reale definito in intervallo de a ad b, cum derivatas usque ad ordine n, tunc differentia inter fb et summa de primos n termine de suo evolutione secundo potestates de $b-a$ vale $(b-a)^n/n!$ per uno ex valores de derivata de ordine n.

Dem. 1

Si $n=1$, Prop. coincide cum theorema de valore medio:

$n=1 \,.\, P22 \,.\supset.\, fb-fa \;\varepsilon\; (b-a)Df`a\frown b$ (1)

Applica isto theorema ad derivata:

$x \varepsilon a\frown b \,.\supset.\, Dfx \;\varepsilon\; Dfa + (x-a)D^2f`a\frown x$

Integra cum regula P26·2:

$fx-fa \;\varepsilon\; (x-a)Dfa + (x-a)^2/2\, D^2f`a\frown x$

que es theorema pro $n=2$. Et ita pro. In generale, si theorema es vero pro aliquo valore n, nos demonstra illo pro $n+1$:

$n \varepsilon N_1 \,.\, x \varepsilon a\frown b \,.\, Dfx \;\varepsilon\; \Sigma[(x-a)^r / r!\, D^r fa | r, 0\cdots(n-1)] +$ $(x-a)^n / n!\, D^n f`a\frown b \,.\, P26 \,.\supset.\, fb-fa \;\varepsilon\; \Sigma[(b-a)^{r+1}/(r+1)!D^{r+1}fa | r, 0\cdots$ $(n-1)] + (b-a)^{n+1}/(n+1)!\, D^{n+1}f`a\frown b$ (2)

Unde, per inductione, seque propositione:

(1) . (2) . Induct $.\supset.$ P

Dem. 2

$k = \{ fb - \Sigma[(b-a)^r / r!\, D^r fa \,|\, r, 0 \cdots (n-1)] \} / (b-a)^n$.
$g = (\{ fx - \Sigma[(x-a)^r / r!\, D^r fa \,|\, r, 0 \cdots (n-1)] - k(x-a)^n \} \,|\, x, a \frown b)$.⊃.
$ga = Dga = D^2 ga = \ldots = D^{n-1} ga = 0 \,.\, gb = 0$.⊃.
$\exists\, (a \frown b) \cap u \ni (D^n gu = 0)$.⊃. $\exists\, (a \frown b) \cap u \ni (D^n fu - n!k) = 0$.⊃.
$k \varepsilon\, (D^n f`a \frown b)/n!$

In facto, si nos voca k quantitate in primo membro diviso per $(b-a)^n$, et si nos pone

$$gx = fx - fa - (x-a)Dfa - \ldots - (x-a)^{n-1}/(n-1)!\, D^{n-1}fa - k(x-a)^n ,$$

id es, si nos voca g functione in secundo membro, ubi x varia in intervallo dato, tunc functione g es nullo, simul cum suo derivatas de ordine 1, 2,... $n-1$, pro $x = a$, et es nullo pro $x = b$. Ergo per theorema de Rolle, suo derivata de ordine n, $D^n fx - n!k$, es nullo pro aliquo valore inter a et b. Unde seque theorema.

Si in Dem. 1 ad P26·2 nos substitue suo Dem., resulta Dem. 2 de P34.

Dem. 3

$h = \{ fb - \Sigma[(b-x)^r / r!\, D^r fx \,|\, r, 0 \cdots (n-1)] \,|\, x, a \frown b \}$. $x \varepsilon\, a \frown b$.⊃.
$Dhx = -(b-x)^{n-1} (D^n fx)/(n-1)!$. P26·1 .⊃. P

Dem. 3 es Dem. dato per Bernoulli, completato.

{ LAGRANGE a.1797, *Th. des Fonctions analytiques* p.49 :
« D'où résulte enfin ce théorème nouveau et remarquable par sa simplicité et généralité, qu'en désignant par u une quantité inconnue, mais renfermée entre les limites 0 et x, on peut développer successivement toute fonction de x et d'autres quantités quelconques suivant les puissances de x, de cette manière :

$$\begin{aligned} \mathrm{f}x &= \mathrm{f}. + x\mathrm{f}'u, \\ &= \mathrm{f}. + x\mathrm{f}'. + \frac{x^2}{2}\mathrm{f}''u, \\ &= \mathrm{f}. + x\mathrm{f}'. + \frac{x^2}{2}\mathrm{f}''. + \frac{x^3}{2.3}\mathrm{f}'''u, \end{aligned}$$

les quantités f., f'., f''., etc. étant les valeurs de la fonction fx et de ses dérivées f'x, f''x, etc., lorsqu'on y fait $x = 0$ ». }

Exemplo.

Si nos pone $fx = e^x$, $a = 0$, $b = x$, resulta :

$$x \varepsilon q \;.\supset.\; e^x \;\varepsilon\; 1 + x + x^2/2 + \ldots + x^{n-1}/(n-1)! + x^n/n!\; e^{(\theta x)}$$

et, pro $n = \infty$, serie de pag. 243 P4·1.

Pro $fx = \sin x$, vel $fx = \cos x$, seque serie de pag. 252 P16·1·2.

Application de theorema præcedente ad functiones $\log(1+x)$ pag. 246, $(1+x)^n$ p. 226, $\tan^{-1} x$ p. 263, es complicato.

✻ 35. SERIE DE POTESTATES.

$u \varepsilon\, \mathrm{q'fN_0} \,.\, r = /\, \max \mathrm{Lm}(\sqrt[n]{\mathrm{mod}u_n}) \,|n \,.\, r>0 \,.$
$f = [\,\Sigma(u_n x^n \,|n, \mathrm{N_0}) \,|x, \mathrm{q'}\frown x\ni(\mathrm{mod}x<r)] \,.\, x\varepsilon \mathrm{q'} \,.\, \mathrm{mod}\, x < r \,.\supset:$

·1 $fx\, \varepsilon \mathrm{q}$

·2 $\mathrm{D}fx = \Sigma[nu_n x^{n-1} \,|n, \mathrm{N_1}]$

·3 $n\varepsilon \mathrm{N_0} \,.\supset.\, u_n = (\mathrm{D}^n f0)/n!$

·4 $fx = \Sigma(x^n/n! \times \mathrm{D}^n f0 \,|n, \mathrm{N_0})$

·5 $h\varepsilon \mathrm{q'} \,.\, \mathrm{mod}x + \mathrm{mod}h < r \,.\supset.$
$f(x+h) = \Sigma[h^n/n!\, \mathrm{D}^n fx \,|n, \mathrm{N_0}]$

u indica successione de quantitates, reale aut imaginario, nos voca r reciproco de maximo limes de radice de indice n de modulo de u_n, dum varia n; et suppone illo non nullo.

Nos considera functione f que pote es repræsentato per serie de potestates de variabile:

$$fx = u_0 + u_1 x + u_2 x^2 + \ldots$$

ubi varia x, in campo de quantitates imaginario minore in valore absoluto de r; nos sume quantitate x in ce campo. Tunc

·1 Serie fx es convergente.

·2 Suo derivata vale: $\mathrm{D}fx = u_1 + 2u_2 x + \ldots$

·3 Coefficiente de potestate n de x vale derivata de ordine n, pro 0, diviso per $n!$. Ergo:

·4 $fx = f0 + x\mathrm{D}f0 + x^2/2\,\mathrm{D}^2 f0 + x^3/3!\,\mathrm{D}^3 f0 + \ldots$

·5 Pro valores de h de modulo $< r - \mathrm{mod}x$, $f(x+h)$ es evolubile secundo potestates de h.

Dem.

Ex definitione de r, seque:

$$\max \mathrm{Lm} \sqrt[n]{(\mathrm{mod}u_n\, r^n)} \,|\, n = 1 \qquad (1)$$

Ergo maximo limes de radice de indice n ex modulo de termine de loco n in serie dato es minore de 1:

$$(1) \,.\supset.\, \max \mathrm{Lm} \sqrt[n]{\mathrm{mod}(u_n x^n)} |n = (\mathrm{mod}x)/r < 1 \qquad (2)$$

Unde, per theorema de Cauchy, serie de modulos de termines de serie dato es convergente:

$$(2) \,.\, \S\mathrm{lim}\ \mathrm{P}24{\cdot}5\ (\mathrm{pag.}\ 224) \,.\supset.\, \Sigma[\mathrm{mod}(u_n x^n) \,|n, \mathrm{N_0}]\ \varepsilon \mathrm{Q} \qquad (3)$$

Et serie dato es convergente, ut Prop. ·1 affirma:

$$(3) \,.\, \S\mathrm{lim}\ \mathrm{P}52{\cdot}1\ (\mathrm{pag.}\ 235) \,.\supset.\, \mathrm{P}{\cdot}1$$

Si $\mathrm{mod}x > r$, maximo limes de termine generale $= \infty$.

Pro $x=0$: $f0 = u_0$. De $fx = u_0+x(u_1+u_2x+...)$, seque :

$\lim(f, \text{Variab}f, 0) = u_0$ (4)

De $(fx-f0)/x = u_1+u_2x+...$, ad limite, per Prop. (4), resulta :

$Df0 = u_1$ (5)

Nunc nos evolve omni termine de serie $f(x+h)$ cum formula de binomio:

$f(x+h) = \Sigma\{\Sigma[C(n,p)u_n x^{n-p}h^p \mid p, N_0] \mid n, N_0\}$ (6)

que es serie de serie, vel serie duplo. Nos commuta summa pro p et summa pro n. Hoc lice, si serie duplo de modulos de termines de serie duplo dato es convergente. Vide P36 infra. Serie duplo de modulos converge :

(3) .⊃. $\Sigma\{\Sigma[C(n,p)(\text{mod}u_n)(\text{mod}x)^{n-p}(\text{mod}h)^p \mid p, N_0] \mid n, N_0\} =$
$\Sigma[(\text{mod}u_n)(\text{mod}x+\text{mod}h)^n \mid n, N_0] \;\varepsilon\; Q$ (7)

(6) . (7) . P·6 .⊃. $f(x+h) = \Sigma\{h^p \Sigma[C(n,p)u_n x^{n-p} \mid n, N_0] \mid p, N_0\}$ (8)

Per (5), coefficiente de h in isto evolutione es derivata $Df(x+h)$, pro $h=0$, id es, Dfx secundo P·2 :

(8) . (5) .⊃. P·2

Nos deriva p vice :

P·2 . $p\varepsilon N_1$.⊃. $D^p fx = p!\, \Sigma[C(n,p)\, u_n x^{n-p} \mid n, N_0]$ (9)

(8) . (9) .⊃. P·5

Serie de Taylor et de MacLaurin.

Plure libro voca serie de P·4·5 « serie de MacLaurin et de Taylor ».

Theorema præcedente ***præsuppone*** que functione es evolubile in serie de potestates.

Serie de MacLaurin, pro functiones e↑x, sin$x$, cos$x$, vale pro omni x. Vide pag. 243 P4·1, pag. 252 P16·1·2.

Functiones $(1+x)$↑m, log$(1+x)$, tang^{-1}x es repræsentato per serie de potestates, si mod$x$ <1. Vide pag. 222, 246, 263.

Serie de Stirling, et serie de potestates per logarithmo integrale es divergente pro omni x. Alio exemplo in D'Arcais a.1899 p.326.

Functione e↑$-/x^2$ habe omni derivata nullo pro $x=0$; serie de MacLaurin es convergente, et habe summa differente de valore de functione, ut nota Cauchy a.1823 s.2 t.4 p.230.

Reproductione sequente ex operas de Auctores expone historia de serie.

Joh. Bernoulli a.1694 t.1 p.126:

« habetur hæc series generalissima :

$$Integr.\, ndz = +nz - \frac{zzdn}{1.2.dz} + \frac{z^3ddn}{1.2.3.dz^2} - \frac{z^4dddn}{1.2.3.4.dz^3} \&c. \text{ »}$$

Si in citatione præcedente nos pone $n = Dfz$, et si nos effectua integratione indicato intra limites b et x, formula fi:

$fx-fb = (x-b)Dfx - (x-b)^2/2\, D^2fx + (x-b)^3/3!\, D^3fx - ...$ (1)

unde serie ·5, si nos pone $b = x+h$.

Demonstratione de Bernoulli es : « differentia de duo membro de (1) habe derivata semper nullo ». In realitate, differentia inter primo membro et summa de primos n termine de secundo habe derivata scripto in P 34, Dem. 3 de theorema de Lagrange.

TAYLOR a.1715 p.21 :

« Sint z et x quantitates duae variabiles, quarum z uniformiter augetur per data incrementa $\dot{z}$, et sit $n\dot{z} = v$ . . . . .

p.23 : . . . quo tempore z uniformiter fluendo fit $z+v$, fiet x,

$$x + \dot{x}\frac{v}{1\dot{z}} + \ddot{x}\frac{v^2}{1.2\dot{z}^2} + \dddot{x}\frac{v^3}{1.2.3\dot{z}^3} + \&c. \text{ »}$$

Taylor deduce isto theorema ex propositione de Mercator pro differentias. Vide supra pag. 131 P4·1, et infra P38·1. Ipse in pag. 38 expone processu identico ad illo de Bernoulli. Unde Bernoulli reclama prioritate, et dice : « Quam eandem seriem postea Taylorus, interjecto plus quam viginti annorum intervallo, in librum, quem edidit A. 1715, *de Methodo incrementorum* transferre dignatus est, sub alio tantum characterum habitu. Vide ejus lib. p. 38 ».

Prof. A. Pringsheim, *Zur Geschichte des Taylorschen Lehrsatzes*, BM. a.1900 s.3 t.1 p.433, expone plure interessante notitia super historia de isto formula, sed frustra, me puta, tenta rehabilitatione de Taylor.

MACLAURIN a.1742 p.610 :

« Suppose that y is any quantity that can be expressed by a series of this form $A + Bz + Cz^2 + Dz^3 + \&c.$ where A, B, C represent invariable coefficients . . . When z wanishes, let E be the value of y, and let $\dot{E}$, $\ddot{E}$, $\dddot{E}$, &c. be then the respective values of $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. z being supposed to flow uniformly. Then

$$y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times 2\times 3\dot{z}^3} + \&c. \text{ »}$$

Isto theorema æquivale ad theorema de Taylor, ut Auctore declara :

p.611 : « This theorem was given by Dr. Taylor. »

p.612 : « which theorem is not materially different from Mr. Bernouilli's. »

Notatione simile ad hodierno es in :

ARBOGAST a.1800 :

$$\text{« } F(a+x) = Fa + \frac{DFa}{1}x + \frac{D^2Fa}{1.2}x^2 + \frac{D^3Fa}{1.2.3}x^3 + \text{etc. »}$$

In conclusione, Bernoulli a. 1694, et Taylor a. 1715 habe scripto serie, sine hypothesi præciso.

Lagrange habe calculato resto sub forma de Prop. 34. Ce theorema in plure libro de Calculo infinitesimale, es vocato « Theorema de Taylor ».

Pro alio expressione de resto, vide Calculo integrale.

Prop. 35 que præcede, es de Cauchy, *Résumés analytiques*, Turin a. 1833 pag. 47 et 113, Œuvres s.1 t.5 p.360.

Plus recente es interpretatione ut serie asymptotico, Prop. 32, que duce ad aliquo resultatu interessante.

✻ 36. Serie duplo.

$$u \varepsilon \mathrm{q'f}(\mathrm{N}_0 \mathbin{;} \mathrm{N}_0) \,.\, \Sigma\{\Sigma[\mathrm{mod}\, u(r,s) | r, \mathrm{N}_0] | s, \mathrm{N}_0\} \;\varepsilon\; \mathrm{Q} \;.\supset.$$
$$\Sigma\{\Sigma[u(r,s) | r, \mathrm{N}_0] | s, \mathrm{N}_0\} = \Sigma\{\Sigma[u(r,s) | s, \mathrm{N}_0] | r, \mathrm{N}_0\}$$

Dem.

§lim P21·0 (p.220) $.\supset.$ » $= \lim \Sigma\{\Sigma[u(r,s)|r,\mathrm{N}_0]|s, 0\cdots n\}|n$

§lim P21·5 (p.222) $.\supset.$ » $= \lim \Sigma\{\Sigma[u(r,s)|s,0\cdots n]|r,\mathrm{N}_0\}|n$

§lim P43·1 (p.233) $.\supset.$ » $= \Sigma(\{\lim \Sigma[u(r,s)|s,0\cdots n]|n\}|r,\mathrm{N}_0)$

§lim P21·0 (p.220) $.\supset.$ » $= \Sigma\{\Sigma[u(r,s)|s,\mathrm{N}_0]|r,\mathrm{N}_0\}$

Si u es quantitate, reale aut imaginario, functione de duo variabile, et si serie de serie de modulos de $u(r,s)$ ubi varia in primo loco r, et in secundo loco s, es convergente, tunc serie de serie de u, ubi varia in primo loco r et in secundo loco s, æqua idem serie, si varia in primo loco s, et in secundo r. Occurre in dem. de Prop. 35.

✻ 37. Rationes incrementale successivo.

$u \varepsilon \mathrm{Cls'q} \,.\, f \varepsilon \mathrm{qF}u \,.\, n \varepsilon \mathrm{N}_1 \;.\supset:$

·1 $x_0, x_1, x_2 \,\varepsilon\, u \;.\supset.\; \mathrm{D}^2(f; x_0, x_1, x_2) = \mathrm{D}[\mathrm{D}(f; x_0, y) | y; x_1, x_2]$ Df

·2 $x \varepsilon u \,\mathrm{f}\, 0\cdots(n+1) \;.\supset.$
$\mathrm{D}^{n+1}\{f; [x, 0\cdots(n+1)]\} = \mathrm{D}^n\{\mathrm{D}(f; x_0, y) | y; [x, 1\cdots(n+1)]\}$ Df

Ratione incrementale $\mathrm{D}(f; x_0, y)$ es definito in Prop. 1·1·2, pagina 275. Suo ratione incrementale, dum varia y, et sume valores x_1, x_2, es « ratione incrementale de ordine duo », et nos indica illo per $\mathrm{D}^2(f; x_0, x_1, x_2)$. Ita pro rationes incrementale de ordine successivo.

Valores $x_0, x_1, x_2, \ldots$ pote es distincto aut plure coincidente.

·3 $x \varepsilon (u \,\mathrm{f}\, 0\cdots n)\mathrm{sim} \;.\supset.\; \mathrm{D}^n[f; (x, 0\cdots n)] =$
$\Sigma\{(f x_r) / \Pi[(x_r - x_s) | s, (0\cdots n) - \iota r] \,|\, r, 0\cdots n\}$

·4 $x \varepsilon (u \,\mathrm{f}\, 0\cdots n)\mathrm{sim} \;.\supset.\; \mathrm{D}^n[f; (x, 0\cdots n)] =$
$\mathrm{Dtrm}\{[(x_r^s | s, 0\cdots(n-1)), f x_r] \,|\, r, 0\cdots n\} \,/\, \mathrm{Dtrm}[(x_r^s | s, 0\cdots n) | r, 0\cdots n]$

Si $x_0\, x_1 \ldots$ es quantitates differente in campo u, tunc ratione incrementale de ordine n de functione f pro valores x vale determinante, que habe pro linea de loco r

$$1,\; x_r,\; x_r^2 \ldots\; x_r^{n-1},\; f x_r$$

ubi r sume valores $0\cdots n$, diviso per determinante de potestates successivo des x.

·5 $x \varepsilon\, u\, \mathrm{f}\, 0\cdots n \,.\, y \varepsilon\, (0\cdots n\, \mathrm{f}\, 0\cdots n)\mathrm{rcp} \,.\supset.$
$\mathrm{D}^n[f; (xy,0\cdots n)] = \mathrm{D}^n[f; (x,0\cdots n)]$

Si x es successione de elementos de classe u, cum indices $0,1,\ldots n$, et y es correspondentia reciproco inter ce indices, vel es permutatione de indices, tunc ratione incrementale de ordine n de f, respondente ad indices permutato, æqua ratione respondente ad indices primitivo.

Functione de plure variabile, que non varia, si nos permuta variabiles, vocare « functione symmetrico ». Ergo ratione incrementale es functione symmetrico de variabiles $x_0, x_1, \ldots x_n$.

·6 $x \varepsilon\, (\mathrm{q}-\iota 0)\, \mathrm{f}\, 0\cdots n \,.\supset.\, \mathrm{D}^n[/; (x,0\cdots n)] = (-1)^n / \Pi(x,0\cdots n)$

·7 $f \varepsilon\, \mathrm{qFN}_0 \,.\, x \varepsilon \mathrm{N}_0 \,.\supset.\, \Delta^n fx = n!\, \mathrm{D}^n(f; x+0\cdots n)$

Relatione inter ratione incrementale, et differentia finito Δ, introducto in pag. 130.

✻ 38. $u \varepsilon\, \mathrm{Cls'q} \,.\, f \varepsilon\, \mathrm{qF}u \,.\, n \varepsilon \mathrm{N}_1 \,.\, x \varepsilon\, u\, \mathrm{f}\, 0\cdots n \,.\supset:$

·0 $\mathrm{Interp}[f; (x,0\cdots n)] =$
$fx_0 + \Sigma\{\Pi[(y-x_r)|r, 0\cdots(s-1)]\mathrm{D}^s[f; (x,0\cdots s)\, |r, 1\cdots n\}\, |y, \mathrm{q}]$ Df

$\mathrm{Interp}[f; (x,0\cdots n)]$, lege « functione interpolante de f, respondente ad $n+1$ valores $x_0, x_1, \ldots x_n$ », indica polynomio

$$fx_0 + (y-x_0)\mathrm{D}(f; x_0, x_1) + (y-x_0)(y-x_1)\mathrm{D}^2(f; x_0, x_1, x_2) + \ldots$$
$$+ (y-x_0)\ldots(y-x_{n-1})\mathrm{D}^n(f; x_0, x_1, \ldots x_n)$$

ubi varia y, in campo de numeros reale.

interpolatione: A.D.F. interpolation, H. interpolacion, I. interpolazione, R. interpolátsiá. ⊂ interpola + -tione.
interpola: A. interpola-te, D. interpol-ieren, F. interpol-er, H. interpola-r, I. interpola-re, R. interpol-irovatı.
⊂ L. inter (vide pag. 159 N. 268) + -pola (elemento de origine non concorde apud linguistas).
« Functione interpolare » de Ampère, GergonneA. a. 1826 t. 16 p. 329 vale « ratione incrementale » $\mathrm{D}^n(f; x_0 x_1 \ldots)$ id es coefficiente in functione interpolante.

·1 $\mathrm{D}^n[f; (x,0\cdots n)]\, \varepsilon \mathrm{q} \,.\, y \varepsilon\, u - x'0\cdots n \,.\supset.$
$fy = \mathrm{Interp}[f; (x,0\cdots n)]y + \Pi[(y-x_r)|r, 0\cdots n]\mathrm{D}^{n+1}[f; (x,0\cdots n), y]$

} Newton a. 1686, t. 3 prop. XL lemma 5 {

Si ratione incrementale de ordine n pro valores x existe, et si y es u differente de omni x, tunc fy vale functione interpolante de f in valores x, pro y, plus termine complementare

$$(y-x_0)(y-x_1)\dots(y-x_n)\mathrm{D}^{n+1}(f;x_0,x_1,x_2,\dots x_n,y)$$

Si valores x es æquidifferente, resulta Prop. 4·1 de pag. 131.

Si valores x es coincidente, resulta evolutione asymptotico de fy secundo potestates de $y-x$, plus generale de P32.

·3 $x\varepsilon(u\,\mathrm{f}\,0\cdots n)\mathrm{sim} \,.\!\supset\!.\, \mathrm{Interp}[f;(x,0\cdots n)] =$
$[\Sigma\}fx_r\Pi[(y-x_s)/(x_r-x_s)\,|s,(0\cdots n)-\iota r]\,|r,0\cdots n\}\,|y,\mathrm{q}]$

Si valores de x es differente, functione interpolante pote sume forma

$$fx_0\times[(y-x_1)\dots(y-x_n)]/[(x_0-x_1)\dots(x_0-x_n)] + \text{alios termine}$$

dato per WARING a. 1776, et LAGRANGE a. 1795 (Oeuvres t. 7, pagina 285).

·4 $y\varepsilon\,\mathrm{q}-x\text{‘}0\cdots n \,.\!\supset\!.\, \mathrm{Interp}[f;(x,0\cdots n)]y =$
$\Pi[(y-x_r)|r,0\cdots n]\;\mathrm{D}^n[fx/(y-x)|x,(x,0\cdots n)]$

✻ 39. $u\varepsilon\,\mathrm{Intv}\,.\,n\varepsilon\mathrm{N}_1\,.\,f\,\varepsilon\,\mathrm{qF}u \,.\!\supset\!:$

·1 $\mathrm{D}^n f\,\varepsilon\,\mathrm{qF}u\,.\,x\varepsilon\,u\,\mathrm{f}\,0\cdots n \,.\!\supset\!.\, \mathrm{D}^n[f;(x,0\cdots n)]\;\varepsilon\;(\mathrm{D}^n f\text{‘}u)/n!$

Dem. $h = f - \mathrm{Interp}[f;(x,0\cdots n)] \,.\!\supset\!.\, hx_0 = hx_1 = hx_2 = \dots = hx_n = 0$.
P21 $.\!\supset\!.\; 0\,\varepsilon\,\mathrm{D}^n h\text{‘}u \,.\!\supset\!.\; 0\,\varepsilon\,\mathrm{D}^n f\text{‘}u - n!\,\mathrm{D}^n[f;(x,0\cdots n)]$

Si campo u de variabilitate es intervallo, et functione f habe derivatas usque ad ordine n in isto intervallo, tunc ratione incrementale de ordine n de f respondente ad valores $x_0\,x_1\dots x_n$ es uno ex valores de derivata de ordine n, diviso per factoriale.

Dem. In vero, differentia h inter functione f et suo interpolante in $x_0,\,x_1\dots\,x_n$, es nullo pro ce $n+1$ valores; ergo, per theorema de Rolle, suo derivata de ordine 1 es nullo pro n valores intermedio; derivata de ordine 2 es nullo pro $n-1$ valores,... et derivata de ordine n es nullo pro uno valore medio inter præcedentes. Ce derivata de ordine n de h vale derivata de f, minus derivata de functione interpolante, que es $n!\times$ coefficiente de termine de gradu maximo.

Si plure valoro x coincide, modificatione es evidente.

} CAUCHY, Œuvres t.5 pag. 409. Demonstratione hic scripto es de H. A. SCHWARZ, TorinoA. a.1882; et de STIELTJES, a.1882 Amsterdam Ak. s.2 t.17 p.239-254 {

·2 $\mathrm{D}^{n+1}f \,\varepsilon\, \mathrm{qF}u \,.\, x \varepsilon u\mathrm{f}0\cdots n \,.\, y\varepsilon u \,.\supset.$
$fy - \mathrm{Interp}[f; (x,0\cdots n)]y \;\varepsilon\; \Pi[(y-x_r)|r, 0\cdots n]\, (\mathrm{D}^{n+1}f`u)\,/(n+1)!$

Dem. P38·1 . P39·1 $.\supset.$ P

Si in intervallo u, functione f habe derivata de ordine $n+1$, tunc differentia inter valore fy de functione dato et valore de functione interpolante de f, in $n+1$ valores $x_0, x_1 \dots x_n$, respondente ad valore y, vale

$$(y-x_0)(y-x_1)\dots(y-x_n)(\mathrm{D}^{n+1}fz)/(n+1)!$$

ubi z es aliquo valore in campo u.

Vale theorema præcedente. Si valores $x_0,x_1 \dots x_n$ es æquale, resulta theorema de Lagrange P 34; pro $n=1$, resulta P 31.

·3 $x\varepsilon u \,.\, \mathrm{D}^{n-1}[\mathrm{D}f; (\iota x, 1\cdots n)] \;\varepsilon \mathrm{q} \,.\, y\varepsilon\, u-\iota x \;.\supset.$
$\mathrm{D}^{n+1}[f; (\iota x, 0\cdots n), y] \;\varepsilon\; \mathrm{D}^{n}[\mathrm{D}f; (\iota x, 1\cdots n), z]/(n+1) \mid z`x^{-}y$

Dem. P38·1 $.\supset.$ $fy = \mathrm{Interp}[\mathrm{D}f; (\iota x {:} 1\cdots n)] + (y-x)^{n}\, \mathrm{D}^{n}[\mathrm{D}f; (\iota x {:} 1\cdots n), y]$
P26·1 $.\supset.$
$fy \;\varepsilon\; \mathrm{Interp}[f; (\iota x {:} 0\cdots n)] + (y-x)^{n+1}/(n+1)\mathrm{D}^{n}[\mathrm{D}f; (\iota x {:} 1\cdots n), z] \mid z`x^{-}y$
P38·1 $\supset$ P

·4 $x\varepsilon u \,.\, \mathrm{D}^{n}[\mathrm{D}f; (\iota x, 0\cdots n)] \;\varepsilon \mathrm{q} \;.\supset.$
$\mathrm{D}^{n+1}[f; (\iota x, 0\cdots(n+1))] = \mathrm{D}^{n}[\mathrm{D}f; (\iota x {:} 0\cdots n)]\,/(n+1)$
Dem. P·3 $\supset$ P·4

·5 $x\varepsilon u \,.\, \mathrm{D}^{n}fx \;\varepsilon \mathrm{q} \;.\supset.\; \mathrm{D}^{n}[f; (\iota x {:} 0\cdots n)] = (\mathrm{D}^{n}fx)/n!$
Dem. P·4 $\supset$ P·5

Si pro valore x in u, derivata de ordine n de f existe, tunc ratione incrementale de ordine n de functione f, respondente ad $n+1$ valores de variabile coincidente in x, vale derivata de ordine n, diviso per factoriale de n. Equivale ad P32 præcedente.

✻ 40. integro (functione integro)

·1 (qFq)integro $=$
$(\mathrm{qFq}) \frown f \ni [\exists(\mathrm{qfN}_0) \frown a \ni \{f = [\Sigma(a_r x^r \,|r, \mathrm{N}_0)\,|x, \mathrm{q}]\}]$ Df

Nos voca functione reale de variabile reale « integro » omni functione expresso, pro omni valore reale de x, per serie de potentias de x.

·2 $f \varepsilon$ (qFq)integro $.\supset.$
gradu $f = \min N_0 \frown n \ni [D^{n+1} f = (\iota 0 \text{`} q)]$ Df

Gradu de functione integro f, es minimo inter valores de n, que redde derivata de ordine $n+1$ de f semper nullo. Ce gradu pote non es finito.

gradu : A. grade degree, D. grad, F. grade degré, H.I. grado, R. gradus. $=$ passu. $\supset$ A.D.F.H.I. gradu-ale gradu-atione.
$\subset$ grad- $=$ i. $\supset$ A.D.F.H.I.R. con-gres-su pro-gres-su pro-gres-sione.
$\subset$ E. gredh-, S. gredhjati, R. gresti.

·3 $n \varepsilon N_1 \,.\, f \varepsilon$ (qFq)integro . gradu $f \leqq n \,.\, a \varepsilon$ (qF$0 \cdots n$)sim .
$x \varepsilon q - a\text{`}0 \cdots n \,.\, c = f a_r / \Pi[(a_r - a_s)|s, 0 \cdots n - \iota r] \,|r \,.\supset.$
$(fx)/\Pi[(x-a_r)\,|r, 0\cdots n] = \Sigma[c_r/(x-a_r)\,|\,r, 0\cdots n]$

Dem. Hyp $.\supset.$ $f =$ Interp[f; $(0 \cdots n)$] . P38·3 $.\supset.$ P

Nos considera functione rationale, quotiente de duo functione integro in x. Denominatore es producto de factores

$$(x-a_0)(x-a_1)...(x-a_n)$$

ubi $a_0, a_1, ... a_n$ es $n+1$ quantitates distincto. Numeratore es de gradu inferiore ad denominatore. Valore x es differente de valores a. Tunc illo functione rationale vale summa

$$c_0/(x-a_0) + c_1/(x-a_1) + ... + c_n/(x-a_n),$$

cum numeratores c_0, c_1, ... constante.

·4 $n \varepsilon N_1 \,.\, f \varepsilon$ (qFq)integro . gradu$f = n$ $.\supset.$ Radices$f =$
$(q\text{'}F1\cdots n) \frown a \ni [f = \{(D^n f0)/n!\, \Pi[(x-a_r)|r, 1\cdots n] \mid x, q\}]$ Df

Si n es numero naturale, et f es functione integro de gradu n, tunc « Radices f » indica successione de n numeros imaginario a tale que, pro omni x, fx vale coefficiente de termine de gradu maximo, multiplicato per factores $(x-a_1)(x-a_2)...(x-a_n)$.
Nos limita ad casu que f es functione reale; sed radices de f debe es sumpto in campo imaginario. Valores $a_1, a_2, ... a_n$ pote es distincto aut non.

Ergo, si f es functione arbitrario, et g es functione integro de gradu n, habe sensu Interp [f; Radices g], que exprime polynomio de gradu $n-1$. Si f es integro, isto polynomio vocare « resto de divisione de f per g ».

·5 $f, g, h \varepsilon$ (qFq)integro . gradu$f <$ grado$g$ + grado$h$. Radices$g \frown$ Radices$h = \wedge$. $p =$ Interp(fx/hx | x; Radices$g$) . $q =$ Interp(fx/gx | x; Radices$h$) . $x \varepsilon$q . $gx \times hx - = 0$.⊃. $(fx)/(gx \times hx) = (px)/(gx) + (qx)/(hx)$

Nos considera fractione $(fx)/[(gx)\times(hx)]$ ubi f, g, h es functione integro, et gradu de numeratore fx es minore de gradu de denominatore. Functiones g et h habe nullo radice commune. Tunc nos calcula duo functione integro px et qx, tale que, pro omni x, que non redde nullo denominatore, fractione dato es decomposito in duo fractiones $px/gx + qx/hx$. Gradu de p es minore de gradu de g, et gradu de q minore de gradu de h.

✻ 41·1 $x, a \varepsilon$q . mod$x$, mod$a$ $\varepsilon \theta$.⊃. $/\surd(1-2ax+a^2) =$
$\Sigma\{a^n/(n!2^n)\, D^n[(x^2-1)^n|x, x]|n, N_0\}$

Coefficiente de $a \uparrow n$, es dicto « polynomio de Legendre ». Plure Auctore indica illo per $X_n\, x$.

·2 $n \varepsilon N_1$.⊃. $B_n = (-1)^{n-1} D^{2n}[x/(e^x-1)\,|x, q-\iota 0]0$

Expressione de numero de Bernoulli (pag. 131) per derivata. Nos applica definitione de derivata dato per P3·2 (pag. 278). Pag. 260 P11·4 ⊃ P.

✻ 42. Serie de Lagrange.

·1 $u \varepsilon$ Cls'q' . f, Df ε q'Fu . $x \varepsilon u$. $a \varepsilon$q' . mod$a <$ l,mod$[(z-x)/fz\,|z`u]$.⊃. Num $u \frown z \ni (z = x + afz) = 1$

·2 Hp·1 . $z \varepsilon u$. $z = x + afz$. g, Dg ε q'Fu .⊃.
$gz = gx + \Sigma\{(a^n/n!) D^{n-1}[(fx)^n(Dgx)|x, u, x]|n, N_1\}$

{ Lagrange a.1768; oeuvres t. 3, p. 25 }

Rouché, a.1861 JP. t. 39 p.193 determina hypothesi in serie de Lagrange.

·3 $e \varepsilon$Q . $e <$ 0·6627434 . $x \varepsilon$q .⊃.
$\imath\, q \frown z \ni (z = x + e\sin z) = x + \Sigma\{(e^n/n!) D^{n-1}[(\sin x)^n|x, x]|n, N_1\}$

Æquatione in primo membro, es « æquatione de Keplero », que occurre in theoria de motu de planetas.

Lagrange id. p.113 da serie. Laplace a.1823, œuvres t. 5 p.473, pone hypothesi $e <$ 0·662.

Vide Levi-Civita LinceiR. a.1904 p.260.

✻ 43. Functione complexo.

Derivata de numero complexo, de vectore et de puncto functione de variabile reale, es definito in pag. 284 P 15.

Si puncto mobile p es functione de « tempore » x, tunc:
$\mathrm{D}px$ = « velocitate de puncto ».
D^2px = « acceleratione ».

Si puncto es materiale, et m es suo massa:
$m\mathrm{D}^2px$ = « vi » vel « fortia » agente super puncto.
$m(\mathrm{D}px)^2/2$ = « energia ».

Si o es puncto fixo, motu de $o+\mathrm{D}px$ es dicto « hodographo ».

acceleratione, A. acceleration, F accélération, H. aceleracion, I. accelerazione. ⊂ ad (p.21) + celer(e) (= I.) + -a (p.18) + -tione (p.19).

massa: A. mass, D.F. masse, H. masa, I.R. massa.
⊂ G. maza = pasta, substantia.

vi (L. classico). ⊃ A.F.H.I. : vi-olatione, vi-olento.

fortia (L. tardo) : A.F. force, H. fuerza, I forza.
⊂ fort(e) (F. fort, H. fuerte, I forte) + -ia (p.68).

energia G. ἐνέργεια : A. energy, D. energie, F. énergie, H.I. energia, R. energiá. ⊂ en + erg- + -ia.

en G. ⊃ A.D.F.H.I.R. : en-demico en-thusiasmo en-tomologia en-tropia el-lipsi em-blema em-pirico. ⊂ E. eni, L.A.D. in.

ergo G. = labore. ⊃ A.D.F.H.I.R. : chir-urgo, dramat-urgo, metall-urgia. Argon (Chemia, = a-ergon), erg (unitate de labore in Physica).
⊂ G.ant. vergo ⊂ E. vergo ⊃ A. work, D. werk = labore.

hodographo, motu considerato per Möbius a.1843 t.4 p.47. Nomen introducto per Hamilton a.1846 DublinT. t.3.
⊂ hodo + graph(e) + -o

hodo G. = via. ⊃ A.D.F.H.I.R. : met-hodo, peri-odo, an-odo, cat-hodo...
⊂ E sodo ⊃ R. chod' = via. R. ischod' =‖ G.A.F.H.I. ex-odo.

graphe G. = scribe.
⊃ A.D.F.H.I.R.: calli-graph-o, epi-graph-ê, litho-graph-o, ortho-graph-ia, panto-graph-o, tele-graph-o, typo-graph-ia, gram-ma (vide pag. 207), ...
⊂ E. gerbhe, A. carve, D. kerbe = sculpe.

-o Indica nomen de agente. Vide tabula de suffixos.
hodo-graph-o = que describe via.

✻ 44. Valore medio pro functione complexo.

·1 $a,b \varepsilon \mathrm{q} \,.\, a -= b \,.\, n \varepsilon \mathrm{N}_1 \,.\, f, \mathrm{D}f \,\varepsilon\, \mathrm{Cx}n \,\mathrm{F}\, a \frown b \,.\supset.$
$(fb - fa)/(b-a) \;\varepsilon\; \mathrm{Medio}\, \mathrm{D}f`(a \frown b)$

·2 (vct | Cx) P·1
(pnt | Cx) P·1

Si f es numero complexo de ordine n, aut puncto, aut vectore, functione definito de variabile reale in intervallo de a ad b, tunc ratione de incremento de functione ad incremento de variabile es « medio » inter valores de derivata. Non es semper uno ex valores de derivata, ut pro functione reale.

Classe « Medio » es definito pro numeros reale, in pag. 133, et pro numeros complexo, pag. 145 Prop. 5·0.

Per exemplo, si x es tempore, et fx es puncto mobile, nos sume punto fixo o, et imagina puncto $o + \mathrm{D}fx$, que habe motu hodographo. Theorema dice que

$$o + \mathrm{D}(f;a,b) \;\varepsilon\; \mathrm{Med}(o + \mathrm{D}f`a \frown b).$$

$o + \mathrm{D}f`a \frown b$ es « arcu de curva descripto per motu hodographo», suo classe medio es minimo figura convexo que contine figura dato. In ce figura convexo jace puncto $o + \mathrm{D}(f;a,b)$.

In casu particulare $fa = fb$, quando positione finale coincide cum initiale, nos deduce que origine o pertine ad minimo figura convexo continente arcu de motu hodographo (sed non es puncto hodographo).

Dem.

$p \varepsilon \mathrm{Cx}n \,.\, h = \{\Sigma[p_r \times (fx)_r \,|\, r, 1\cdots n] \,|\, x, a \frown b\} \,.\, \mathrm{P}22 \,.\supset.\, \mathrm{D}(h;a,b) \,\varepsilon\, \mathrm{D}h`a \frown b$
$.\supset.\, \Sigma[p_r \times \mathrm{D}(f_r\,;a,b) \,|\, r,1\cdots n] \,\varepsilon\, \Sigma[p_r \times (\mathrm{D}f_r`a \frown b) \,|\, r,1\cdots n].\ \mathrm{Def\ Med.} \supset.\ \mathrm{P}$

Me expone demonstratione scripto in symbolos, pro casu de vectore functione de variabile reale. Sume vectore p ad arbitrio. Tunc producto $hx = p \times (fx)$ es quantitate reale functione de variabile reale x. Ad illo es applicabile theorema de valore medio, et seque
$\mathrm{D}(p \times fx \,|\, x;a,b) \,\varepsilon\, p \times \mathrm{D}f`a \frown b$ vel $p \times \mathrm{D}(f;a,b) \,\varepsilon\, p \times \mathrm{D}f`a \frown b$.

Nunc, si u es vectore, v es classe de vectores, et pro omni vectore p resulta $p \times u \,\varepsilon\, p \times v$, seque (non que $u \varepsilon v$), sed, per definitione de classe Medio, que $u \,\varepsilon\, \mathrm{Med}\, v$.

·3 $\mathrm{Hyp}\cdot 1 \,.\, m \varepsilon \mathrm{N}_1 \,.\, \mathrm{D}^m f \,\varepsilon\, \mathrm{Cx}n \,\mathrm{F}\, a \frown b \,.\, x \varepsilon a \frown b \,.\, h \varepsilon a \frown b - x \;.\supset.$
$f(x+h) - \Sigma[(h^r/r!\, \mathrm{D}^r fx) \,|\, r, 0\cdots(m-1)] \;\varepsilon\; h^m/m!\ \mathrm{Medio}\ \mathrm{D}^m f`(x + \theta h)$

·4 Hyp·1 . $m \varepsilon N_1$. $D^m f \varepsilon CxnFa{-}b$. $x \varepsilon a{-}b f 0 \cdots m$.⊃.
$D^m[f;(x,0\cdots m)] \varepsilon$ Medio$(D^m f`a{-}b)/m!$

Expressione de resto, per Lagrange, de evolutione in serie de potestates (pag. 300), et expressione de ratione incrementale successivo (pag. 307 P39) subsiste pro functione complexo, introducto valore medio de derivata.

✻ 45. rectaT = RECTA TANGENTE.

$k\varepsilon$ Cls'q . $p\varepsilon$ pFk . $x\varepsilon k\frown\delta k$.⊃.
·0 rectaTpx = lim[recta(px, py) | y, $k\frown y\ni(py -= px)$, x] Df

Si k es classe de quantitates, et p es puncto functione definito in classe k, tunc « rectaTpx », lege « recta tangente ad trajectoria de p, pro valore x », es limite de recta per px, et per altero puncto py, dum y varia, in classe k, ubi sume valores que redde py differente de px, et tende ad x.

Classe k es aut classe q, aut Q, aut es intervallo, etc., ut in Df de derivata. Symbolo rectaTpx vale (rectaTp)x, et non rectaT(px), id es, nos determina recta tangente ad traiectoria de p, pro valore x de variabile, et non tangente ad puncto px, que non habe sensu.

tangente D.F.H.I., A tangent, R tangens'. ⊂ tange + -nte (142).
tange H.I., D. tang-ieren. ⊂ tage (242) + -n- (341).

E u c l i d e 1.3 Df 2, dice que recta es tangente « ἐφάπτεσθαι » ad circulo (1.1 Df15) si habe uno solo puncto commune cum circulo.

Nos pote applica idem Df ad ellipsi, etc.; sed non ad omni curva.

D e s c a r t e s, *La Geometrie* a.1637 Œuvres, t.6, p.418 dice que tangente es recta que seca curva in duo puncto « ioins en vn »; id es, si æquatione que determina ce punctos de intersectione habe duo « racines entierement ésgales ».

Df considerato se transforma in P·0, si nos considera recta per duo puncto « juncto in uno », ut limite de recta per duo puncto distincto.

·1 Dpx ε v -ι0 .⊃. rectaTpx = recta(px, Dpx)

Si derivata de puncto mobile p, pro valore x de variabile, es vectore determinato, non nullo, tunc recta tangente in px es recta per puncto px, et parallelo ad derivata Dpx.

Dem. Df rectaT $.\supset.$ rectaT $px = \lim [\,\mathrm{recta}(px, py) | y, k, x]$
p.180 P1·4 $.\supset.$ » $=$ » » $(px, py-px) | y,k,x]$
» ·11 $.\supset.$ » $=$ » » » $(py-px)/(y-x) | y, k, x]$
p.237 §lim P71·2 $.\supset.$ » $= \mathrm{recta}\,[\,px, \lim($ »
Df D $.\supset.$ » $= \mathrm{recta}(px, \mathrm{D}px)$

In facto, recta tangente es limite de recta per px et py; vel de recta per px, et parallelo ad vectore $py - px$, ad que me substitue vectore parallelo $(py - px)/(y - x)$. Limite de ce vectore es derivata $\mathrm{D}px$; unde seque theorema.

·2 $n \varepsilon \mathrm{N}_1 \,.\, \mathrm{D}px = \mathrm{D}^2px = \ldots = \mathrm{D}^n px = 0 \,.\, \mathrm{D}^{n+1}px \,\varepsilon\, \mathrm{v}\text{-}\iota 0 \;.\supset.$
rectaT$px = \mathrm{recta}(px, \mathrm{D}^{n+1}px)$

Dem. Df rectaT . §D P32 $.\supset.$ rectaT$px = \lim[\mathrm{recta}(px,py)|y,k,x]$
$= \lim \mathrm{recta}(px,\{py-px-\Sigma[(y-x)^r/r!\,\mathrm{D}^r px\,|r,1\cdots n]\}/(y-x)^{n+1})|y,k,x]$
$= \mathrm{recta}[px,\mathrm{D}^{n+1}px/(n+1)!] = \mathrm{recta}(px,\mathrm{D}^{n+1}px)$

Si pro valore x, derivata de puncto es nullo, tunc recta tangente habe directione de primo inter derivatas sequente, que non es nullo.

In facto nos evolve $py-px$ in serie asymptotico secundo potestates de $y-x$, usque ad ordine $n+1$; mane solo ultimo termine; ergo tangente habe directione de primo termine non nullo.

✻ 46. planN = Plano normale.

Hyp P45 $.\supset.$ planN$px = \mathrm{p}\frown y \ni [\mathrm{proj}(\mathrm{rectaT}\,px)y = px]$ Df
$=$ « plano normale ad trajectoria de p »

« planNpx », lege « plano normale ad curva p, pro valore x », es plano perpendiculare ad tangente in px; vel, expresso per ideas præcedente, es loco de punctos y tale que projectione super recta tangente ad px, de y, coincide cum px.

·1 $\mathrm{D}px \,\varepsilon\, \mathrm{v}\text{-}\iota 0 \;.\supset.$ planN$px = \mathrm{p}\frown y \ni [(y-px) \times \mathrm{D}px = 0]$
$= \mathrm{p}\frown y \ni \{\mathrm{D}[\mathrm{mod}(y-pz)|z, k]x = 0\}$
$= \mathrm{p}\frown y \ni [\mathrm{real}(y-px)/\mathrm{D}px = 0] = \mathrm{plan}(px, \mathrm{I}\mathrm{D}px)$

Plano normale in px es loco de punctos y que satisfac æquatione $(y-px)\times \mathrm{D}px=0$; vel que redde derivata de distantia de y ad pz nullo pro $z=x$; vel que redde nullo parte reale de quaternione $(y-px)/\mathrm{D}px$; vel es plano de puncto px, et de bivectore indice de $\mathrm{D}px$.

✻ 47. planO = Plano osculatore.

Hp P45 $.\supset$: $\text{planO}px =$
$\lim[\text{plan}(\text{rectaT}px, py) \mid y, k \frown y\ni(py -\varepsilon\ \text{rectaT}px), x]$ Dt

« planO px » lege « plano osculatore ad linea p, pro valore x » es limite de plano per recta tangente in px, et per py, dum y varia, in campo k, et sume valores que redde plano de recta et de puncto determinato, vel in modo que py non es super recta tangente in px, et verge ad x.

Notione de plano osculatore occurre in Tinseau, « Solutions de quelques problèmes relatifs à la théorie des surfaces courbes », Mém. Sav. Etrang. t. 9, a.1781.

osculatore I, A osculatory, F osculateur. $\subset$ oscula- + -tore.
oscula- = basia, combasia. $\supset=$ A. oscula-te, D. —lieren, F. —ler.
$\subset$ oscul(o) + -a (92).
osculo H. = basio, parvo bucca. $\subset$ os + -culo.
os, ore = bucca. $\supset$ A.D.F.H.I.R. or-atore or-atorio or-aculo.
$\subset$ E. ôs, S as, || (secundo Van.) L es (1), in sensu « respira ».
-culo $\supset$ se-culo, arti-culo, mole-cul-a, os-culo. $\subset$ -c(o) (201) + -ulo (224).

·1 $\text{D}px\ \varepsilon\ \text{v}-\iota 0\ .\ \text{D}^2px\ \varepsilon\ \text{v}-\text{qD}px\ .\supset.$
$\text{planO}\ px = \text{plan}(px, \text{D}px, \text{D}^2px)$

[$\text{planO}px = \lim\{\text{plan}[\text{rectaT}px, py] \mid y, k, x\}$
P45·1 $.\supset.$ » $[\text{recta}(px, \text{D}px), py] \mid y, k, x\}$
» $px, \text{D}px, [py - px - (y-x)\text{D}px]/(y-x)^2 \mid y, k, x\}$
§D P32 $.\supset.$ » $= \text{plan}(px, \text{D}px, \text{D}^2px)$]

Si, pro valore x de variabile, derivata de puncto p es vectore non nullo, et si derivata de ordine duo de p es vectore non parallelo ad derivata de ordine uno, tunc plano osculatore es plano per px, et parallelo ad derivata primo et secundo.

Vel: plano osculatore ad trajectoria de puncto p, pro tempore x, es plano de puncto, de suo velocitate, et de suo acceleratione, si ce elementos determina plano.

Vel: plano osculatore contine fortia que move puncto.

In facto, per definitione de plano osculatore, et per theorema super recta tangente, plano osculatore es limite de plano per px, $\text{D}px$, et py. Vectore $py - px$ jace in ce plano; ergo plano osculatore es limite de plano per px et $\text{D}px$, et parallelo ad vectore $py - px - (y - x)\text{D}px$, et si diviso per $(y-x)^2$. Per theorema de Taylor, limite de isto vectore es $\text{D}^2px/2$, unde seque theorema.

·2 $m, n \varepsilon N_1 . Dpx = D^2px = ... = D^{m-1}px = 0 . D^m px \varepsilon v\text{-}\iota 0 .$
$D^{m+1}px, ... D^{m+n-1}px \varepsilon qD^m px . D^{m+n}px \varepsilon v\text{-} qD^m px .\supset.$
$\text{planO}px = \text{plan}(px, D^m px, D^{m+n}px)$

Si m—1 derivata successivo de puncto p, pro valore x considerato, es nullo et derivata de ordine m non es nullo, et si n—1 derivata de ordine post m es parallelo ad derivata de ordine m, et derivata de ordine $m+n$ non es parallelo ad derivata de ordine m, tunc plano osculatore es plano determinato per puncto px, et per suo derivatas de ordine m et $m+n$.

·3 $\text{Hp·1} .\supset. \text{T}px = \text{UD}px \quad . \quad \text{N}px = \text{UDT}px \quad .$
$\text{B}px = \{\text{I}(\text{T}px)\alpha(\text{N}px)\}$ Df

Tpx, Npx, Bpx es vectores unitario parallelo ad « tangente », « normale principale », « binormale », de linea p.

·4 $\text{rectaN}\,px = \text{planN}px \frown \text{planO}px = \text{recta}(px, \text{N}px)$ Df
= « normale principale ».

·5 $\text{rectaB}px = \text{recta}(px, \text{B}px)$ Df
= « binormale » (Saint Venant a.1845).

❋ 48. Ax Cc Rc

$k \varepsilon \text{Cls'q} . p \varepsilon \text{pF}k . x \varepsilon k \frown \delta k .\supset.$

·0 $\text{Ax}px = \lim[(\text{planN}px \frown \text{planN}py) | y, k, x]$ Df

Axpx vocare « axi de plano osculatore ad curva », vel « axi de curva » (Monge, *Géométrie descriptive*, a.VII, p.106). Es intersectione de duo plano normale consecutivo.

axi, A.L. axis, D.F. axe, I. asse, H. eje, F. essieu.
⊂ E. acsi ⊃= D. achse, G. axon axon-o-metria, R osı̦, S acs'a.

·1 $\text{Cc}px = \imath[\text{Ax}px \frown \text{planO}px]$ Df
» = « centro de curvatura ». Es puncto de intersectione de axi cum plano osculatore.

curvatura H.I., A. curvature, F. courbure.
⊂ curva (verbo) + -t(o) + -ura (153).
curva (verbo) = fac curvo. ⊂ curv(o) + -a (4).
curva (nomen) H.I., A. curve, D. kurve, F. courbe.
⊂ curv(o) + -a (126, linea).
curvo H.I. ⊂ E. qervo ⊃ R crivi-tı̦, criva-á liniá = linea curva.
⊂ cur- (⊃ cor-ona, cir-co, ci-n-ge,...) + -vo (214).

·2 $\text{Rc}px = \text{d}(px , \text{Cc}px)$ Df
= « radio de curvatura ».

$\mathrm{D}px \, \varepsilon \, \mathrm{v}\text{-}\iota 0 \, . \, \mathrm{D}^2px \, \varepsilon \, \mathrm{v}\text{-}\mathrm{qD}px \; .\supset.$

·3 $\mathrm{Ax}px = \mathrm{planN}px \cap z\ni[(z-px)\times(\mathrm{D}^2px) = (\mathrm{D}px)^2]$

$[\mathrm{Hp} \, . \, f = |(z-py)\times \mathrm{D}py \,|\, y, k] \; .\supset. \; \mathrm{Ax}px = \lim\{p\cap z\ni[fx = 0 \, . \, fy = 0] | y, k, x\}$
$= \lim\{p\cap z\ni[fx = 0 \, . \, (fy-fx)/(y-x) = 0] \,|\, y, k, x\}$
$= p\cap z\ni\{fx = 0 \, . \lim[(fy-fx)/(y-x) \,|\, y, k, x] = 0\} = p\cap z\ni[fx=0 \, . \, \mathrm{D}fx=0]$
$= \mathrm{planN}px \cap z\ni[-(\mathrm{D}px)^2 + (z-px)\times \mathrm{D}^2px = 0] \;]$

Axi de curva p pro x, es intersectione de plano normale in px, vel loco de punctos z que redde

$$(z-px)\times \mathrm{D}px = 0,$$

cum plano loco de punctos z que satisfac æquatione:

$$(z-px)\times \mathrm{D}^2px = (\mathrm{D}px)^2,$$

que resulta ex præcedente, post derivatione pro x.

In vero, me pone $fy = (z-py)\times \mathrm{D}py$, ubi z es puncto. Tunc $fy=0$ es æquatione in z, satisfacto per punctos de plano normale pro valore y. Tunc axi es limite de intersectione de planos de æquatione $fx=0$, $fy=0$. Isto systema vale $fx=0$ et $\mathrm{D}(f \, ; x,y)=0$, et ad limite, $fx=0$ et $\mathrm{D}fx=0$.

·4 $\mathrm{Cc}\, px = px - \mathrm{D}px \,/\, \mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px)$

$[\; z = \mathrm{Cc}px \; .\supset. \; \mathrm{real}(z-px)/\mathrm{D}px = 0 \, . \, \mathrm{D}[\mathrm{real}(z-px)/\mathrm{D}px] \,|\, x, k, x] = 0$ (1)
$\mathrm{real}\, \mathrm{D}[(z-px)/\mathrm{D}px|x, k, x] = \mathrm{real}\{-1-[(z-px)/\mathrm{D}px][\mathrm{D}^2px/\mathrm{D}px]\} =$
$= -1 - \mathrm{real}[(z-px)/\mathrm{D}px]\mathrm{real}[\mathrm{D}^2px/\mathrm{D}px]$
$-\mathrm{Imag}[(z-px)/\mathrm{D}px]\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px)$ (2)
(1) . (2) $.\supset. \; 1+\mathrm{Imag}[(z-px)/\mathrm{D}px]\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px) = 0 \; .\supset.$
$\mathrm{Imag}[(z-px)/\mathrm{D}px] = -/\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px)$ (3)
$(z-px)/\mathrm{D}px = \mathrm{real}[(z-px)/\mathrm{D}px] + \mathrm{Imag}[(z-px)/\mathrm{D}px]$. (1) . (3) $.\supset.$
$(z-px)/\mathrm{D}px = -/\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px) \; .\supset.$
$z-px = -\mathrm{D}px/\mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px) \; .\supset. \; \mathrm{P} \;]$

Centro de curvatura de curva p, pro valore x, vale px minus derivata de px diviso per imaginario de ratione de derivata de ordine duo ad derivata ordine uno.

Ratione de duo vectore es quaternione (pag. 185). Si vectores considerato in calculo es in plano fixo, tunc quaterniones es repræsentato per numeros imaginario (pag. 268).

·5 $\mathrm{Rc}\,px = (\mathrm{D}px)^2/\mathrm{mod}[(\mathrm{cmp}\perp\mathrm{D}px)\mathrm{D}^2px]$

$[\; \mathrm{Df\, Cc} \, . \, \mathrm{P}{\cdot}3 \; .\supset. \; (\mathrm{Cc}px - px)\times(\mathrm{D}^2px) = (\mathrm{D}px)^2 \; .\supset. \; \mathrm{P} \;]$

Radio de curvatura vale quadrato de velocitate diviso per componente normale de acceleratione, considerato in valore absoluto. Componente normale de acceleratione vocare sæpe «acceleratione normale».

Constructione graphico de centro de curvatura, dato px, Dpx, D^2px:
Construe punctos $px+Dpx$, $px+Dpx+(\text{cmp}\perp Dpx)D^2px$; per puncto $px+Dpx$ duce, in plano osculatore, normale ad recta$[px, px+Dpx+(\text{cmp}\perp Dpx)D^2px]$, que seca normale ad curva in $Ccpx$.

·6 $Rcpx = (\text{mod } Dpx)^3 / \text{mod}(Dpx \,\alpha\, D^2px)$

·7 $Axpx = \text{recta}(Ccpx, Bpx)$

·8 $D^2px = (D \text{ mod } Dpx)Tpx + (Dpx)^2/(Rc\, px)\, Npx$

Dem. $Dpx = (\text{mod}Dpx)\, Tpx$. Oper D .Ɔ. P

Derivata de ordine 2, vel acceleratione de puncto es summa de vectore derivata de modulo de velocitate, directo secundo tangente ad trajectoria, plus quadrato de velocitate diviso per radio de curvatura, directo secundo normale principale ad curva.

·9 $Ccpx = px+(Dpx)^2[(D^2px)(Dpx)^2-(Dpx)(Dpx \times D^2px)]/[(Dpx)^2(D^2px)^2-(Dpx \times D^2px)^2]$

Expressione de centro de curvatura ope solo producto interno de duo vectore.

✻ 49. curvatura

Hyp P48 .Ɔ. curvatura $px = D\,Tpx / \text{mod}Dpx$ Df
$= Npx / Rc\, px$

«Curvatura» es vectore directo secundo normale principale, et reciproco de radio de curvatura.

✻ 50. torsio

$k \varepsilon$ Cls'q . $p \varepsilon$ pFk . $x \varepsilon k \frown \delta k$.Ɔ.

·1 torsio $px = D\,B\,px \times Npx / \text{mod } D\,px$ Df

«Torsione» es quantitate cum signo. Torsione es positivo in luppulo et in cucurbita, es negativo in viti vinifera, et in viti commune de Mechanica constructo in Europa. Viti constructo in Sina et in Nippon, habe torsione positivo.

·2 torsio $px = -(Dpx \,\alpha\, D^2px \,\alpha\, D^3px /\psi) / (Dpx \,\alpha\, D^2px)^2$

Torsione vale minus producto alterno de derivatas de ordine 1, 2, 3, diviso per trivectore unitate (pag. 198), diviso per quadrato de producto alterno de derivatas de ordine 1 et 2.

·3 $\mathrm{D\,B}px \,/\, \mathrm{mod\,D}px = (\mathrm{torsio}px)\mathrm{N}px$

·4 $\mathrm{DN}px \,/\mathrm{mod\,D}\,px = -(/\,\mathrm{Rc}\,px)\mathrm{T}px - (\mathrm{torsio}px)\,\mathrm{B}px$

Derivata de vectores unitario T (Prop. 49), N, B, definito in Prop. 47.

} FRENET, *Sur les courbes à double courbure*, Thèse 10 juillet 1847, JdM. t.17 a.1851.

$/$ torsio px = radio de torsione.

Cc px — (/ torsio px)(D Rc px)B px = « centro de sphæra osculatrice » = « puncto de inviluppo de planN px ».

recta;px, (Rc px) Tpx — (/ torsio px) Bpx; = « generatrice de superficie rectificante inviluppo de plan(px, I N px) ».

(o+Tpx)|x describe indicatrice de tangentes.
(o+Npx)|x » » normales principale.
(o+Bpx)|x » » binormales.

torsione (L. a. +100), A.D.F. torsion, H. tortijon, I. torsione.
⊂ tortione (L. classico) ⊂ tort(o) + -ione.
torto I., A.D.F. tort, H. tuerto. ⊂ torq(ue) + -to.
torque, H.I. torce, F. tordre.

✻ 51·1 $k\varepsilon\,\mathrm{Cls'q}\,.\,p\varepsilon\,\mathrm{pF}k\,.\,z_1,z_2\varepsilon k\;.\supset.$
$pz_1\,a\,pz_2 = (z_2-z_1)\,pz_1\,a\,\mathrm{D}(p;\,z_1,\,z_2)$

Expressione de producto alterno de duo puncto de curva.

·2 $\mathrm{Hp}{\cdot}1\,.\,\mathrm{D}p\;\varepsilon\;(\mathrm{vF}k)\mathrm{cont}\,.\,x\varepsilon k\;.\supset.$
$\lim[(pz_1\,a\,pz_2)/(z_2-z_1)\,|z,\,(k\mathrm{f}\,1\cdots2)\mathrm{sim},\,(x,x)] = px\,a\,\mathrm{D}px$

·3 $\mathrm{Hp}{\cdot}2\;.\supset.\;\lim[\mathrm{d}(pz_1,\,pz_2)/(z_2-z_1)\,|z,\,\ldots] = \mathrm{modD}px$

·4 $\mathrm{Hp}{\cdot}2\,.\,\mathrm{D}px\,-\!\!=\!0\;.\supset.$
$\lim[\mathrm{recta}(pz_1,pz_2)\,|z,\,(k\mathrm{f}\,1\cdots2)\mathrm{sim},\,(x,x)] = \mathrm{recta}(px,\,\mathrm{D}px)$

Recta tangente, in uno puncto de curva, per definitione, es positione limite de recta que uni ce puncto ad alio puncto de curva, dum secundo puncto tende ad primo. Theorema dice que, si puncto habe derivata de ordine 1 continuo et non nullo, tunc recta tangente es positione limite de recta que uni duo puncto differente de curva, dum ambo puncto varia et tende ad puncto dato.

Si hypothesi non es vero, thesi pote es in defectu. P. ex. recta tangente ad curva descripto per puncto $px = o + x^2a + x^3b$, ubi $o\varepsilon\mathrm{p}\,.\,a,b\varepsilon\mathrm{v}\,.\,aab-\!\!=\!0$, que vocare « parabola de ordine 3/2 », habe rectaT $p0$ = recta(0,a). Positione limite de recta(px, p—x), que uni duo puncto de curva, que tende ad $p0$ dum x tende ad 0, es recta(0,b), differente de præcedente.

$\divideontimes$ 52·1 Hp P51 . $z_1, z_2, z_3 \varepsilon k$.⊃.
$pz_1 \, a \, pz_2 \, a \, pz_3 = (z_2-z_1)(z_3-z_1)(z_3-z_2) pz_1 \, a \, \mathrm{D}(p; z_1, z_2) \, a \, \mathrm{D}^2(p; z_1, z_2, z_3)$

Producto alterno de tres puncto de curva, expresso per rationes incrementale de puncto.

·2 Hp·1 . $\mathrm{D}^2 p \, \varepsilon \, (\mathrm{vF}k)\mathrm{cont}$.⊃. $\lim\{(pz_1 \, a \, pz_2 \, a \, pz_3)/[(z_2-z_1)(z_3-z_1)(z_3-z_2)] \,|\, z, (k\mathrm{f}\, 1\cdots 3)\mathrm{sim}, (x,x,x)\} = px \, a \, \mathrm{D}px \, a \, \mathrm{D}^2 px / 2$

Limite de triangulo (producto alterno) de tres puncto de curva, diviso per producto de differentias de variabile, si tres puncto verge ad idem puncto px.

·3 Hp·2 . $\mathrm{D}px \mathrel{-}= 0$.⊃. $\lim \mathrm{d}[pz_1, \mathrm{rectaT}(p, z_2)]/(z_2-z_1)^2 \,|\, z, \ldots] = \mathrm{mod}(\mathrm{D}px \, a \, \mathrm{D}^2 px)/(2 \mathrm{mod} \mathrm{D}px)$

Distantia de uno puncto de curva ab tangente in puncto successivo.

·4 Hp·2 .⊃. $\lim[\mathrm{ang}(\mathrm{D}pz_1, \mathrm{D}pz_2)/(z_2-z_1) \,|\, z, \ldots] = \mathrm{mod}(\mathrm{D}px \, a \, \mathrm{D}^2 px)/(\mathrm{mod}\mathrm{D}px)^2$

Angulo de duo tangente successivo.

·5 $\mathrm{D}^2 p \, \varepsilon \, (\mathrm{vF}k)\mathrm{cont}$. $\mathrm{D}px \, a \, \mathrm{D}^2 px \mathrel{-}= 0$.⊃.
$\lim[\mathrm{plan}(pz_1, pz_2, pz_3) \,|\, z, (k\mathrm{f}\, 1\cdots 3)\mathrm{sim}, (x,x,x)] = \mathrm{plan}(px, \mathrm{D}px, \mathrm{D}^2 px)$

Si puncto habe derivatas de ordine 1 et 2 continuo et non parallelo, tunc limite de plano per tres puncto de curva vergente ad idem puncto, es plano de duo derivata, vel coincide cum plano osculatore.
Si hypothesi non es satisfacto, thesi pote es in defectu.

$\divideontimes$ 53·1 Hp P51 . $z_1, z_2, z_3, z_4 \varepsilon k$.⊃.
$pz_1 \, a \, pz_2 \, a \, pz_3 \, a \, pz_4 = (z_2-z_1)(z_3-z_1)(z_3-z_2)(z_4-z_1)(z_4-z_2)(z_4-z_3) \, pz_1 \, a \, \mathrm{D}(p; z_1, z_2) \, a \, \mathrm{D}^2(p; z_1, z_2, z_3) \, a \, \mathrm{D}^3(p; z_1, z_2, z_3, z_4)$

Producto alterno, vel tetrahedro cum signo, de quatuor puncto consecutivo de curva.

Hp·1 . $\mathrm{D}^3 p \, \varepsilon \, (\mathrm{vF}k)\mathrm{cont}$.⊃.

·2 $\lim\{(pz_1 \, a \, pz_2 \, a \, pz_3 \, a \, pz_4)/[(z_2-z_1)(z_3-z_1)(z_3-z_2)(z_4-z_1)(z_4-z_2)(z_4-z_3)] \,|\, z, (k\mathrm{f}\, 1\cdots 4)\mathrm{sim}, (x,x,x,x)\} = px \, a \, \mathrm{D}px \, a \, \mathrm{D}^2 px \, a \, \mathrm{D}^3 px / 12$

·3 $\mathrm{D}px \, a \, \mathrm{D}^2 px \mathrel{-}= 0$.⊃. $\lim\{\mathrm{d}[pz_1, \mathrm{planO}(p, z_2)]/(z_2-z_1)^3 \,|\, z, \ldots\} = \mathrm{mod}(\mathrm{D}px \, a \, \mathrm{D}^2 px \, a \, \mathrm{D}^3 px)/[6 \, \mathrm{mod}(\mathrm{D}px \, a \, \mathrm{D}^2 px)]$

Distantia de uno puncto de curva ab plano osculatore in puncto successivo.

·4 $\mathrm{D}px -= 0 .\supset. \lim\{\sin(\mathrm{D}pz_1, \mathrm{D}pz_2, \mathrm{D}pz_3)/[(z_2-z_1)(z_3-z_1)(z_3-z_2)] \mid z, ...\} = [(\mathrm{D}px\, \alpha\, \mathrm{D}^2px\, \alpha\, \mathrm{D}^3px)/\psi]/[2(\mathrm{mod}\, \mathrm{D}px)^3]$

Sinu de trihedro de tres tangente successivo. Sinu de trihedro es definito in pag. 199 Prop. 22·2.

·5 $\mathrm{D}px\, \alpha\, \mathrm{D}^2px -= 0 .\supset.$
$\lim\{\mathrm{d}[\mathrm{rectaT}(p,z_1), \mathrm{rectaT}(p,z_2)]/(z_2-z_1)^3 \mid z, ...\} =$
$\mathrm{mod}(\mathrm{D}px\, \alpha\, \mathrm{D}^2px\, \alpha\, \mathrm{D}^3px)/[12\, \mathrm{mod}(\mathrm{D}px\, \alpha\, \mathrm{D}^2px)]$

Distantia de duo recta tangente successivo.

·6 $\mathrm{D}px\, \alpha\, \mathrm{D}^2px -= 0 .\supset. \lim \mathrm{ang}[\mathrm{rectaT}(p,z_1), \mathrm{planO}(p,z_2)]/(z_2-z_1)^2 = \mathrm{mod}(\mathrm{D}px\, \alpha\, \mathrm{D}^2px\, \alpha\, \mathrm{D}^3px)/[2\, \mathrm{mod}(\mathrm{D}px\, \alpha\, \mathrm{D}^2px)]$

·7 $\mathrm{D}px\, \alpha\, \mathrm{D}^2px -= 0 .\supset.$
$\lim\{\mathrm{ang}[\mathrm{planO}(p,z_1), \mathrm{planO}(p,z_2)]/(z_2-z_1) \mid z, ...\} =$
$\mathrm{mod}\, \mathrm{D}px\, \mathrm{mod}(\mathrm{D}px\, \alpha\, \mathrm{D}^2px\, \alpha\, \mathrm{D}^3px)/[\mathrm{D}px\, \alpha\, \mathrm{D}^2px]^2$

Angulo de duo plano osculatore successivo.
Vide meo libro *Applicazioni geometriche* a.1887 p.110.

✻ 54. Coordinatas

$o \varepsilon \mathrm{p} . a,b \varepsilon \mathrm{v} . a^2 = b^2 = 1 . a \times b = 0 . i = b/a . k \varepsilon \mathrm{Cls'q} . f \varepsilon \mathrm{qF}k .$
$p = (o + xa + fx\, b) \mid [x, k] . x \varepsilon k \frown \delta k . y = fx . y' = \mathrm{D}fx . y'' = \mathrm{D}^2fx .\supset:$

·0 $\mathrm{D}px = a + y'b \quad . \quad \mathrm{D}^2px = y''b$

Nos considera puncto o, et duo vectores unitario et orthogonale a et b. Nos voca i unitate imaginario que fer a in b.

Tunc $o+xa+yb$ es puncto de coordinatas chartesiano x et y. Nos suppone y dato per x; puncto describe curva, de que nos determina plure elemento.

·1 $\mathrm{rectaT}px = [o + (x+z)a + (y+zy')b] \mid z\text{'q}$

Dem. $[\ \mathrm{rectaT}\, px = \mathrm{recta}(px, \mathrm{D}px) = \mathrm{recta}(o+xa+yb, a+y'b) = ...\ ]$

·2 $y' \varepsilon \mathrm{q} - \iota 0 .\supset. o + (x - y/y')a \ \varepsilon\ \mathrm{rectaT}px$

Dem. $[\ \mathrm{P}{\cdot}1 . z = -y/y' .\supset. \mathrm{P}\ ]$

Intersectione de recta tangente cum axi oa.
Vectore $-y/y'a$ vocare « subtangente » de linea descripto per puncto de coordinatas cartesiano x et y in $\mathrm{plan}(o,a,b)$.

·3 $o + (y - xy')b\ \varepsilon\ \mathrm{rectaT}\, px$

Intersectione de recta tangente cum axi ob.

·4 $o+(x+yy')a \; \varepsilon$ rectaN px

Dem. [$[(o+xa+yy'a)-(o+xa+yb)]\times(a+y'b)=0$]

Intersectione de recta normale cum axi oa.

Vectore $yy'a$ es « subnormale ».

·5 $y'' \varepsilon \, \mathrm{q}\text{-}\iota 0 \;.\supset.\; \mathrm{Cc}px = o+\{x+iy+(1+y'^2)i(1+iy')/y''\}a$

Dem. [$\mathrm{Cc}px = o+(x+iy)a - (1+iy')a \,/\, \mathrm{Imag}[iy''/(1+iy')]$
$= \quad » \quad / \mathrm{Imag}[iy''(1-iy')/(1+y'^2)]$
$= \quad » \quad / [iy''/(1+y'^2)] = \ldots$]

✻ 55.

$o\varepsilon \mathrm{p} \,.\, a,b\varepsilon \mathrm{v} \,.\, a^2=b^2=1 \,.\, a\times b=0 \,.\, i=b/a \,.\, k\varepsilon\, \mathrm{Cls'q} \,.\, r\varepsilon\, \mathrm{qF}k \,.$
$p=\{[o+(rt)\mathrm{e}^{it}a]|t,k\} \,.\, t\varepsilon\, k\frown\delta k \,.\, \mathrm{D}rt, \mathrm{D}^2rt\, \varepsilon \mathrm{q} \;.\supset:$

·0 $\mathrm{D}pt = (\mathrm{D}rt+irt)\mathrm{e}^{it}a \quad . \quad \mathrm{D}^2pt = (\mathrm{D}^2rt+2i\mathrm{D}rt-rt)\mathrm{e}^{it}a$

·1 $\mathrm{rectaT}pt = [o+(rt+z\mathrm{D}rt)\mathrm{e}^{it}a+z(rt)\mathrm{e}^{it}ia]|z\text{'q}$

·2 $\mathrm{D}rt\, \varepsilon\, \mathrm{q}\text{-}\iota 0 \;.\supset.\; o-(rt)^2/(\mathrm{D}rt)\, i\mathrm{e}^{it}a \;\varepsilon$ rectaTpt

·3 $o+(\mathrm{D}rt)i\mathrm{e}^{it}a \;\varepsilon$ rectaNpt

·4 $\mathrm{Cc}pt = o + r\mathrm{e}^{it}a + [(\mathrm{D}rt)^2 + (rt)^2](i\mathrm{D}rt-rt)\mathrm{e}^{it}a/$
$[(rt)^2 + 2(\mathrm{D}rt)^2-(rt)(\mathrm{D}^2rt)]$

$o + r\mathrm{e}^{it}a$ es puncto de coordinatas polare r et t in systema dato. Si nos da r in functione de t, formulas præcedente da derivatas de puncto, tangente ad trajectoria, etc.

$-(rt)^2/(\mathrm{D}rt)\, i\mathrm{e}^{it}a$ es « subtangente polare ».
$(\mathrm{D}rt)i\,\mathrm{e}^{it}a$ es « subnormale polare ».

✻ 56. $o\varepsilon \mathrm{p} \,.\, a,b,c\varepsilon \mathrm{v} \,.\, a^2=b^2=c^2=1 \,.\, a\times b=b\times c=c\times a=0 \,.\, k\varepsilon$ Cls'q $.\, x,y,z\; \varepsilon\; \mathrm{qF}k \,.\, p=[o+(xt)a+(yt)b+(zt)c \mid t,k] \,.\, t\varepsilon\, k\frown\delta k \;.\supset:$

Si o es puncto origine, et a,b,c es vectore unitario et orthogonale, et si k es campo de variabilitate, ubi es definito tres functione reale x,y,z, et si nos voca pt puncto de coordinatas xt,yt,zt, et si t es valore in k et prope alios k, tunc:

·1 $\mathrm{D}pt = (\mathrm{D}xt)a+(\mathrm{D}yt)b+(\mathrm{D}zt)c$

·2 $\mathrm{mod}\, \mathrm{D}pt = \sqrt{[(\mathrm{D}xt)^2+(\mathrm{D}yt)^2+(\mathrm{D}zt)^2]}$

·3 $\mathrm{D}xt, \mathrm{D}yt, \mathrm{D}zt\; \varepsilon\; \mathrm{q}\text{-}\iota 0 \,.\, X,Y,Z\varepsilon \mathrm{q} \;.\supset:\; o+Xa+Yb+Zc\; \varepsilon$ rectaT$pt \;.=.\; (X-xt)/\mathrm{D}xt = (Y-yt)/\mathrm{D}yt = (Z-zt)/\mathrm{D}zt$

Si derivatas de x,y,z non es nullo, tunc conditione ut puncto de coordinatas X,Y,Z es super recta tangente ad curva in puncto px, es que differentias $X-xt$, $Y-yt$, $Z-zt$ es proportionale ad $\mathrm{D}xt$, $\mathrm{D}yt$, $\mathrm{D}zt$.

·4 $\mathrm{D}pt\ \varepsilon\ \mathrm{v}\text{-}\iota 0\ .\ X,Y,Z\varepsilon\mathrm{q}\ .\supset:\ o+Xa+Yb+Zc\ \varepsilon\ \mathrm{planN}px\ .=.$
$(X-xt)(\mathrm{D}xt)+(Y-yt)(\mathrm{D}yt)+(Z-zt)(\mathrm{D}zt)=0$

Æquatione de plano normale ad curva.

·5 $\mathrm{D}^2pt=(\mathrm{D}^2xt)a+(\mathrm{D}^2yt)b+(\mathrm{D}^2zt)c$

·6 $\mathrm{D}pt\ \varepsilon\ \mathrm{v}\text{-}\iota 0\ .\ \mathrm{D}^2pt\ \varepsilon\ \mathrm{v}\text{-}\mathrm{qD}pt\ .\ X,Y,Z\varepsilon\mathrm{q}\ .\supset:\ o+Xa+Yb+Zc$
$\varepsilon\ \mathrm{planO}px\ .=.\ \mathrm{Determ}[X-xt,\ Y-yt,\ Z-zt;$
$\mathrm{D}xt,\ \mathrm{D}yt,\ \mathrm{D}zt;$
$\mathrm{D}^2xt,\ \mathrm{D}^2yt,\ \mathrm{D}^2zt]=0$

Æquatione de plano osculatore.

❋ 57. ÆQUATIONE DIFFERENTIALE LINEARE.

$a\varepsilon\mathrm{q}\ .\supset:\ f\varepsilon\,\mathrm{qFq}\ .\ \mathrm{D}f=af\ .=.\ f=[(f0)\,\mathrm{e}\urcorner(ax)|x,\mathrm{q}]$

Dem. $\mathrm{D}f=af\ .=.\ [(\mathrm{D}fx-afx)|x,\mathrm{q}]=(\iota 0{:}\mathrm{q})$
$.=.\ \mathrm{D}(\mathrm{e}^{-ax}fx|x,\mathrm{q})=$ »
$.=.\ (\mathrm{e}^{-ax}fx|x,\mathrm{q})\ \varepsilon\ (\mathrm{qFq})\,\mathrm{const}$
$.=:\ x\varepsilon\mathrm{q}\ .\supset_x.\ \mathrm{e}^{-ax}fx=[(\mathrm{e}^{-ax}fx)|x]0=f0$

Si a es quantitate dato, tunc functione f reale de variabile reale, que pro omni valore de x satisfac æquatione

$$\mathrm{D}fx=a(fx)$$

es expresso per

$$fx=(f0)\mathrm{e}^{ax}$$

ubi varia x in campo de numeros reale.

In vero, æquatione dato vale, pro omni x, $\mathrm{D}fx-afx=0$. Nunc derivata de $\mathrm{e}^{-ax}fx$ es $\mathrm{e}^{-ax}(\mathrm{D}fx-afx)$; ergo æquatione dato dice que derivata de isto functione es semper nullo. Tunc isto functione habe valore constante, æquale ad suo valore pro $x=0$. Ergo, pro omni x es $\mathrm{e}^{-ax}fx=f0$, vel $fx=(f0)\mathrm{e}^{-ax}$.

* * *

Æquatione inter functione f et aliquo suo derivata vocare « æquatione differentiale ». Suo ordine es ordine maximo de derivatas que occurre in æquatione.

Æquatione differentiale es dicto « lineare », si es de primo gradu in functione et suo derivatas. Nos « resolve » vel « integra » æquatione differentiale, si nos determina functione f que satisfac illo, pro omni valore de variabile.

Æquatione supra considerato es « differentiale de primo gradu, lineare ». Coefficiente de f es constante, et non habe termine independente de f et $\mathrm{D}f$. Formula da integrale de æquatione.

✻ 58·1 $a \varepsilon \mathrm{q}{-}\iota 0 \,.\, b \varepsilon \mathrm{q} \,.\supset:\, f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}f = [(afx+b)|x, \mathrm{q}] \,.=.$

$\mathrm{D}[(fx+b/a) \mid x, \mathrm{q}] = [a(fx+b/a) \mid x, \mathrm{q}] \,.=.$

$[(fx+b/a) \mid x, \mathrm{q}] = [(f0+b/a)\mathrm{e}^{ax} \mid x, \mathrm{q}] \,.=.$

$f = [(f0)\mathrm{e}^{ax} + b/a(\mathrm{e}^{ax}-1)|x, \mathrm{q}]$

Calculo de f, functione reale de variabile reale, que pro omni valore de x, satisfac æquatione $\mathrm{D}fx = afx + b$, ubi a et b es quantitate constante.

·2 $a,b,m \varepsilon \mathrm{q} \,.\, m {-}{=} a \,.\supset:\, f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}f = [(afx+b\mathrm{e}^{mx})|x, \mathrm{q}] \,.=.$

$\mathrm{D}\{[\mathrm{e}^{-ax}fx - b\mathrm{e}^{(m-a)x}/(m-a)] \mid x, \mathrm{q}\} = (\iota 0 \,{:}\, \mathrm{q}) \,.=.$

$\{[\mathrm{e}^{-ax}fx - b\mathrm{e}^{(m-a)x}/(m-a)] \mid x, \mathrm{q}\} \,\varepsilon\, (\mathrm{qFq})\,\mathrm{const} \,.=:$

$x \varepsilon \mathrm{q} \,.\supset_x .\, \mathrm{e}^{-ax}fx - b\mathrm{e}^{(m-a)x}/(m-a) = f0 - b/(m-a) :=.$

$f = \{[(f0)\mathrm{e}^{ax} + b\mathrm{e}^{mx}/(m-a) - b\mathrm{e}^{ax}/(m-a)]|x, \mathrm{q}\}$

✻ 59. $m \varepsilon \mathrm{q} \,.\, n \varepsilon \mathrm{q}{-}\iota 1 \,.\, k \varepsilon \mathrm{Intrv} \,.\, a \varepsilon k \,.\supset:$

$f \varepsilon \mathrm{QF}k \,.\, \mathrm{D}f = [m(fx)^n|x, k] \,.=.$

$[(fx)^{-n}\mathrm{D}fx - m\, |x, k] = (\iota 0 \,{:}\, k) \,.=.$

$\mathrm{D}[(fx)^{-n+1}/(-n+1) - mx\, |x, k] = (\iota 0 \,{:}\, k) \,.=.$

$[(fx)^{-n+1}/(-n+1) - mx\, |x, k] \,\varepsilon\, (\mathrm{qF}k)\mathrm{const} \,.=.$

$[(fx)^{-n+1}/(-n+1) - (fa)^{-n+1}/(-n+1) - m(x-a)\, |x, k] = (\iota 0 \,{:}\, k) \,.=.$

$f = \{[(fa)^{-n+1} + (-n+1)m(x-a)]\upharpoonleft/(-n+1)|x, k\} \,.$

$k \supset \mathrm{q} \frown x \ni [(fa)^{-n+1} + (-n+1)m(x-a) > 0]$

Nos quære functione f positivo definito in aliquo intervallo k, que pro omni valore de x in k, satisfac æquatione $\mathrm{D}fx = m(fx)^n$, id es, que habe derivata proportionale ad aliquo potestate, diverso de 1, de functione. Æquatione vale $(fx)^{-n}\mathrm{D}fx - m = 0$; primo membro es derivata de functione scripto in linea 4, que resulta constante, vel æquale ad suo valore pro $x = 0$, unde resulta valore de functione f. Intervallo k contine solo valores de x, que redde basi positivo. Nam nos habe definito potestate cum exponente q, pro basi positivo.

Casu $n = 1$ es tractato in P57.

✻ 60. MOTU DE PUNCTO GRAVE.

$g \varepsilon \mathrm{v} \,.\supset:\, p \varepsilon \mathrm{pFq} \,.\, \mathrm{D}^2 p = (\iota g{:}\mathrm{q}) \,.=.\, p = [(p0 + t\mathrm{D}p0 + t^2 g/2)|t, \mathrm{q}]$

$[\ \ldots \,.=.\, \mathrm{D}p = [(\mathrm{D}p0 + tg)|t, \mathrm{q}] \,.=.\ \ldots\ ]$

Si g es vectore dato, tunc puncto mobile p, que pro omni valore de tempore t habe acceleratione constante g:

$$\mathrm{D}^2 pt = g$$

es dato per

$$pt = p0 + t(\mathrm{D}p0) + t^2 g/2,$$

et describe « parabola ».

Puncto materiale grave, in vacuo, prope superficie de Terra, habe acceleratione constante, dicto « gravitate ».

Dem. Æquatione: $\mathrm{D}^2pt = g$, pro omni t, post integratione (Prop. 24 pagina 289), fi: $\mathrm{D}pt = \mathrm{D}p0 - gt$; que post novo integratione sume forma scripto.

Galilei, *Dialoghi*, a.1638 ; Opere t. 13 p. 222 :

« Projectum, dum fertur motu composito ex horizontali æquabili, et ex naturaliter accelerato deorsum, lineam semiparabolicam describit in sua latione. »

❋ 61. Motu centrale.

$$p \varepsilon \mathrm{pFq} \,.\, o \varepsilon \mathrm{p} \,.\supset\therefore$$

$$x \varepsilon \mathrm{q} \,.\supset_x.\, \mathrm{D}^2px \,\varepsilon\, \mathrm{q}(px - o) \;:=.\; o \,\alpha\, p \,\alpha\, \mathrm{D}p \;\varepsilon\; (\mathrm{p}^3\mathrm{fq})\mathrm{const}$$

Dem. $x\varepsilon\mathrm{q} \,.\supset.\, \mathrm{D}[(o \,\alpha\, px \,\alpha\, \mathrm{D}px)|x, \mathrm{q}]x = o \,\alpha\, px \,\alpha\, \mathrm{D}^2px$

p es puncto functione de variabile reale, id es, p es puncto mobile; et o es punto fixo. Si pro omni valore de tempore x, semper es acceleratione de puncto parallelo ad vectore $px - o$, vel fortia que move puncto es semper directo verso centro o, tunc tripuncto $o \,\alpha\, px \,\alpha\, \mathrm{D}px$, id es triangulo de vertices o, px, $px + \mathrm{D}px$, considerato in magnitudine, plano suo, et in sensu, habe valore constante, dum varia tempore x. Et viceversa.

Newton, Principia l. 1 P 1:

« Areas, quas corpora in gyros acta radiis ad immobile centrum virium ductis describunt, et in planis immobilibus consistere et esse temporibus proportionales ».

❋ 62. Puncto grave in medio resistente.

$$a \varepsilon \mathrm{Q} \,.\, g \varepsilon \mathrm{v} \,.\supset:\, p \varepsilon \mathrm{pFq} \,.\, \mathrm{D}^2p = g - a\mathrm{D}p \,.=.$$

$$\mathrm{D}p = [(\mathrm{D}p0)\mathrm{e}^{-ax} - g(\mathrm{e}^{-ax} - 1)/a|x, \mathrm{q}] \,.=.$$

$$\mathrm{D}p = \mathrm{D}\{[-(\mathrm{D}p0)\mathrm{e}^{-ax}/a + g\mathrm{e}^{-ax}/a^2 + gx/a]|x, \mathrm{q}\} \,.=.$$

$$p = \{[p0 - \mathrm{D}p0\mathrm{e}^{-ax}/a + g\mathrm{e}^{-ax}/a^2 + gx/a + \mathrm{D}p0/a - g/a^2]|x, \mathrm{q}\}$$

$$.=.\; p = \{[p0 + (\mathrm{D}p0)(1 - \mathrm{e}^{-ax})/a + g(\mathrm{e}^{-ax} - 1 + ax)/a^2]|x, \mathrm{q}\}$$

$$.=.\; p = \{[(p0 + \mathrm{D}p0/a - g/a^2) + (g/a)x + (g/a^2 - \mathrm{D}p0/a)\mathrm{e}^{-ax}]|x, \mathrm{q}\}$$

Nos suppone que p es puncto functione de quantitate, que nos voca tempore, id es, p es puncto mobile. Nos suppone que suo acceleratione D^2p habe forma $g - a\mathrm{D}p$, ubi g es vectore dato, et a es quantitate positivo. Si puncto p es grave, et g es acceleratione gravitate, et si puncto move se in medio resistente, ubi resistentia es opposito et proportionale ad velocitate $\mathrm{D}p$, tunc D^2p habe forma scripto. Pro calcula p, nos nota que

æquatione es in Dp lineare et de primo ordine (58·1) unde nos deduce Dp in functione de tempore; nos calcula functione que habe pro derivata functione præcedente, et que pro x=0 habe valore $p0$. Ita nos habe p.

Trajectoria de puncto p es « linea exponentiale » vel « logarithmica ».

Si nos pone $qx = p0 + \mathrm{D}p0/a - g/a^2 + gx/a$,
seque $\lim[(px-qx)|x, \mathrm{q}, \infty] = 0$; qx describe recta, que es asymptoto de logarithmica.

{ Newton, a.1687 libro 2, Prop. 4. }

✻ 63. Æquatione lineare de ordine duo.

$$a \varepsilon \mathrm{q'}{-}\iota 0 \,.\supset\!:\, f \varepsilon \mathrm{q'Fq'} \,.\, \mathrm{D}^2 f = a^2 f \,.{=}.$$
$$f = \{[(f0)(e^{ax}+e^{-ax})/2 + (\mathrm{D}f0)(e^{ax}-e^{-ax})/(2a)] \,|x, \mathrm{q'}\}$$

Dem. $[\, f\varepsilon\, \mathrm{q'Fq'} \,.\, \mathrm{D}^2 f = a^2 f \,.{=}.\, [e^{ax}(\mathrm{D}^2 fx - a^2 fx) \mid x, \mathrm{q'}] = (\iota 0 \,!\, \mathrm{q'})$
$.{=}.\, \mathrm{D}[(e^{ax}\mathrm{D}fx - ae^{ax}fx) \mid x, \mathrm{q'}] = (\iota 0 \,!\, \mathrm{q'})$
$.{=}.\, [(e^{ax}\mathrm{D}fx - ae^{ax}fx) \mid x, \mathrm{q'}] \,\varepsilon\, (\mathrm{q'Fq'})\mathrm{const}$
$.{=}\!:\, x\varepsilon \mathrm{q'} \,.\supset_x .\, e^{ax}\mathrm{D}fx - ae^{ax}fx = \mathrm{D}f0 - af0$
$.{=}.\, \mathrm{D}f = [afx + (\mathrm{D}f0 - af0)e^{-ax} \mid x, \mathrm{q'}]$
$.{=} \ldots\,]$

Si a es quantitate dato, tunc omni functione de variabile, reale aut imaginario, que pro omni valore de x satisfac æquatione

$$\mathrm{D}^2 fx = a^2 fx,$$

id es, que habe derivata secundo proportionale ad functione, habe forma scripto.

✻ 64. Motu harmonico.

$$a\varepsilon \mathrm{Q} \,.\, o\varepsilon \mathrm{p} \,.\supset\!:\, p\varepsilon\, \mathrm{pFq} \,.\, \mathrm{D}^2 p = -a^2(p-o) \,.{=}.$$
$$p = \{[o + (p0-o)\cos ax + (\mathrm{D}p0)\sin(ax)/a] \,|x, \mathrm{q}\}$$

Dem. $(p-o,\, ia) \mid (f,\, a)\ \mathrm{P}63 \,.\supset.\ \mathrm{P}$

Si p es puncto functione de quantitate reale, que me voca tempore, id es, si p es puncto mobile, et suo acceleratione D^2p habe forma $-a^2(p-o)$, ubi a es quantitate positivo, et o es puncto fixo, id es, si D^2p es directo verso puncto o, et proportionale ad vectore $p-o$, tunc p habe expressione scripto.

Puncto describe ellipsi de centro o, et de hemi-diametros conjugato vectores $p0-o$ et $(\mathrm{D}p0)/a$. Motu habe periodo $2\pi/a$, vel $px = p(x+2\pi/a)$. Periodo non depende de $p0$ et $\mathrm{D}p0$: motu es isochrono.

Puncto p pertinente ad systema elastico, posito ex suo positione de æquilibrio o, tende ad o cum fortia proportionale (pro parvo deformatione) ad vectore $p-o$.

NEWTON a.1687 ; Principia l. 1, Prop. X :

« si vis sit ut distantia, movebitur corpus in ellipsi centrum habente in centro virium ».

❋ 65. SYSTEMA DE ÆQUATIONES DIFFERENTIALE LINEARE.

$n\varepsilon \mathrm{N}_1 \,.\, a\varepsilon\, \mathrm{Subst}n \;.\supset:$

$x\varepsilon\, \mathrm{Cx}n\, \mathrm{F}\, \mathrm{q} \,.\, \mathrm{D}x = ax \;.=.\; x = [(x0)\mathrm{e}\backslash(at)|t, \mathrm{q}]$

[(Subst | q) Dem 57]

Si n es numero, et a es substitutione de ordine n, tunc complexo x de ordine n, functione de numero reale, que satisfac æquatione differentiale $\mathrm{D}x = ax$, habe forma

$$xt = (x0)\mathrm{e}\backslash(at),$$

ubi t es variabile in campo de numeros reale.

Formula et demonstratione es identico ad theorema præcedente P57.

Æquatione $\mathrm{D}x = ax$ repræsenta systema de n æquatione differentiale lineare homogeneo ad coefficientes constante :

$$\mathrm{D}x_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n$$
$$\mathrm{D}x_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n$$
$$\ldots\ldots\ldots\ldots$$
$$\mathrm{D}x_n = a_{n1}y_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \,.$$

Exponentiale de substitutione es definito in V § e P11·0 pag. 249 :

$$\mathrm{e}^a = 1 + a + a^2/2! + a^3/3! + \ldots$$

Potestate n, et successivos, de a, pote es expresso per præcedentes, ope æquatione characteristico (pag. 151 Prop.5·4). Ergo $\mathrm{e}\backslash a$ es polynomio de gradu $n-1$ in a :

$$\mathrm{e}^a = \mathrm{Interpolante}\{\mathrm{e}^x|x;\ \mathrm{Radices}[\mathrm{Determinante}(a-h)|h,\mathrm{q}]\}a$$

« $\mathrm{e}\backslash a$ es functione interpolante (p.306) de exponentiale, calculato super radices de functione characteristico $\mathrm{Dtrm}(a-h)$, ubi h es reale, pro a.

✻ 66. DERIVATAS PARTIALE.

$u,v \varepsilon$ Intv . $f \varepsilon$ qF$(u \mathbin{:} v)$. $a \varepsilon u$. $b \varepsilon v$.⊃.

·0 $D_1 f(a,b) = D[f(x,b)|x,u]a$. $D_2 f(a,b) = D[f(a,y)|y,v]b$ Df

Si u et v es intervallo, et f es quantitate functione de duo variabile, primo in u, secundo in v, et si nos sume a in u et b in v, tunc $D_1 f(a,b)$ indica derivata de $f(x,b)$, ubi varia x in campo u, pro valore a. Ita $D_2 f$ indica derivata pro secundo variabile.

Plure auctore indica ce derivatas, dicto « derivatas partiale », per notationes :

$$D_1 f(x,y) = D_x f(x,y) = f'_x(x,y) = \frac{df(x,y)}{dx} = \frac{\partial f(x,y)}{\partial x}$$

$$D_2 f(x,y) = D_y f(x,y) = f'_y(x,y) = \frac{df(x,y)}{dy} = \frac{\partial f(x,y)}{\partial y}$$

In primo notatione, que nos seque, lice, in loco de x et y, pone duo valore determinato, p. ex. 0 et 0; nos habe derivatas

$$D_1 f(0,0) \qquad D_2 f(0,0).$$

Cetero notationes es incompleto, et debe es completato per lingua commune. Per ex. quod in primo notatione es indicato per $D_1 f(0,0)$, in secundo notatione debe es indicato per $D_x f(0,0)$ aut per $D_0 f(0,0)$? Et quod in primo notatione es indicato per $D_1 f(y,x)$, in secundo notatione debe es indicato per $D_x f(y,x)$ aut per $D_y f(y,x)$? Vide in pag. 278 citatione de Jacobi.

·1 $D_1 f, D_2 f \varepsilon$ qF$(u \mathbin{:} v)$. $x \varepsilon u$. $y \varepsilon v$.⊃. $f(x,y) - f(a,b) \varepsilon$ $(x-a) D_1 f`(a \frown x \mathbin{:} b \frown y) + (y-b) D_2 f`(a \frown x \mathbin{:} b \frown y)$

Dem. $f(x,y) - f(a,b) = [f(x,y) - f(a,y)] + [f(a,y) - f(a,b)]$ (1)

$f(x,y) - f(a,y) \varepsilon (x-a) D_1 f`(a \frown x, y)$ (2)

$f(a,y) - f(a,b) \varepsilon (y-b) D_2 f`(a, b \frown y)$ (3)

(1) (2) (3) ⊃ P

Si functione f habe ambo derivatas partiale in intervallos dato, et si nos sume alios valore x in u, et y in v, tunc differentia de duo valores de functione f vale incremento de x, per derivata de f pro primo variabile, plus incremento de y, per derivata de f pro secundo variabile, ubi derivatas es calculato per valores medio inter considerato.

$\cdot 2$ $\mathrm{D}_1 f, \mathrm{D}_2 f \,\varepsilon\, (\mathrm{q\,F}\, u \mathbin{:} v)\mathrm{cont} \,.\, w \varepsilon\, \mathrm{Intv} \,.\, x \varepsilon\, u \mathrm{F} w \,.\, y \varepsilon\, v \mathrm{F} w \,.\, t \varepsilon w \,.$
$\mathrm{D}xt, \mathrm{D}yt \varepsilon \mathrm{q} \;.\supset.\; \mathrm{D}[f(xt,yt)|t,w]t = \mathrm{D}_1 f(xt,yt)\,\mathrm{D}xt + \mathrm{D}_2 f(xt,yt)\,\mathrm{D}yt$

Dem. $\mathrm{P}\cdot 1 \,.\, t_1 \varepsilon w \;.\supset.\; f(xt_1,yt_1) - f(xt,yt) \;\varepsilon\; (xt_1 - xt)\mathrm{D}_1 f(xt \frown xt_1 \mathbin{\vdots} yt \frown yt_1)$
$+ (yt_1 - yt)\mathrm{D}_2 f($ » » $)$
$.\supset\colon \mathrm{D}[f(xt,yt)|t;t,t_1] = \mathrm{D}_1 f(\ldots)\,\mathrm{D}(x;t,t_1) + \mathrm{D}_2 f(\ldots)\mathrm{D}(y;t,t_1) \;.\supset.\; \mathrm{P}$

Si derivatas partiale de f es continuo, et si w es intervallo, et x es functione definito in campo w, que sume valores in u, et y es functione definito in campo w, que sume valores in v, et si t es valore in w, et existe derivatas de x et de y pro t, tunc derivata de $f(xt,yt)$, ubi varia t (litera t es apparente), pro t, vale derivata de f, pro primo variabile, multiplicato per derivata de x pro t, plus derivata de f, pro secundo variabile, multiplicato per derivata de y pro t.

Functione $f(xt,yt)$ vocare « functione composito » de t.

Exemplo. Derivata de producto $(xt)(yt)$ pro t, vale derivata pro primo factore xt, vel yt, per derivata de primo $\mathrm{D}xt$, plus derivata pro secundo factore yt, vel xt, per derivata $\mathrm{D}yt$, id es

$$(yt)(\mathrm{D}xt) + (xt)(\mathrm{D}yt),$$

secundo regula P6 de pag. 280.

Derivata de $(xt) \curlywedge (yt)$ vale derivata de potestate, pro basi variabile, vel $(yt)) \times (xt) \curlywedge (yt - 1)$, multiplicato per derivata de basi $\mathrm{D}xt$, plus derivata de potestate pro exponente variabile, vel $(xt) \curlywedge (yt) \times \log(xt)$ per derivata de exponente $\mathrm{D}yt$. Seque formula P13·4 de pag. 283.

$\cdot 3$ $\mathrm{D}_1 f, \mathrm{D}_2 f, \mathrm{D}_2\mathrm{D}_1 f \,\varepsilon\, \mathrm{qF}(u \mathbin{:} v) \,.\, x \varepsilon u \,.\, y \varepsilon v \;.\supset.$
$f(x,y) - f(a,y) - f(x,b) + f(a,b) \;\varepsilon\; (x-a)(y-b)\mathrm{D}_2\mathrm{D}_1 f(a \frown x \mathbin{\vdots} b \frown y)$

Dem. $f(x,y) - f(x,b) - f(a,y) + f(a,b) = \Delta \mathbin{;} [\Delta f(h,k)\,|k;b,y]\,|h;a,x \mathbin{;}$
$\varepsilon\; (x-a)\Delta[\mathrm{D}_1 f(h,k)\,|k;b,y] \mid h \text{`} a \frown x$
$\varepsilon\; (x-a)(y-b)\mathrm{D}_2\mathrm{D}_1 f(h,k) \mid k \text{`} b \frown y \mid h \text{`} a \frown x$

$\cdot 4$ $\mathrm{D}_1 f, \mathrm{D}_2 f, \mathrm{D}_2\mathrm{D}_1 f \,\varepsilon\, \mathrm{qF}(u \mathbin{:} v)\,\mathrm{cont} \;.\supset.\; \mathrm{D}_1\mathrm{D}_2 f = \mathrm{D}_2\mathrm{D}_1 f$

Dem. $\mathrm{D}_1\mathrm{D}_2 f(a,b) = \lim \mathbin{;} [\mathrm{D}_2 f(x,b) - \mathrm{D}_2 f(a,b)]/(x-a)\,|x,u,a \mathbin{;}$
$= \lim\lim\,[f(x,y) - f(a,y) - f(x,b) + f(a,b)]/[(x-a)(y-b)]\,|y,v,b\,|x,u,a$
$= \lim\lim\,\mathrm{D}_2\mathrm{D}_1 f(a \frown x, b \frown y)\,|y,v,b\,|x,u,a = \mathrm{D}_2\mathrm{D}_1 f(a,b)$

Si functione f de duo variabile habe derivatas $\mathrm{D}_1 f$, $\mathrm{D}_2 f$, et $\mathrm{D}_2\mathrm{D}_1 f$ continuo, tunc lice inverte ordine de derivationes, id es, lice commuta operationes D_1 et D_2.

Ex demonstratione resulta que nos debe suppone existentia et continuitate de $\mathrm{D}_2\mathrm{D}_1 f$. Tunc es necessario existentia de $\mathrm{D}_1 f$, que es continuo pro secundo variabile (pag. 279, P3·3), sed non es necesse suo continuitate pro

primo variabile. Nos suppone existentia de $D_2 f(a,y)$, pro aliquo valore a de primo variabile, et pro omni valore de secundo variabile y; tunc seque existentia de $D_2 f(x,y)$, et de $D_2 D_1 f(x,y)$ pro omni valore de x et de y.

$$\cdot 5 \quad h,k \varepsilon \mathrm{q} \;.\supset.\; (hD_1+kD_2)f(a,b) = hD_1 f(a,b)+kD_2 f(a,b) \qquad \mathrm{Df}$$

$$\cdot 6 \quad n \varepsilon N_1 : r,s \varepsilon N_0 \,.\, r+s \overline{\gtrless} n \;.\supset_{r,s}.\; D_1{}^r D_2{}^s f \,\varepsilon\, \mathrm{qF}(u{:}v)\mathrm{cont} \,.\, h \varepsilon u-a \,.\, k \varepsilon v-b \;.\supset.\; f(a+h,b+k) = \Sigma[(hD_1+kD_2)^r f(a,b)/r! \,|r, 0\cdots(n-1)] + (hD_1+kD_2)^n f(a+zh,b+zk)/n! \,|z{'}\theta$$

Si n es numero naturale, et pro omni dyade de numeros r et s, de summa non superiore ad n, semper existe derivata de ordine r pro primo variabile de derivata de ordine s pro secundo variabile, et es continuo, et si nos sume duo quantitate h et k, in modo que $a+h$ es u, et $b+k$ es v, tunc $f(a+h,b+k)$ es evolubile secundo potestates de h et de k, plus termine complementare. Operatione $(hD_1+kD_2)^r$ resulta definito per Prop. ·5, et pote es evoluto ut potestate de binomio.

✻ 67. Derivata de functione de numero complexo.

$$m,n \varepsilon N_1 \,.\, u \varepsilon \mathrm{Cls'Cx}n \,.\, f \varepsilon \mathrm{Cx}m\,\mathrm{F}\,u \,.\, x \varepsilon u \frown \delta u \;.\supset.\; Dfx = \iota\,(\mathrm{Cx}m\,\mathrm{f}\,\mathrm{Cx}n)\mathrm{lin} \frown g\ni[\lim\{[fy-fx-g(y-x)]/\mathrm{mod}(y-x)\,|y, u,x\} = 0] \qquad \mathrm{Df}$$

Derivata de functione reale de variabile reale (pag. 275), et de numero complexo, vectore, puncto, functione de variabile reale, es, per definitione, limite de ratione de incremento de functione ad incremento de variabile:

$$Dfx = \lim(fy-fx)/(y-x).$$

Si nos habe functione reale aut complexo de variabile complexo, non habe sensu ratione de duo numero complexo, et definitione præcedente non es applicabile. Sed pro numeros reale, derivata satisfac conditione:

$$\lim[fy-fx-(y-x)\times Dfx]/\mathrm{mod}(y-x) = 0,$$

que nos sume per definitione de derivata, si variabile independente es complexo.

Derivata de f, numero complexo de ordine m, functione de complexo de ordine n, in campo u, pro valore x prope u, es illo transformatio lineare g, tale que incremento de functio $fy-fx$ minus g de incremento de variabile $y-x$ diviso per $\mathrm{mod}(y-x)$, tende ad 0, quando y, in u, tende ad x.

Derivata es trasformatione que habe ut elementos derivatas partiale de coordinatas de functione pro coordinatas de variabile.

Derivata de functione de numero complexo es considerato per:

Jacobi a.1841, opera t.3 p.421. Grassmann a.1862, t. 2, p.295.

Vide in Geometria, derivata de potentiale P71.

❋ 68. Tang (Figura tangente).

$u \varepsilon$ Cls'p . $x \varepsilon$ p .⊃.

·0 Tang(u,x) = Lm{[$x+h(u-x)$] | h, Q, ∞}
= Lm[Homot$(x,h)u$ | h, Q, ∞] Df

Si u es figura, et x puncto, tunc Tang(u,x), lege « figura tangente ad u in puncto x », indica limite de figura homothetico de u cum centro de homothetia in x, quando ratione de homothetia h cresce ad infinito.

·1 $x \varepsilon$ ex$u$.⊃. Tang(u,x) = ⋀

Si x es puncto externo ad u, tunc figura tangente ad u in puncto es classe nullo.

·2 $x \varepsilon u$-δu .⊃. Tang(u,x) = ιx

Si x es puncto isolato de u, figura tangente contine solo puncto x.

·3 $x \varepsilon \delta u$.⊃. ∃ Tang(u,x)

Si x es puncto prope alios u, tunc semper existe punctos de figura tangente ad u in puncto x.

·4 $x \varepsilon$ in$u$.⊃. Tang(u,x) = p

Si x es interno ad u, figura tangente es toto spatio.

·5 $y \varepsilon$ Tang(u,x) -ιx .⊃. x+Q$(y-x)$ ⊃ Tang(u,x)

Si in figura tangente ad u, in puncto x, nos sume aliquo puncto y, differente de x, toto radio de origine x, et que i trans y, pertine ad figura tangente.

·6 $v \varepsilon$ Cls'p . v ⊃ u .⊃. Tang(v,x) ⊃ Tang(u,x)

Si figura v continere in u, et figura tangente ad v in puncto x continere in figura tangente ad u.

·7 $v \varepsilon$ Cls'p .⊃. Tang$(u \cup v, x)$ = Tang(u,x) ∪ Tang(v,x)

Operatione « Tang » es distributivo ad « ∪ ».

·8 Tang[Tang(u,x),x] = Tang(u,x)

❋ 69. $a,p \varepsilon$ p . $r \varepsilon$ Q . d(p,a) = r .⊃.

·1 Tang{p ∩ x∍[d(x,a) = r], p} = plan[p, I$(p-a)$]

Nos considera loco de punctos que dista ab puncto dato a per quantitate dato r, id es superficie de sphæra de centro a et de radio r. Tunc figura tangente ad superficie dicto, in suo puncto p, es plano per puncto p, et normale ad vectore $p-a$.

·2 $\text{Tang}\{\text{p} \frown x\ni[\text{d}(x,a) \leqq r], p\} = \text{plan}[p, \text{I}(p-a)] + \text{Q}_0(a-p)$

Figura tangente ad solido sphærico in puncto de superficie es semispatio limitato per plano præcedente.

·3 $a,x \,\varepsilon\, \text{p} \,.\, m,n \,\varepsilon\, \text{q} \,.\, m\text{U}(x-a) + n\text{U}(x-b) -= 0 \,.\supset.$
$\text{Tang}\{\text{p} \frown y\ni[m\text{d}(y,a) + n\text{d}(y,b) = m\text{d}(x,a) + n\text{d}(x,b)]\} =$
$\text{p} \frown z\ni\{(z-x) \times [m\text{U}(x-a) + n\text{U}(x-b)] = 0\}$

Tangente ad ovales de Descartes.

Descartes, *La Geometrie*, Leyde a. 1637, Œuvres t. 6, p. 429.

·4 $u \,\varepsilon\, \text{Intv} \,.\, p \,\varepsilon\, (\text{pF}u)\text{cont} \,.\, x \,\varepsilon\, \text{in}u \,.\, px -\varepsilon\, p\text{'}(u-\iota x) \,.\, \text{D}px \,\varepsilon\, \text{v}-\iota 0$
$.\supset.\ \text{Tang}(p\text{'}u, px) = \text{rectaT}(p, x)$

Puncto mobile p es functio definito et continuo in intervallo u, x es interno ad isto intervallo: puncto px non es multiplo de curva $p\text{'}u$, et habe derivata non nullo; tunc figura tangente (Tang) ad curva $p\text{'}u$ es recta tangente (rectaT), ante considerato.

✻ 70. Plano tangente ad superficie.

·1 $u,v \,\varepsilon\, \text{Intv} \,.\, p \,\varepsilon\, \text{pF}(u\text{:}v) \,.\, x \,\varepsilon\, \text{in}u \,.\, y \,\varepsilon\, \text{in}v \,.\, p(x,y) -\varepsilon\, p\text{'}(u\text{:}v)-\iota(x,y).$
$\text{D}_1 p, \text{D}_2 p \,\varepsilon\, [\text{vF}(u\text{:}v)]\text{cont} \,.\, \text{D}_1 p(x,y) -= 0 \,.\, \text{D}_2 p(x,y) -\varepsilon\, \text{q}\, \text{D}_1 p(x,y)$
$.\supset.\ \text{Tang}[p\text{'}(u\text{:}v), p(x,y)] = \text{plan}[p(x,y), \text{D}_1 p(x,y), \text{D}_2 p(x,y)]$

Dem. $\text{Hp} \,.\, u',v' \,\varepsilon\, \text{Intv} \,.\, u' \supset u \,.\, v' \supset v \,.\, x \,\varepsilon\, \text{in}\, u' \,.\, y \,\varepsilon\, \text{in}\, v' \,.\supset:$
$a \,\varepsilon\, \text{Tang}[p\text{'}(u\text{:}v), p(x,y)] \,.=.\, a \,\varepsilon\, \text{Tang}[p\text{'}(u'\text{:}v'), p(x,y)]$
$.=.\ \lim \text{dist}\{a, p(x,y) + h[p(u',v') - p(x,y)] | h, \text{Q}, 0\} = 0$
$.=.\ \lim \text{dist}[a, p(x,y) + h(u'-x)\,\text{D}_1 p(u',v') + h(v'-y)\text{D}_2 p(u',v')] = 0$
$.=.\ \lim \text{dist}\{a, \text{plan}[p(x,y), \text{D}_1 p(u',v'), \text{D}_2 p(u',v')]\} = 0$
$.=.\ \text{dist}\{a, \text{plan}[p(x,y), \text{D}_1 p(x,y), \text{D}_2 p(x,y)]\} = 0$
$.=.\ a \,\varepsilon\, \text{plan}[p(x,y), \text{D}_1 p(x,y), \text{D}_2 p(x,y)]$

Es dato duo intervallo u et v, et puncto mobile p, functione de duo variabile, in dicto intervallos. Tunc puncto p describe « superficie », et duo variabile es dicto « coordinatas curvilineo de puncto in superficie ».

Nos sume valore x interno ad u, et y interno ad v, et suppone que $p(x,y)$ es puncto simplice de superficie descripto per puncto p; tunc si puncto habe derivatas pro ambo variabile, continuo, non nullo et non parallelo, figura tangente ad superficie loco de punctos p, in puncto considerato, es plano per puncto et que contine derivatas de puncto pro ambo variabile. Ce plano vocare « plano tangente ».

Si in superficie es descripto curva que habe tangente in suo puncto p, tunc tangente ad curva jace in plano tangente ad superficie. Seque de Prop. 68·6.

Plure Auctore sume ce proprietate ut definitione. « Plano tangente ad superficie in suo puncto p » es definito ut « plano que contine recta tangente in p ad omni curva, descripto in superficie, et que i trans p ».

Tunc, si per puncto p, in superficie dato, nos duce linea sine tangente (ut spira mirabile in suo polo, loxodromia in suos polo, etc.), plano tangente contine tangente ad linea, que non habe tangente; quod es contradictorio.

Aliquo Auctore corrige definitione præcedente, et voca plano tangente « plano que contine tangente ad dicto curvas, que habe tangente ». Tunc omni plano es tangente ad superficie, que contine nullo linea cum tangente. Es tale superficie genito per rotatione de curva $y = x \sin 1/x$, circa oy, in puncto o.

Vide alio Df de plano tangente in Formul. t.4 p.295.

·2 $k \varepsilon \mathrm{Intv} \,.\, a \varepsilon \mathrm{pF}k \,.\, u \varepsilon (\mathrm{v} - \iota 0)\mathrm{F}k \,.\, x \varepsilon k \,.\, y \varepsilon \mathrm{q} \,.\, \mathrm{D}ax, \mathrm{D}ux \,\varepsilon \mathrm{v} \,.$
$\mathrm{D}ax + y\mathrm{D}ux \,-\varepsilon\, \mathrm{q}\, ux \;.\supset.\; \mathrm{Tang}[\bigcup \mathrm{recta}(ax,ux) \,|x`k,\, ax + yux]$
$= \mathrm{plan}(ax,\, ux,\, \mathrm{D}ax + y\mathrm{D}ux)$

Dem. $\mathrm{plan}[\bigcup \mathrm{recta}(ax,ux)|x`k,\, au + yux]$
$= \mathrm{plan}[(ax + yux)|(x;y)`(k:\mathrm{q}),\, ax + yux]$
$= \mathrm{plan}\{ax + yux,\, \mathrm{D}_1[(ax + yux)|(x,y)](x,y),\, \mathrm{D}_2[(ax + yux)|(x,y)](x,y)\}$
$= \mathrm{plan}(ax + yux,\, \mathrm{D}ax + y\mathrm{D}ux,\, ux)$
$= \mathrm{plan}(ax,ux,\, \mathrm{D}ax + y\mathrm{D}ux)$

Nos considera puncto a et vectore non nullo u, ambo functione dato in intervallo k.

Es dato x in intervallo k, et quantitate reale y. Nos suppone existentia de derivatas de a et de u, et que vectore $\mathrm{D}ax + y\mathrm{D}ux$ non es parallelo ad ux. Superficie loco de rectas per ax et parallelo ad ux, ubi x varia in intervallo k, es expresso per $\bigcup \mathrm{recta}(ax,ux)|x`k$.

Theorema dice que figura tangente ad ce superficie in suo puncto $ax + yux$ es plano per ax, et parallelo ad vectores ux et $\mathrm{D}ax + y\mathrm{D}ux$.

Superficie loco de recta mòbile = F. surface reglée = D. Regelfläche = I. superficie rigata.

F. règle =|| D. Regel ⊂ L. regula.

I. riga ⊂ Germanico : riga ⊃ D. Reihe, A. row.

Ce superficie habe in Germania nomen Latino, et in Italia nomen Germanico.

❋ 71. DERIVATA DE POTENTIALE.

$u \varepsilon$ qFp . $x \varepsilon$p .⊃. Dux =
 ι v ⌒ $v \ni$[lim{[$uy - ux - v\times(y-x)$]/mod$(y-x)$ |y, p, x{ = 0] Df

Quantitate reale functione de positione de puncto in spatio vocare « potentiale », nam uno suo interpretatione es « potentiale » de Mechanica.

Si u es potentiale, et x es puncto, tunc Dux, lege « derivata de functione u in puncto x », indica illo vectore v tale que differentia $uy-ux$ de duo valore de functione, minus producto interno de vectore v per vectore $y-x$ differentia de duo positione de puncto, diviso per mod$(y-x)$, tende ad 0, quando puncto y, in spatio, verge ad x.

Ce definitione es analogo ad definitione de derivata de functione de numero complexo, dato in P67·0. Derivata g de P67·0 responde ad $v\times$ de præsente definitione.

Lamé (JdM. a.1840 t.5 p.316) voca « parametro differentiale de primo ordine de functio u in puncto x » valore absoluto de Dux.

Hamilton considera illo ut vectore, quem indica per ∇, et voca Nabla. Vide IrishT. t. 3, Quaternions t. 2, p. 432.

Nabla, G νάβλα ⊂ Hebraico; instrumento musico, in forma de ∇.

Idem vectore in plure libro (Gans) vocare « gradiente ».

Du es « vi respondente ad Functio de vi u (Hamilton), vel ad potentiale $-u$ », et « fluxu de calore pro temperatura $-u$ ».

« Potentiale » es considerato per Laplace. Green a.1828 introduce vocabulo.

·1 $a,x\varepsilon$p . $i\varepsilon$v .⊃. D[$i\times(x-a)$ |x, p]$x = i$

Dem. $y\varepsilon$p .⊃. $i\times(y-a) - i\times(x-a) = i\times(y-x)$.⊃. P

·2 $a,x\varepsilon$p .⊃. D[$(x-a)^2$ | x, p]$x = 2(x-a)$

Dem. $(y-a)^2-(x-a)^2 = (y-x)^2 + 2(x-a)\times(y-x)$.⊃.
[$(y-a)^2-(x-a)^2-2(x-a)\times(y-x)$]/mod$(y-x)$ = $(y-x)\times$U$(y-x)$.⊃
lim[$(y-a)^2-(x-a)^2-2(x-a)\times(y-x)$]/mod$(y-x)$|$y$, p, x] = 0

·3 $a\varepsilon$p . $x\varepsilon$ p-ιa .⊃. D[d(x,a) |x, p]x = U$(x-a)$

Dem. D[d(x,a)|x, p]x = D[mod$(x-a)$|x, p]x = D[$\sqrt{(x-a)^2}$|x, p]x =
$2(x-a)/[2\sqrt{(x-a)^2}] = (x-a)/mod(x-a)$ = U$(x-a)$]

Derivata de distantia de puncto mobile x ad puncto a, si x es differente de a, vale vectore unitario de a ad x.

·4 $a \varepsilon\, p^2 \cup p^3 \,.\, x \varepsilon\, p - a \;.\supset.\; D[d(x,a) | x, p]x = U[x - (\mathrm{proj}\; a)x]$

Si a indica recta vel plano, et x es puncto ex a, tunc derivata de distantia de x ad figura a es vectore unitario secundo projectione super a de x ad x.

·5 $k \varepsilon\, \mathrm{Cls}'p \,.\, k = \lambda k \,.\, x \varepsilon\, p - k \,.\, a \varepsilon k \,.\, d(x,a) = d(x,k) : y \varepsilon\, k - \iota a \,.\supset_y .$
$d(x,a) < d(x,y) :\supset.\; D[d(x,k) | x, p]x = U(x - a)$

Derivata de distantia de puncto x ab figura k es vectore unitario in directione de distantia, in hypothesi scripto.

·6 $a \varepsilon p \,.\, i \varepsilon\, v - \iota 0 \,.\, x \varepsilon\, p - (a + q i) \;.\supset.\; D[\mathrm{ang}(x - a, i) | x, p]x =$
$U\{[\mathrm{cmp} \perp (x - a)] i\} / d(x,a)$

✻ 72.

·1 $u \varepsilon\, qFp \,.\, x \varepsilon p \,.\, ux = \max u\text{‘}p \,.\, Dux\, \varepsilon v \;.\supset.\; Dux = 0$

·11 (min | max) P·1

Si potentiale u es maximo pro aliquo puncto x, pro que existe derivata de u, tunc ce vectore derivata vale 0.

Idem pro minimo.

·2 $u \varepsilon\, qFp \,.\, k \varepsilon\, \mathrm{Cls}'p \,.\, x \varepsilon k \,.\, ux = \min u\text{‘}k \,.\, Dux\, \varepsilon v \,.$
$y \varepsilon\, \mathrm{Tang}(k,x) \;.\supset.\; (Dux) \times (y - x) \geqq 0$

Si nos considera solo punctos de aliquo figura k, et si potentiale sume pro puncto x, valore minimo inter valores respondente ad punctos de figura k, et derivata de u in x es vectore determinato, tunc producto de ce vectore per omni variatione $y - x$ de puncto in figura tangente ad k in x, es positivo aut nullo.

·3 $u \varepsilon\, qFp \,.\, k \varepsilon\, \mathrm{Cls}'p \,.\, x \varepsilon k \,.\, ux = \max u\text{‘}k \,.\, Dux\, \varepsilon v \,.$
$y \varepsilon\, \mathrm{Tang}(k,x) \;.\supset.\; (Dux) \times (y - x) \leqq 0$

Regula simile subsiste pro maximo.

·4 $u \varepsilon\, qFp \,.\, x \varepsilon p \,.\, Dux\, \varepsilon\, v - \iota 0 \;.\supset.$
$\mathrm{Tang}[p \cap y \ni (uy = ux), x] = \mathrm{plan}(x, \mathrm{I}Dux)$

Figura loco de punctos y, que satisfac conditione $uy = ux$, vocare «superficie æquipotentiale» que i trans x.

Si nos considera solo punctos in plano dato, in loco de superficie, occurre « linea æquipotentiale ».

Si derivata de potentiale u, in puncto x, es vectore non nullo, tunc figura tangente ad superficie æquipotentiale que transi per x, es plano per x, et normale ad vectore $\mathrm{D}ux$. Id es, $\mathrm{D}ux$ es vectore « normale » ad superficie æquipotentiale.

Applicationes.

1. Normale ad loco de punctos x que redde constante $\mathrm{d}(x,a)+\mathrm{d}(x,b)$, ubi a et b es puncto dato (ellipsi de foco a et b), es directo secundo $\mathrm{U}(x-a)+\mathrm{U}(x-b)$, vel secundo bisectrice de radios focale. (Apollonio).

2. Si es constante summa de distantias de x ad plure puncto fixo a_1,... normale es directo secundo vectore $\mathrm{U}(x-a_1)+\ldots$
(Leibniz, Math.S a.1693 t.6 p.233).

3. Si es constante functione $f(r_1, r_2)$, ubi $r_1 = \mathrm{d}(x, a_1)$, $r_2 = \mathrm{d}(x,a_2)$, vectore $\mathrm{D}_1 f(r_1,r_2)\mathrm{U}(x-a_1) + \mathrm{D}_2 f(r_1,r_2)\mathrm{U}(x-a_2)$ es normale ad loco.
(Poinsot a.1806, p.206).

4. Puncto que redde minimo summa de distantias ab plure puncto dato es in æquilibrio sub actione de fortias æquale inter se, et directo ad punctos dato. (Steiner t. 2, p.95).

Vide demonstratione et alio applicationes de propositiones præcedente in meo libro a. 1887, p.131-151.

Vide etiam Hurwitz MA. t. 22, p.231, Wetzig JfM. t. 62, p. 346, Baker AmerJ. t. 4 p.327, Sturm JfM. t. 96 p.36, t. 97 p.49.

✻ 73. Relatione inter potentiale et energia.

$$u \varepsilon \mathrm{qFp} \,.\, p \varepsilon \mathrm{pFq} \,.\, \mathrm{D}^2 p = -\mathrm{D}up \;.\supset.\; \{[(\mathrm{D}pt)^2/2 + upt] \mid t, \mathrm{q}\} \;\varepsilon\; \mathrm{const}$$

Dem. $\mathrm{D}[(\mathrm{D}p)^2/2+up] = \mathrm{D}p\times\mathrm{D}^2p+\mathrm{D}up\times\mathrm{D}p = \mathrm{D}p\times(\mathrm{D}^2p+\mathrm{D}up) = 0$

Si u es quantitate functione de positione de puncto, vel potentiale, et si p es puncto mobile, vel puncto materiale cum massa 1, et si acceleratione de puncto vale vi respondente ad potentiale u, tunc summa de energia cum potentiale, dum varia tempore, es constante.

·1 $u \varepsilon \mathrm{qFp} \,.\, p \varepsilon \mathrm{pFq} \,.\, (\mathrm{D}^2 p + \mathrm{D}up)\times\mathrm{D}p = (\iota 0{:}\mathrm{q}) \;.\supset.$ Ths

Idem fi, si puncto p move se, in modo que suo velocitate es semper normale ad $\mathrm{D}^2p + \mathrm{D}up$, id es si vi D^2p que move illo es summa de $-\mathrm{D}u$, vi de potentiale u, plus vi normale ad trajectoria de puncto. Ce casu se præsenta, si puncto es mobile super linea dato, aut super superficie dato, sine attrito.

VII

CALCULO INTEGRALE

VII. CALCULO INTEGRALE.

Primo idea de integrale se præsenta ad mathematicos in mensura de area et de volumen.

Si nos nosce area de polygonos, et si nos habe, in dato plano, figura (non polygonale), tunc nos voca « polygono circumscripto » polygono que contine in suo interno figura dato; et « polygono inscripto », polygono interno ad figura dato. Ce vocabulos habe valore etymologico, et non ut in Geometria elementare.

Limite infero de areas de polygonos circumscripto vocare « area supero de figura ». Limite supero de polygonos inscripto vocare « area infero ». Si area supero æqua infero, valore commune de duo area vocare « area de figura dato ».

Nos exprime per symbolos de Analysi ideas præcedente, et nos habe definitione de integrale.

❋ 1. s′ = Polygono circumscripto.

$u \varepsilon$ Intv . $f \varepsilon$ q f u . l′f‘u, l$_{\prime}f$‘u εq . ⊃:

Nos suppone: u es intervallo; f indica quantitate functione in intervallo u; limite supero et limite infero de valores de f in u es quantitate finito.

·0 s′(f,u) = (Long$u$) (l′f‘u) Df s′

s′(f,u), que nos lege « rectangulo circumscripto ad figura diagramma de functione f, super basi u », indica producto de amplitudo de intervallo u per limite supero de valores de f in u.

·1 $h \varepsilon \mathrm{Cls}'u \,.\, \mathrm{l}'u, \mathrm{l}_{\prime}u \,\varepsilon h \,.\, \mathrm{Num}h \,\varepsilon \mathrm{N}_1 \,.\supset.$
$\mathrm{s}'(f,u,h) = \Sigma\{\mathrm{s}'[f, (\min_r h)\overline{}(\min_{r+1} h)] \,|r, 1\cdots(\mathrm{Num}h-1)\}$ Df

Si h es classe de numeros in intervallo u, continente $\mathrm{l}'u$ $\mathrm{l}_{\prime}u$ extremos de u, et in numero finito, tunc $\mathrm{s}'(f,u,h)$, lege « poly gono circumscripto ad figura f, super basi u diviso in partes per valores h », indica summa de rectangulos $\mathrm{s}'(f,x\overline{}y)$, ubi x es uno ex numeros h, et y es numero successivo.

$\min_r h$ « minimo de loco r de h » indica elemento que habe loco r in classe h de numeros, disposito in ordine crescente, et es definito in pag. 120.

·2 $h \varepsilon \mathrm{Cls}'u \,.\, \mathrm{Num}h \,\varepsilon \mathrm{N}_0 \,.\supset.\, \mathrm{s}'(f,u,h) = \mathrm{s}'(f,u,h \cup \iota \mathrm{l}_{\prime}u \cup \iota \mathrm{l}'u)$ Df

Si h non contine extremos de intervallo u, adde illos.

✻ 2. $\mathrm{s}_{\prime}$ = Polygono inscripto.

$u \varepsilon \mathrm{Intv} \,.\, f \varepsilon \mathrm{q}fu \,.\, \mathrm{l}'f`u, \mathrm{l}_{\prime}f`u \,\varepsilon \mathrm{q} \,.\supset:$
$(\mathrm{s}_{\prime}, \mathrm{l}_{\prime}) \mid (\mathrm{s}', \mathrm{l}')$ P1·0·1·2

$\mathrm{s}_{\prime}(f,u,h)$, lege « polygono inscripto in figura f, super basi u, diviso per h », es definito ut s', mutato limite supero in infero

·1 $h \varepsilon \mathrm{Cls}'u \,.\, \mathrm{Num}h \,\varepsilon \mathrm{N}_0 \,.\supset.$
$\mathrm{s}'(f,u) \geqq \mathrm{s}'(f,u,h) \geqq \mathrm{s}_{\prime}(f,u,h) \geqq \mathrm{s}_{\prime}(f,u)$

Rectangulo circumscripto super basi u supera polygono circumscripto super basi diviso per h, que supera polygono inscripto, que supera rectangulo inscripto; æqualitate es incluso

Dem. $x,y \varepsilon h \,.\supset.\, \mathrm{l}'f`u \geqq \mathrm{l}'f`x\overline{}y \geqq \mathrm{l}_{\prime}f`x\overline{\phantom{x}}y \geqq \mathrm{l}_{\prime}f`u$

·2 $h,k \varepsilon \mathrm{Cls}'u \,.\, \mathrm{Num}h, \mathrm{Num}k \,\varepsilon \mathrm{N}_0 \,.\supset.$
$\mathrm{s}'(f,u,h) \geqq \mathrm{s}'(f,u,h \cup k) \geqq \mathrm{s}_{\prime}(f,u,h \cup k) \geqq \mathrm{s}_{\prime}(f,u,k)$
[P·1 ⊃P]

Si h et k es duo divisione de intervallo u, in numero finito, tunc polygono circumscripto pro divisione h supera aut æqua polygono circumscripto pro divisione $h \cup k$, que es subdivisione de divisione h. Ita polygono inscripto pro subdivisione $h \cup k$ de divisione k supera polygono inscripto pro divisione k.

✻ 3. S′ = Integrale supero. $S_{\prime}$ = Integrale infero.

$u \varepsilon$ Intv . $f \varepsilon$ qf$u$. $l'f`u$, $l_{\prime}f`u$ εq .⊃:

·0 $S'(f,u) = l_{\prime}[s'(f,u,h) | h$ ‘ Cls′$u \frown h\ni$(Num$h$ εN_0)] Df S′

·1 $S_{\prime}(f,u) = l'[s_{\prime}$— — ——————————— Df $S_{\prime}$

Si u et f habe sensu ut in P1, tunc $S'(f,u)$, lege « integrale supero de functione f, in intervallo u », indica limite infero de valores que sume $s'(f, u, h)$, polygono circumscripto, ubi h indica omni divisione de u, in numero finito de partes.

$S_{\prime}(f,u)$, « integrale infero », es limite supero de polygonos inscripto.

·2 $S'(f,u)$, $S_{\prime}(f,u)$ εq . $S'(f,u) \geqq S_{\prime}(f,u)$

Dem. Df S′ $S_{\prime}$. p.116 §q P10·1 .⊃. P

Integrale supero et integrale infero habe valore determinato et finito; integrale supero es majore aut æquale ad integrale infero.

✻ 4. S = Integrale.

u ε Intv . $f \varepsilon$ qf$u$. $l'f`u$, $l_{\prime}f`u$ εq .⊃:

·0 $S(f,u) = \imath[\iota S'(f,u) \frown \iota S_{\prime}(f,u)]$ Df S

Si functione f, dato in intervallo u, habe limite supero et infero de suo valores finito, tunc $S(f,u)$, lege « integrale de f, in intervallo u », indica valore commune de integrale supero et de integrale infero.

Ergo $S(f,u)$ habe sensu, si integrale supero et infero coincide secundo Df de $\imath$, pag. 13. Propositione $S(f,u)$ ε q « integrale es quantitate » vel « existe in campo de quantitates », es sæpe indicato per « integrale existe », vel « functione es integrabile in intervallo u ». Vocabulo « existe » in tale expressione, et in similes, ut « limite existe », « derivata existe », habe valore de symbolo ε, diverso de ∃, que præcede semper nomen de classe.

·1 $S(f,u)\,\varepsilon\,\mathrm{q} \;.=.\; S'(f,u) = S_{,}(f,u)$

·2 ——— $.\supset.$ —————— $= S(f,u)$

Ut functione f es integrabile in intervallo u, es necesse et suffice que integrale supero æqua integrale infero.

Valore commune de illos es valore de integrale.

·3 $S(f,u)\,\varepsilon\,\mathrm{q} \;.=:\; m\,\varepsilon\,\mathrm{Q} \,.\supset_m.\, \exists h \ni [h\,\varepsilon\,\mathrm{Cls}'u \,.\, \mathrm{Num}h\,\varepsilon\,\mathrm{N}_0 \,.$
$s'(f,u,h) - s_{,}(f,u,h) < m]$

Dem. P·1 . Def S′ S, ⊃ P

Ut integrale de functione f in campo u existe, es necesse et suffice que, pro omni quantitate positivo (parvo ad arbitrio) m, responde divisione h de intervallo u, in numero finito, ita ut differentia inter polygono circumscripto et polygono inscripto es minore de m.

Æquivale ad conditione præcedente, post substitutione de integrales supero et infero per definitiones.

·4 $S(f,u) = \imath y \ni \{h\,\varepsilon\,\mathrm{Cls}'u \,.\, \mathrm{Num}h\,\varepsilon\,\mathrm{N}_1 \,.\, 1_{,}u, 1'u\,\varepsilon\,h \,.\supset_h.\, y\,\varepsilon$
$\Sigma[(\min_{r+1}h - \min_r h)\mathrm{Med}f`(\min_r h \frown \min_{r+1}h) | r, 1\cdots(\mathrm{Num}h-1)]\}$ Dfp

Definitione possibile de integrale, independente de S′ et $S_{,}$.

Si nos divide intervallo in partes, et multiplica amplitudo de intervallos partiale per valores medio de functione in illos, et summa ce productos, nos habe classe de quantitates, que contine integrale; et integrale es solo numero que habe ce proprietate, pro omni divisione de intervallo.

Historia.

Integrale, sub nomen de area et de volumen, occurre in Euclide et in Archimede. Vide Prop. 16.

Kepler a. 1605, Cavalieri a. 1639, Wallis a.1655 considera integrale ut « summa valores de functione ».

Leibniz (A. Erud. a.1686) in loco de « summa » introduce ∫, suo initiale sub forma typographico de illo tempore.

Jac. Bernoulli, AErud. a.1690. voca illo « integrale » nomen hodie commune inter nationes:

integrale, A.D.H. integral, F. intégrale, I. integrale, R. integral'.
⊂ integra + -le (p. 19 N. 6).

integra = fac integrale. A. integra-te, D. integr-ieren, F. intégre-r, I. integra-re, H. —r, R. integr-ir-ovatį. ⊃ A.D.F. integra-tion.
⊂ integro (p. 157 N. 240) — -o + -a (p.18 N.4).

Cauchy a.1823, Œuvres s.2 t.4 p.122, defini integrale ut limite, per P5·2.
Darboux a. 1875 (vide Prop. 5, et in idem anno Ascoli et Thomæ) considera S′ S, ut limite de summas s′ s,.

Substitutione de limite supero de classe (l′) ad limite de functione (lim) reduce definitiones ad forma præcedente simplicissimo, sed pauco noto. Vide meo scripto *Sulla definizione di integrale*, Annali di Matem. a.1895.

« Integrale de functione f in in intervallo de a ad b » es indicato in generale per $\int_a^b f(x)dx$.

Parenthesi circa x in $f(x)$ es inutile, ut es dicto in pag. 74, et produce confusione cum usu de parenthesi commune in Arithmetica et in Algebra. Ergo integrale pote es scripto $\int_a^b fx\, dx$.

Signo dx indica que variabile de integratione es x. Ergo signo « d » habe in integrales, valore de signo de variabilitate « | » (pag. 77), que jam occurre in Σ (pag. 120), Π, D,... Tunc integrale sume forma $\int_a^b fx|x$.

Nunc $fx|x = f$, ut resulta de definitione, pag. 77. Ergo integrale sume forma $\int_a^b f$; et si nos scribe literas variabile super uno linea, $\int(f,a,b)$.

Integrale, ut es definito, depende de functione f, et de campo de integratione u. Si campo es intervallo de a ad b, tunc es functione symmetrico de extremos: $a\overline{\ }b = b\overline{\ }a$ (vide pag. 118). Sed campo u pote es campo arbitrario.

Ergo, si nos scribe elementos que occurre in integrale, resulta notatione S(f,u), que nos adopta.

Signo « d » in commune notatione habe officio de signo « | » ; illo vocare « differentiale », et in suo definitione Auctores non es concorde. Secundo Leibniz (vide pag. 277), dx es quantitate arbitrario, et lice pone $dx = 1$. Plure Auctore dice que dx es « infinitesimo », id es « quantitate variabile que verge ad 0 ». Sed omni quantitate indicato per litera x es constante in idem formula, et variabile cum formula; distinctione de quantitates in constante et variabile care sensu. Alio Auctore dice que dx es « infinitesimo constante » ; et discussione cum methodo de Metaphysica super infinitesimo produce multo obscuritate. Vide discussiones in Mem. Bologna a. 1895 p. 399, in Bulletin de l'Académie de Belgique a. 1901 p. 549-589, et plure alio. Substitutione de signo d per « | » elimina omni ambiguitate.

Vide *Encyclopædie* t. 2 p. 69.

❋ 5·1 $u \,\varepsilon\, \mathrm{Intv} \,.\, f \varepsilon\, \mathrm{q}f u \,.\, \mathrm{l}'f`u, \mathrm{l}_{,}f`u \,\varepsilon\, \mathrm{q} \,.\, m \varepsilon \mathrm{Q} \,.\supset.$

$\exists\, \mathrm{Q} \frown n \ni [h \varepsilon\, \mathrm{Cls}'u \,.\, \mathrm{Num}h\, \varepsilon \mathrm{N}_0 \,.\, \mathrm{l}'u, \mathrm{l}_{,}u \varepsilon h : r \varepsilon\, 1 \cdots (\mathrm{Num}h - 1) \,.\supset_r.$
$\mathrm{min}_{r+1} h - \mathrm{min}_r h < n \,:\supset_h .\, \mathrm{s}'(f,u,h) - \mathrm{S}'(f,u) < m]$

{ Darboux a.1875 p.66 }

$S'(f,u)$, que nos defini ut limite infero de valores de $s'(f,u,h)$, es etiam limite de $s'(f,u,h)$, quando varia lege de divisione h, in modo que maximo amplitudo de intervallos partiale verge ad 0. Ce limite non es « lim » de successione, definito in p.214; et non es limite de uno vel plure variabile continuo, considerato in p.230 et sequentes. Nam lege de divisione h depende de punctos de divisione, et Num$h$ cresce sine limite, quando amplitudo de omni intervallo verge ad 0. Ergo vocabulo « limite » habe novo valore. Nos elimina suo definitione, cum enuntiato sequente de theorema de Darboux:

« Si f es functione reale dato in intervallo u, limitato supra et infra, tunc, si nos fixa quantitate positivo m (parvo ad arbitrio), semper existe alio quantitate positivo n, ita ut, pro omni divisione h de intervallo u, in numero finito de partes, si amplitudo de omni parte es minore de n, tunc differentia inter s' respondente ad divisione h, et integrale supero, semper es minore de m ».

Idea « limite » pote etiam reducto ad limite de successione, per §lim P42·7 (p.233). Ut casu particulare, si nos divîde intervallo in n partes æquale, tunc:

·2 Hp·1 . $a = 1_{,}u$. $b = 1'u$.Ɔ. $S'(f, a\frown b) =$
$\lim s'[f, a\frown b, a+(0\cdots n)(b-a)/n]|n$ Dfp

·3 Hp·2 : $S(f,a\frown b)\,\varepsilon\,q$.Ɔ. $S(f,a\frown b) =$
$(b-a)\lim \Sigma\{f[a+r(b-a)/n]|r, 0\cdots(n-1)\}/n\,|n$

Integrale, si existe, diviso per longore de intervallo, es limite de valore medio arithmetico inter n valore de functione, respondentes ad valores de variabile in progressione arithmetico. Plure auctore voca isto limite « valore medio de functione ».

✻ 6. Decompositione de intervallo basi.

$a,b\,\varepsilon\,q$. $a<b$. $f\,\varepsilon\,q\,f\,a\frown b$. $1'f`a\frown b, 1_{,}f`a\frown b\;\varepsilon\;q$. $c\,\varepsilon\,a{}^{-}b$.Ɔ:

·0 $S'(f, a\frown b) = S'(f, a\frown c) + S'(f, c\frown b)$

a et b es quantitate, a es minore de b; f es quantitate functione dato in intervallo ab a ad b, et habe limite supero et infero de suo valores, ambo finito; et c es valore interno ad intervallo ab a ad b. Tunc integrale supero de f, ab a ad b, vale integrale supero de f ab a ad c, plus integrale supero de f ab c ad b.

Dem. $h \varepsilon \mathrm{Cls}'a\frown b \,.\, \mathrm{Num}h \,\varepsilon\, \mathrm{N}_0 \,.\, c\varepsilon h \;.\supset.$
$\mathrm{s}'(f, a\frown b, h) = \mathrm{s}'(f, a\frown c, h\frown a\frown c) + \mathrm{s}'(f, c\frown b, h\frown c\frown b)$.
Oper l, $.\supset.$ P

In vero, nos sume divisione h de intervallo totale, in numero finito, que contine puncto c. Tunc polygono circumscripto, super basi ab a ad b, diviso per h, vale polygono super basi $a\frown c$, plus polygono super basi $c\frown b$, diviso per punctos de h pertinente ad duo intervallo partiale.

Nos opera per limite infero, et seque theorema.

·1 $(\mathrm{S}_, \mid \mathrm{S}')$ P·0

Ita pro integrale infero.

·2 $\mathrm{S}(f, a\frown b)\, \varepsilon \mathrm{q} \;.=.\; \mathrm{S}(f, a\frown c), \mathrm{S}(f, c\frown b)\, \varepsilon \mathrm{q}$

·3 $\mathrm{S}(f, a\frown b)\, \varepsilon \mathrm{q} \;.\supset.\; \mathrm{S}(f, a\frown b) = \mathrm{S}(f, a\frown c) + \mathrm{S}(f, c\frown b)$
[P·0·1 $.\supset.$ P]

Si functione f es integrabile in intervallo ab a ad b, tunc es integrabile in ambo intervallo ab a ad c et ab c ad b; et viceversa. Et integrale ab a ad b vale integrale ab a ad c, plus integrale ab c ad b.

✻ 7. Integrale de summa.

$u \varepsilon \mathrm{Intv} \,.\, f,g \varepsilon \mathrm{qf}u \,.\, \mathrm{l}'f`u, \mathrm{l}_,f`u, \mathrm{l}'g`u, \mathrm{l}_,g`u\; \varepsilon \mathrm{q} \;.\supset:$
·0 $\mathrm{S}'[(fx+gx)|x, u] \leq \mathrm{S}'(f,u) + \mathrm{S}'(g,u)$.
$\mathrm{S}_, \text{———————} \gtreqless \mathrm{S}_, \text{———} \mathrm{S}_, \text{——}$

Si in intervallo u es dato duo functione f et g, ambo limitato supra et infra, tunc integrale supero de summa $fx + gx$, ubi varia x in u, es minore aut æquale ad summa de integrales.

Dem. $a,b\varepsilon u \;.\supset.\; (fx+gx)|x`a\frown b \supset f`a\frown b + g`a\frown b \;.\supset.$
$\mathrm{l}'(fx+gx)|x`a\frown b \leq \mathrm{l}'f`a\frown b + g`a\frown b \;.\supset.$ P

·1 $\mathrm{S}(f,u), \mathrm{S}(g,u)\, \varepsilon \mathrm{q} \;.\supset.\; \mathrm{S}[(fx+gx)|x, u] = \mathrm{S}(f,u) + \mathrm{S}(g,u)$
[P·0 $\supset$ P]

Si ambo functione es integrabile, tunc integrale de summa vale summa de integrales.

Si uno solo termine es integrabile, tunc:

·2 $\mathrm{S}(f,u)\, \varepsilon \mathrm{q} \;.\supset.\; \mathrm{S}'[(fx+gx)|x, u] = \mathrm{S}(f,u) + \mathrm{S}'(g,u)$.
$\mathrm{S}_, \text{———————} = \mathrm{S}(f,u) + \mathrm{S}_,(g,u)$

·3 $m \varepsilon Q \;.\supset.\; S'(mf, u) = mS'(f,u) \;\; . \;\; S_{,}(mf, u) = mS_{,}(f,u)$

Lice commuta integrale supero et infero cum factore positivo.

·4 $S'(-f, u) = -S_{,}(f, u)$

Si nos muta signo ad functione, integrale supero fi infero.

·5 $S(f,u) \varepsilon q \,.\, m \varepsilon q \;.\supset.\; S(mf,u) = mS(f,u)$

Lice transporta factore constante ex signo integrale.

·6 $S(f, a \frown b) = (b-a) S\{ f[a+(b-a)z] \mid z, \Theta \}$

·7 $S(f, a \frown b) = S[f(-x) \mid x, -b \frown -a]$

✻ 8. cres S

·1 $u \varepsilon \mathrm{Intv} \,.\, f \varepsilon (\mathrm{qf}u)\mathrm{cres}_0 \;.\supset.\; S(f,u) \varepsilon q$

Omni functione reale, crescente dum varia, es integrabile.

Dem. $a = l_{,}u \,.\, b = l'u \,.\, n \varepsilon N_1 \;.\supset.$

$s_{,}[f, a \frown b, a+(b-a)(0\cdots n)/n] = (b-a)\Sigma\{f[a+r(b-a)/n] \mid r, 0\cdots(n-1)\}/n \,.$

$s'[\quad » \quad] = (b-a)\Sigma\{\quad » \quad, 1\cdots n\}/n \,.$

$s'(\,»\,) - s_{,}(\,»\,) = (b-a)(fb-fa)/n$ (1)

(1) $.\supset.\; l_{,}[(b-a)(fb-fa)/n \mid n \text{'} N_1] = 0$ (2)

(2) $.\supset.\; S'(f, a \frown b) = S_{,}(f, a \frown b) \;.\supset.\; P$

Si functione es crescente (cres_0), rectangulo inscripto $s_{,}(f,u)$, ubi u es intervallo dato, vel suo parte, habe altore æquale ad primo valore de functione; et rectangulo circumscripto $s'(f,u)$ habe altore æquale ad ultimo.

Nos divide intervallo dato $a \frown b$ in n partes æquale, cum valore $a+(b-a)\times(0\cdots n)/n$. Tunc polygono circumscripto minus polygono inscripto vale differentia de valores extremo de functione multiplicato per long-ore de intervallo partiale $(b-a)/n$.

Limite infero de differentia $s'-s_{,}$, ubi varia n, es zero.

Ergo integrale supero æqua integrale infero, et functione es integrabile.

·2 $u \varepsilon \mathrm{Intv} \,.\, f \varepsilon (\mathrm{qf}u)\mathrm{decr}_0 \;.\supset.\; S(f,u) \varepsilon q$

Idem pro functione decrescente. Ergo si functione es tale que intervallo de integratione pote es diviso in partes in numero finito, ita ut in omni parte functione es monotono, id es vel crescente vel decrescente, tunc functione es integrabile.

✻ 9. cont S

$u \varepsilon \text{Intv} \,.\, f \varepsilon (\text{qf}u)\text{cont} \,.\supset.\, \text{S}(f,u) \,\varepsilon \text{q}$

Dem. §cont P1·3 (pag. 239) . $h \varepsilon \text{Q} \,.\supset.\, \exists \text{N}_1 \frown n \ni [r \varepsilon 0 \cdots (n-1) \,.\supset_r.$
$\text{l}'f`[a+(r+\theta)(b-a)/n] - \text{l}_,f`[a+(r+\theta)(b-a)/n] < h \,] \,.\supset.$
$\exists \text{N}_1 \frown n \ni \{ \text{s}'[f, a\overline{\ }b, a+(0 \cdots n)(b-a)/n] - \text{s}_,[\ldots] < (b-a)h \}$ (1)
(1) $.\supset.\, \text{S}'(f, a\overline{\ }b) - \text{S}_,(f, a\overline{\ }b) = 0 \,.\supset \text{P}$

Omni functione continuo es integrabile.

In vero, pro theorema de continuitate æquabile, dato numero positivo h (parvo ad arbitrio), nos pote divide intervallo $a\overline{\ }b$ in numero satis magno n de partes æquale, ut differentia inter limite supero et limite infero de valores de functione (vel suo oscillatione) in omni intervallo partiale, es minore que h. Tunc differentia inter polygonos circumscripto et inscripto s′ et s, es minore de $(b-a)h$, quantitate parvo ad arbitrio. Seque æqualitate de integrales supero et infero.

✻ 10. Theorema de valore medio.

$u \varepsilon \text{Intv} \,.\, f \varepsilon \text{qf}u \,.\, \text{l}'f`u, \text{l}_,f`u \,\varepsilon \text{q} \,.\supset:$
·0 $\text{S}(f,u) \,\varepsilon \text{q} \,.\supset.\, \text{S}(f,u) \,\varepsilon\, (\text{Long}u) \times \text{Med}f`u$

Integrale vale longore de intervallo multiplicato per valore medio inter valores de functione.

·1 $g \varepsilon \text{Qf}u \,.\, \text{l}'g`u \,\varepsilon \text{q} \,.\, \text{S}(g,u) \,\varepsilon \text{Q} \,.\supset.$
$\text{l}'f`u \times \text{S}(g,u) \geqq \text{S}'(fx \times gx | x,u) \geqq \text{S}_,(fx \times gx | x,u) \geqq \text{l}_,f`u \times \text{S}(g,u)$

Dem. $x \varepsilon u \,.\supset.\, \text{l}'f`u \geqq fx \geqq \text{l}_,f`u \,.\, (\text{l}'f`u)gx \geqq fx \times gx \geqq (\text{l}_,f`u) \times gx$

Si in intervallo u es dato alio functione g positivo limitato et integrabile, tunc integrale supero, et integrale infero de producto $fx \times gx$, ubi varia x, in u, vale integrale de g per valore medio inter valores de f.

Es vocato « theorema de valore medio de integrale ».

Cauchy, a.1823 s.2 t.4 p.138, et Dirichlet, a.1837 t.1 p.138, enuntia illo pro functio continuo.

Vide MA. a.1874 t.7 p.605, JdM. a.1874 s.2 t.3 p.293, et Pringsheim MünchenA. a.1900 p.209.

·2 $g \varepsilon$ Qfu . l'g'n εq .⊃.
(l'f'u)×S'(g,u) ⋚ S'(fx×gx|x,u) ⋚ S,(fx×gx|x,u) ⋚ (l,f'u)×S,(g,u)

·3 a,$b \varepsilon$q . a<b . $f \varepsilon$ qfa⌐b . l'f'a⌐b, l,f'a⌐b εq . $g \varepsilon$(qfa⌐b)crs$_0$
.⊃. ∃ a⌐b ⌒ c∋[S'(f×g, a⌐b) = (ga)S'(f, a⌐c)+(gb)S'(f, c⌐b)]

·4 (S, | S')P·3 } Bonnet JdM. a.1849 p.249;
P·3 = « secundo theorema de valore medio ».

✻ 11. Integrale inter limites.

$u \varepsilon$ Intv . $f \varepsilon$ (qfu)coct .⊃.

·0 x,$y \varepsilon u$. x<y .⊃. S(f; x,y) = S(f, x⌐y) Df
S(f; y,x) = −S(f, x⌐y) Df
S(f; x,x) = 0 Df

Si in u functione f es continuo, tunc S(f; x,y), lege: « integrale de f ab x ad y » indica integrale de f in intervallo x⌐y, si x<y, et idem integrale, cum signo mutato, si x>y.

Integrale ita definito non es functione de intervallo de integratione, sed es functione alterno de duo valores x et y, que vocare « limites de integratione ». In loco de S(f;x,y) in libros commune es scripto $\int_x^y fz\, dz$.

·1 x,$y \varepsilon u$.⊃. S(f; y,x) = −S(f; x,y)

·2 x,y,$z \varepsilon u$.⊃. S(f; x,y) + S(f; y,z) + S(f; z,x) = 0

·3 x,$y \varepsilon u$. x−=y .⊃. S(f; x,y)/(y−x) ε Medio$f$'x⌐y

✻ 12. Relatione inter derivata et integrale.

·1 $u \varepsilon$ Intv . $f \varepsilon$ (qfu)cont . a,$x \varepsilon u$.⊃. D[S(f; a,z) |z, u]x = fx

Dem. D[S(f;a,z)|z,u]x = lim{[S(f;a,z) − S(f;a,x)] /(z−x) |z, u, x}
= lim[S(f;x,z) /(z−x) |z, u, x]
= lim(Med$f$'x⌐z |z, u, x) = fx

Si in intervallo u es dato f functione continuo, et duo valore a et x, tunc derivata de integrale de f, ab a ad z, ubi varia z in u, pro valore x, vale fx.

Ergo omni functione continuo es derivata de alio functione, id es de suo integrale ab limite fixo ad limite variabile.

Plure Auctore voca « integrale indefinito » integrale de functione ab limite fixo ad limite variabile.

·2 $u\varepsilon$ Intv . f, Df ε (qf$u$)cont . $a,b\varepsilon u$.⊃. S(Df; a,b) = Δ(f; a,b)

Dem. $x\varepsilon u$. P·1 .⊃. D[S(Df; a,z)|z,u]x = Dfx .⊃. S(Df;a,x) = $fx-fa$

Si in intervallo u es dato functione f cum derivata continuo, et duo valore a et b, tunc integrale de derivata de f ab a ad b vale incremento de functione.

Ce theorema da importante regula sequente, pro calculo de S(f;a,b). Quære functione g, tale que Dg=f. Tunc integrale vale $gb-ga$. Determinatione de functione g pote es facile, aut minus secundo natura de functione f.

✻ 13·1 $a,b\varepsilon$q . $a<b$. f,Df ε qFa⌐b . l'Df‘a⌐b, l,Df‘a⌐b εq .⊃. S,(Df, a⌐b) ≦ $fb-fa$ ≦ S'(Df, a⌐b)

Si functione f habe derivata in intervallo de a ad b, et systema de valores de derivata es finito, tunc incremento de functione es inter integrale infero et integrale supero de derivata.

Dem.

$x,y\varepsilon a$⌐b . §D P22 .⊃. $(y-x)$l,Df‘x⌐y ≦ $fy-fx$ ≦ $(y-x)$l'Df‘x⌐y

$h\varepsilon$ Cls'a⌐b . Num$h$ εN_0 .⊃. s,(Df, a⌐b, h) ≦ $fb-fa$ ≦ s'(Df, a⌐b, h)

·2 Hp·1 . S(Df, a⌐b) εq .⊃. S(Df, a⌐b) = $fb-fa$

[P·1 ⊃ P]

Si derivata de f es integrabile, tunc integrale de derivata vale incremento de functione.

✻ 14. Integratione per partes.

$a,b\varepsilon$q . f, g, Df, Dg ε (q F a⌐b)cont .⊃.

S(f×Dg; a,b) = (fb)(gb) − (fa)(ga) − S(g×Df; a,b)

Dem. $x\varepsilon a$⌐b .⊃. D(fx×gx |x, a⌐b)x = fx×Dgx + gx×fx .⊃.

Δ(fx×gx |x ; a,b) = S(fx×Dgx|x ; a,b) + S(gx×Dfx|x; a,b) .⊃. P

Decompone functione integrando in producto de functione f per derivata de functione g. Tunc integrale consta ex duo

parte: incremento de producto fx—gx, ubi x varia inter limites de integrale, minus integrale de producto de g per derivata de f. Resulta ex derivata de producto.

·1 $f,g\,\varepsilon\,\mathrm{qf}\Theta\,.\,\mathrm{l}'f\text{‘}\Theta, \mathrm{l}_{,}f\text{‘}\Theta, \mathrm{l}'g\text{‘}\Theta, \mathrm{l}_{,}g\text{‘}\Theta\,\varepsilon\mathrm{q}\,.\,\mathrm{S}(f,\Theta), \mathrm{S}(g,\Theta)\,\varepsilon\mathrm{q}\,.\supset.$
$\mathrm{S}[gx\,\mathrm{S}(f, \Theta x)\,|x, \Theta] + \mathrm{S}[fx\,\mathrm{S}(g, \Theta x)\,|x, \Theta] = \mathrm{S}(f,\Theta) \times \mathrm{S}(g,\Theta)$

{ J. Thomae a.1875 Zm. t.20 p.475 }

Integratione per partes sine signo D.

❋ 15. Substitutione in integrale.

$a,b\varepsilon\mathrm{q}\,.\,g, \mathrm{D}g\;\varepsilon\,(\mathrm{qF}a{\frown}b)\mathrm{cont}\,.\,f\varepsilon\;[\mathrm{qF}(g\text{‘}a{\frown}b)]\mathrm{cont}\,.\supset.$
$\mathrm{S}(f;\, ga, gb) = \mathrm{S}(fgx \times \mathrm{D}gx\;|x;\, a,b)$

Dem. $x\varepsilon\, a{\frown}b\,.\,\S\mathrm{D}\;9\;.\supset.\;\mathrm{D}[\mathrm{S}(f;\, ga,gx)\,|x, a{\frown}b]x$
$= \mathrm{D}[\mathrm{S}(f;\, ga,y)\,|y, g\text{‘}a{\frown}b]gx \times \mathrm{D}gx = fgx \times \mathrm{D}gx$

Dato duo quantitate a et b, et g, functione reale definito in intervallo ab a ad b, cum derivata continuo, et functione f definito in campo de valores sumpto per g in intervallo $a{\frown}b$, et continuo, tunc integrale de functione f, ab ga ad gb vale integrale de producto $fgx \times \mathrm{D}gx$, ubi varia x ab a ad b.

Id es, in calculo de $\mathrm{S}(fy\,|y;\, ga, gb)$, substitue $y = gx$, multiplica fgx per $\mathrm{D}gx$, et integra, pro x, inter limites a et b, respondentes in relatione $y{=}gx$ ad limites de antiquo integrale.
Resulta de derivata de functione de functione, pag. 281.

·1 $f\varepsilon\;(\mathrm{Qf}\Theta)\mathrm{cont}\,.\,g\varepsilon\;\mathrm{qf}[\,\Theta\mathrm{S}(f,\Theta)]\mathrm{cont}\,.\supset.$
$\mathrm{S}[g,\, \Theta\mathrm{S}(f,\Theta)] = \mathrm{S}\{fx \times g[\mathrm{S}(f,\Theta x)]\;|x,\, \Theta\}$

Integratione « per substitutione », scripto sine signo D.

❋ 16. Integrale de potestate.

$m\varepsilon\mathrm{Q}_0\;.\supset.\;\mathrm{S}(x^m|x,\, \Theta) = /(m+1)$

Si m es numero positivo aut nullo, integrale de x^m, ubi varia x, de 0 ad 1 vale $1/(m+1)$.
Primo integrale importante, que occurre in Mathematica.

Dem. 1 $S(x^m|x, \Theta) = S[Dx^{m+1}/(m+1)|x, \Theta]$
$= \Delta[x^{m+1}/(m+1)|x, \Theta] = 1/(m+1)$

In vero, integrale de x^m vale integrale de derivata de $x^{m+1}/(m+1)$, ergo vale incremento de isto functione.

Dem. 2
$n \varepsilon N_1 . \S\Sigma\, P7{\cdot}1 . \supset . s'[x^m|x, \Theta, (0\cdots n)/n] = \Sigma(1\cdots n)^m /n^{m+1} > 1/(m+1) .$
$s_{,}[x^m|x, \Theta, (0\cdots n)/n] < 1/(m+1) . s'(\ldots)-s_{,}(\ldots) = 1/n . \supset . P$

Divide intervallo Θ in n partes æquale.

Summa s' de rectangulos circumscripto ad figura vale summa de potestates m de numeros naturale de 1 ad n, diviso per n ad $m+1$, que supera $1/(m+1)$, per theorema de Algebra (pag. 123).

Summa $s_{,}$ de rectangulos inscripto in figura vale summa de potestates m de numeros de 0 ad $n-1$, diviso per n ad $m+1$, que es minore de $1/(m+1)$.

Differentia de polygono circumscripto et inscripto vale $1/n$, quantitate parvo ad arbitrio.

Ergo integrale, minore de primo summa, et majore de secundo, vale $1/(m+1)$.

Historia.

Pro $m=1$, $S(x|x, \Theta) = 1/2$ dice que area de triangulo vale producto de basi per altitudine, diviso per 2. Euclide, l.1 P34, pro demonstra isto regula, divide parallelogrammo in duo triangulo æquale per diagonale.

Nos pote traduce in symbolos analytico ce Dm ut seque:

$P7{\cdot}6{\cdot}7 . \supset . \quad S(x|x, \Theta) = S[(1-x)|x, \Theta]$ (1)
$P7{\cdot}1 . \supset . \quad S(x|x, \Theta) + S[(1-x)|x, \Theta] = 1$ (2)
$(1) . (2) . \supset . \quad S(x|x, \Theta) = /2$

Lege: « Area de triangulo (0,0), (1,0), (1,1), vale area de triangulo (0,0), (0,1), (1,1), ut resulta de superpositione, vel de regula de Calculo integrale. Summa de duo area vale 1. Ergo integrale dato vale /2 ».

Pro $m=2$, $S(x^2|x, \Theta) = 1/3$ dice que volumen de pyramide vale producto de basi per altitudine, diviso per 3. Euclide, l.12 P7, pro demonstra regula decompone prisma in tres pyramide.

$S(x^2|x, \Theta) = S[(1-x)^2|x, \Theta] = 2S[x(1-x)|x, \Theta]$ (1)

Summa de tres pyramide vale prisma totale:

$S(x^2|x, \Theta) + S[(1-x)^2|x, \Theta] + 2S[x(1-x)|x, \Theta] = 1$ (2)
$(1) . (2) . \supset . S(x^2|x, \Theta) = /3$]

Archimede stude idem integrale in calculo de area de suo spirale, de barycentro de triangulo, et de area de parabola. Suo ratiocinio es circa ut seque (περὶ ἑλικῶν P10-24):

[$S(x^2|x, \Theta) = \lim\Sigma[(r/n)^2|r, 1\cdots n]/n\,|n = \lim\Sigma(1\cdots n)^2/n^3\,|n$
$= \lim n(n+1)(2n+1)/(6n^3)\,|n = \lim(1+/n)[1+/(2n)]/3\,|n = /3$]

| B. CAVALIERI a.1639 p.524: « se in un parallelogrammo, descritto il diametro, intenderemo tirate parallele ad un lato di esso quante se ne possono tirare, indefinitamente di qua e di là prolungate, la parte di esse che resta nel parallelogrammo, cioè (per parlare nella lingua usata in essa geometria) tutte le linee del parallelogrammo saranno doppie di tutte le linee comprese in uno dei fatti triangoli. Tutti i quadrati del parallelogrammo saranno tripli di tutti i quadrati dello stesso triangolo. Tutti i cubi saranno quadrupli di tutti i cubi. Tutti i biquadrati saranno quintupli di tutti i biquadrati (intendo sempre quelli del parallelogrammo di quelli del detto triangolo). Donde argomento probabilmente che tutti li quadricubi saranno sestupli di tutti i quadricubi. Tutti i cubicubi saranno settupli di tutti i cubicubi, e così in infinito secondo i numeri continuamente susseguenti. » |

| FERMAT t.1 p.195; edito per Mersenne a.1644 |

Fermat da ce theorema cum hypothesi $m \varepsilon \mathrm{R}$, in calculo de area de parabola $y_n = x^m$. Area de parallelogrammo circumscripto es ad parabola « ut summa exponentium ambarum potestatum ad exponentem potestatis applicatarum... [$\mathrm{S}[x \uparrow (m/n) \,|x, \Theta] = n/(m+n)$] ».

❊ 17. INTEGRALE DE FUNCTIONE INTEGRO.

·3 $n \varepsilon \mathrm{N}_1 \,.\, a \varepsilon \,\mathrm{qf}0\cdots n \,.\, x \varepsilon \mathrm{q} \,.\supset.$
$\mathrm{S}\{\Sigma(a_r x^{n-r} \mid r, 0\cdots n)|x; 0, x\} = \Sigma[a_r x^{n-r+1}/(n-r+1) \mid r, 0\cdots n]$

Integrale de functione integro de gradu n es functione integro de gradu $n+1$.

❊ 18.

·1 $u \varepsilon \,\mathrm{Intv} \,.\, f \varepsilon \,\mathrm{qf}u \,.\, \mathrm{l}'\mathrm{mod}f\text{‘}u \;\varepsilon \mathrm{Q} \,.\supset.\; \mathrm{mod}\,\mathrm{S}'(f,u) \leqq \mathrm{S}'(\mathrm{mod}f, u)$

·2 » » » $\mathrm{mod}\,\mathrm{S}_{\prime}(f,u) \leqq$ »

·3 $a,b\, \varepsilon \mathrm{q} \,.\, a<b \,.\supset.\; \mathrm{S}(\mathrm{sgn}, a \frown b) = \mathrm{mod}b - \mathrm{mod}a$

·4 $a \varepsilon \mathrm{Q} \,.\supset.\; \mathrm{S}(\mathrm{E}, \Theta a) = [\beta a + (\mathrm{E}a - 1)/2]\mathrm{E}a$

·5 $a \varepsilon \mathrm{Q} \,.\supset.\; \mathrm{S}(\beta, \Theta a) = [(\beta a)^2 + \mathrm{E}a]/2$

·6 $\mathrm{S}'[(\lim \mathrm{sgn}\, \beta\, n!x \mid n) \mid x, \Theta] = 1$
$\mathrm{S}_{\prime}$ » » » $= 0$

Functione que, per x rationale vale 0, et per x irrationale vale 1, habe integrales supero et infero differente, et non es integrabile. (Dirichlet t. 1 p.132). Pro expressione analytico de functione, vide §lim 16·2·21 p.219.

✻ 19. Integrale de functione rationale.

·0 $x \varepsilon Q . \supset . S(/; 1, x) = \log x$

Dem. $S(/; 1,x) = S(D \log; 1,x) = \Delta(\log; 1,x) = \log x$

Si x es quantitate positivo, integrale de functione reciproco, vel $1/x$, ubi varia x, ab 1 ad x, vale logarithmo naturale de x. Pote es sumpto ut definitione de logarithmo.

Gregorius a S. Vincentio in *Opus Geometricum*, Antverpiae a.1647 p.594 vide proprietate characteristico de area de hyperbola $[S(/,1 \frown a)]$, que seque: $a,m \varepsilon Q . \supset . S(/, 1 \frown a^m) = mS(/, 1 \frown a)$.

Alfonso de Sarasa in opusculo: Solutio problematis a R. P. Marino Mersennio minimo propositi, Antverpiae a.1649 p.7, gno que areas de hyperbola responde ad logarithmos.

·1 $a,b \varepsilon q . a -= 0 . b/a > 0 . \supset . S(/; a,b) = \log(b/a)$

Integrale de functione / « reciproco » in intervallo que non contine valore 0, ubi functione non es finito.

·2 $a,b,x,y \varepsilon q . a -= b . (x-a)(y-a)(x-b)(y-b) > 0 . \supset .$
$S\{/[(x-a)(x-b)]|x; x,y\} =$
$/(b-a) \log\{[(y-a)/(y-b)]/[(x-a)/(x-b)]\}$

Quantitate sub signo log vocare « biratione » de quatuor numero x,y,a,b.

Dem. $1/[(x-a)(x-b)] = [1/(x-b)-1/(x-a)]/(b-a)$

In vero, nos decompone fractione integrando in summa de duo fractione, ut es exposito in pag. 309.

·3 $x \varepsilon q . \supset . S[1/(1+x^2)|x; 0,x] = \text{tang}^{-1}x$

Dem. 1 §DP18·4 (pag. 285) $\supset$ P

Dem. 2 $S[1/(1+x^2)|x; 0,x] = S\{1/[(x-i)(x+i)]|x; 0,x\} =$
$[1/(2i)]\log[(1+xi)/(1-xi)] = \text{tang}^{-1}x.$

·4 $n \varepsilon N_1+1 . a,x \varepsilon q . a -= 0 . x/a > 0 . \supset .$
$S(1/x^n|x; a,x) = (1/a^{n-1} - 1/x^{n-1})/(n-1)$

Dato functione rationale, vel quotiente de duo functione integro, si nos nosce radices de denominatore, fractione es decomponibile in fractiones de forma $c/(x-a)^m$. Vide pag. 309. Si $m>1$, suo integrale es dato per Prop. ·4 et es rationale. Si $m = 1$, suo integrale, dato per Prop. ·1 es logarithmo. Ergo omni functione rationale habe functione primitivo, que pote es expresso per functiones rationale et per logarithmos.

✻ 20. Integrales improprio.

Si intervallo de integratione es infinito, aut si functione fi infinito in intervallo de integratione, definitione de integrale dato in Prop. 1-4 non vale; nam uno termine de summa s′, vel $s_{,}$ es infinito.

In isto casu, Auctores pone definitiones sequente:

·1 $a\varepsilon q \,.\, f\varepsilon\, q\,f(a+Q_0) \,.\supset.$
$S(f, a+Q_0) = \lim[S(f, a\frown b)|b, a+Q, \infty]$ Df

Si f es functione reale de valores de a ad ∞, integrale de f, extenso de a ad ∞ es limite de integrale de a ad b, ubi varia b, sume valores supra a, et verge ad ∞.

·2 $b\varepsilon q \,.\, f\varepsilon\, q\,f(b-Q_0) \,.\supset.$
$S(f, b-Q_0) = \lim[S(f, a\frown b)|a, b-Q, -\infty]$ Df

·3 $f\varepsilon\, qfq \,.\supset.\, S(f,q) = \lim[S(f, a+Q_0)|a, q, -\infty]$ Df

·4 $a\varepsilon q \,.\, f\varepsilon\, Qf(a+Q_0) \,.\, S(f, a+Q_0)\,\varepsilon Q \,.\supset.\, 0\varepsilon\, Lm(f, a+Q, \infty)$

·5 Hp P·4 $.\supset.\, 0\varepsilon\, Lm(xfx\,|x, a+Q, \infty)$

·51 $a\varepsilon Q \,.\, S(f,q)\,\varepsilon q \,.\supset.\, S[f(ax)|x, q] = S(f,q)\,/a$

·52 $a\varepsilon q \,.\, S(f,q)\,\varepsilon q \,.\supset.\, S[f(a+x)|x, q] = S(f,q)$

·6 $a,b\varepsilon q \,.\, a<b \,.\, u\varepsilon\, qf(a\frown b) \,.\, 1'\mathrm{mod}\, u`(a\frown b) = \infty : c\varepsilon\, a\frown b \,.\supset_c.$
$1'\mathrm{mod}\, u`(c\frown b)\,\varepsilon Q :\supset.\, S(u, a\frown b) = \lim[S(u, c\frown b)|c, a\frown b, a]$ Df

·7 —
$1'\mathrm{mod}\, u`(a\frown c)\,\varepsilon Q$ — — — — — — — — — — $a\frown c$ — — — $b]$ Df

·8 — — — — — — — — — — — — — — — — — — $c,d\varepsilon\, a\frown b \,.\supset_{c,d}$
$1'\mathrm{mod}\, u`(c\frown d)\,\varepsilon Q :\supset.$ Ths P·6 Df

Integrales definito per P·1·2·3·6·7·8 vocare « integrales singulare (Cauchy a. 1823), aut improprio », et indicare per:

$$\int_a^{\infty} fx\,dx \qquad \int_{-\infty}^{b} fx\,dx \qquad \int_{-\infty}^{+\infty} fx\,dx$$

Non semper isto integrales existe; si illo existe, et habe valore finito, es dicto convergente.

Quando Df·3·6 ne es applicabile, Cauchy (Œuvres s.1 t.1 p.477) considera « valore principale de integrale », que, secundo Riemann p.241, ne habe grande utilitate.

✻ 21·1 $S(/, 1+Q) = \infty$

·2 $n \varepsilon 1+Q . \supset . S(x^{-n} | x, 1+Q) = /(n-1)$

} TORRICELLI a.1644, pro $n=2$; FERMAT t.2 p.338; WALLIS a.1655 t.1 p.409 {

✻ 22. π S

·0 $S[/(1+x^2) | x, \Theta] = \pi/4$

$S[/(1+x^2) | x, Q] = \pi/2$. $S[/(1+x^2) | x, q] = \pi$

·1 $a \varepsilon Q . \supset . S[/(a^2+x^2) | x, q] = \pi/a$

[$[/(a^2+x^2) | x, q] = D[/a\, t^{-1}(x/a) | x, q]$]

·2 $a \varepsilon Q . b, c \varepsilon q . ac - b^2 > 0 . \supset .$

$S[/(ax^2+2bx+c) | x, q] = \pi/\surd(ac-b^2)$

[$x \varepsilon q . \supset . /(ax^2+2bx+c) = a/[(ax+b)^2+ac-b^2]$

$= D\{/\surd(ac-b^2)\, t^{-1}[(ax+b)/\surd(ac-b^2)] | x, q, x\}$]

·3 $S[/(1+x^3) | x, Q] = S[x/(1+x^3) | x, Q] = 2\pi/(3\surd 3)$

·4 $S[/(1+x^4) | x, q] = S[x^2/(1+x^4) | x, q] = \pi\surd 2/2$

·5 $a, b \varepsilon Q . \supset . S\{/[(a^2+x^2)(b^2+x^2)] | x, q\} = \pi/[ab(a+b)]$

[$/[(a^2+x^2)(b^2+x^2)] = [/(a^2+x^2) - /(b^2+x^2)]/(b^2-a^2)$]

·6 $a, b \varepsilon Q . \supset . S\{x^2/[(a^2+x^2)(b^2+x^2)] | x, q\} = \pi/(a+b)$

[$x^2/[(a^2+x^2)(b^2+x^2)] = [b^2/(b^2+x^2) - a^2/(a^2+x^2)]/(b^2-a^2)$]

·7 $a, b, c \varepsilon Q . \supset . S[/(a+bx^2+cx^4) | x, q] = S[x^2/(ax^4+bx^2+c) | x, q]$

$= \pi/\surd[ab+2a\surd(ac)]$ } PLANA TurinM. a.1820 {

·8 $S[/(1+x^6) | x, Q] = 2S[x^2/(1+x^6) | x, Q] = S[x^4/(1+x^6) | x, Q]$

$= \pi/3$

} ·2-·5 EULER *Calc. Int.* a.1768 t.1 §353 t.4 s.4 §105 {

·9 $a \varepsilon Q . m \varepsilon N_1 . \supset . S[/(a^2+x^2)^{m+1} | x, Q] =$

$\Pi\{(2r-1)/(2r) | r, 1 \cdots m\} \pi/(2a^{2m+1})$

✻ 23·0 $f \varepsilon (QfQ_0)\text{decr} . \supset : \Sigma(f, N_0)\, \varepsilon Q . = . S(f, Q_0)\, \varepsilon Q$

} MACLAURIN a.1742 p.289 {

[$Hp . n \varepsilon N_0 . \supset . \Sigma(f, 0 \cdots n) > S[f, 0 \frown (n+1)] > \Sigma[f, 0 \cdots (n+1)] - f0 . \supset .$ Ths]

Si f es functione positivo et decrescente de variabile inter 0 et $+\infty$, tunc conditione necessario et sufficiente pro convergentia de serie $f0 + f1 + f2 + \ldots$, es convergentia de integrale de f de 0 ad $+\infty$.

In vero, summa $f0 + f1 + \ldots + fn$ supera integrale de 0 ad $n+1$, que supera summa $f1 + f2 + \ldots + f(n+1)$. Ergo, si summa de serie es finito, tunc integrale, que es minore de summa de serie, es finito. Et si integrale es finito, serie $f1 + f2 + \ldots$ habe valore finito, minore de integrale; et addito $f0$, seque convergentia de serie dato.

Plure tractato de Calculo tribue isto theorema ad Cauchy, a. 1821.

Exemplo. Si in loco de f nos pone $/(1+x) | x$, de $\mathrm{S}(/, 1+\mathrm{Q}) = \infty$, Prop 21·1, seque $\Sigma(/, \mathrm{N}_1) = \infty$ (§lim P23·1, pag. 223), et viceversa.

Si $f = /(1+x)^n | x$, ubi $n \varepsilon 1+\mathrm{Q}$, de P21·2 seque $\Sigma \mathrm{N}_1{}^{-n} \varepsilon \mathrm{Q}$ (§lim P23·3 p.223).

✻ 24. Functiones irrationale.

·1 $a,x \varepsilon \mathrm{q} \;.\supset.\; \mathrm{S}[/\sqrt{(a^2+x^2)} | x; 0,x] = \log\{[x+\sqrt{(a^2+x^2)}]/a\}$

[§D P14 (p.284) $\supset$ P]

·2 $a,x \varepsilon \mathrm{q} \;.\supset.\; \mathrm{S}[\sqrt{(a^2+x^2)} | x; 0,x] =$
$x\sqrt{(a^2+x^2)}/2 + a^2 \log\{[x+\sqrt{(a^2+x^2)}]/a\}/2$

[$\mathrm{D}[x\sqrt{(a^2+x^2)} | x, \mathrm{q}]x = \sqrt{(a^2+x^2)} + x^2/\sqrt{(a^2+x^2)} =$
$\sqrt{(a^2+x^2)} + (a^2+x^2)/\sqrt{(a^2+x^2)} - a^2/\sqrt{(a^2+x^2)} = 2\sqrt{(a^2+x^2)} - a^2/\sqrt{(a^2+x^2)}$
$.\supset.\; 2\mathrm{S}[\sqrt{(a^2+x^2)} | x; 0,x] = x\sqrt{(a^2+x^2)} + a^2 \mathrm{S}[/\sqrt{(a^2+x^2)} | x; 0,x]$. P·1 $.\supset.$ P]

·3 $a \varepsilon \mathrm{Q} \,.\, y \varepsilon \Theta a \;.\supset.\; \mathrm{S}[/\sqrt{(a^2-y^2)} | y, \Theta y] = \mathrm{s}^{-1}(y/a)$

·4 $a \varepsilon \mathrm{Q} \,.\, y \varepsilon \Theta a \;.\supset.\; \mathrm{S}[\sqrt{(a^2-y^2)} | y, \Theta y] =$
$[y\sqrt{(a^2-y^2)} + a^2 \mathrm{s}^{-1}(y/a)]/2$

·5 $\mathrm{S}[\sqrt{(1-x^2)} | x, \Theta] = \pi/4$ { Wallis a.1655 p.417 :

« Circulus ad Quadratum Diametri, (vel etiam Ellipsis quælibet ad Parallelogrammum sibi circumscriptum), eam habet rationem, quam habent Radices Quadraticæ universales Residuorum seriei infinitæ Æqualium serie Secundanorum mulctatæ, ad seriem illam Æqualium. » }

Versione: Ratione de circulo ad quadrato de diametro, vel $\pi/4$, æqua summa universale (integrale) de radice quadratico de 1 minus potestate secundo de variabile, vel de $\sqrt{(1-x^2)}$.

·6 $a \varepsilon \mathrm{Q} \,.\, y \varepsilon \Theta a \;.\supset.\; \mathrm{S}\{\sqrt{[(a-y)/y]} | y, \Theta y\} =$
$\sqrt{[y(a-y)]} + a/2\, \mathrm{c}^{-1}[(a-2y)/a]$

✻ 25. Integrale Euleriano.

$m,n\,\varepsilon\,Q-1\ .\supset:$ ·0 $S[x^m(1-x)^n|x,\Theta]\ \varepsilon\ Q$

Si m et n es quantitate superiore ad -1, integrale considerato habe valore positivo determinato et finito.

Euler stude ce integrale in TorinoM. t. 3 a.1762, PetrNC. a.1772, PetrNA. a.1779.

Legendre, Exerc. t.1 p.221, t.2 p.3, voca illo « integrale Euleriano de primo specie »; quem Binet, a.1839 JP. c.27, indica per $B(m+1,n+1)$.

·1 $S[x^m(1-x)^n|x,\Theta] = S[x^n(1-x)^m|x,\Theta]$
[$(1-x)|x\ .\supset.$ P]

Integrale non varia si nos permuta m et n. Resulta de substitutione $1-x$ in loco de x.

·2 $S[x^{m+1}(1-x)^n|x,\Theta] = (m+1)/(m+n+2)\ S[x^m(1-x)^n|x,\Theta]$
[$D[x^{m+1}(1-x)^{n+1}|x,\Theta]x = (m+1)x^m(1-x)^{n+1} - (n+1)x^{m+1}(1-x)^n$
$= (m+1)x^m(1-x)^n - (m+n+2)x^{m+1}(1-x)^n$ (1)
(1) $.\supset.\ (m+1)S[x^m(1-x)^n|x,\Theta] - (m+n+2)S[x^{m+1}(1-x)^n|x,\Theta]$
$= \Delta[x^{m+1}(1-x)^{n+1}|x;0,1] = 0\ .\supset.$ P]

Formula de reductione. Exprime integrale, ubi exponentes es $m+1$ et n, per idem integrale, respondente ad exponentes m et n.

·3 $n\varepsilon N_0\ .\supset.\ S[x^m(1-x)^n|x,\Theta] = n!\,/\,\Pi(m+1+0\cdots n)$

Dem. 1 $S[x^m(1-x)^n|x,\Theta] = S\{\Sigma[(-1)^r C(n,r)x^{m+r}|r,0\cdots n]\,|\,x,\Theta\}$
$= \Sigma[(-1)^r C(n,r)\,/\,(m+r+1)\,|r,0\cdots n]$
III § Π P6·2 (p.130) $.\supset.$ P

Dem. 2 P·2 . P20·1 $.\supset.$ P

Si uno de exponentes, p. ex. n es integro, integrale habe valore rationale. Hoc resulta ex Algebra, vel ex regula de reductione præcedente; nam post subtractione de 1 unitate, plure vice, nos redde illo $=0$, et nos habe integrale de potentia.

·4 $m,n\varepsilon N_0\ .\supset.\ S[x^m(1-x)^n|x,\Theta] = m!\,n!\,/\,(m+n+1)!$
$= 1\,/\,[(m+n+1)\ C(m+n,m)]$

} Wallis a.1655 p.428-433; Stirling a.1730 p.125 {

Si ambo exponente es integro, formula habe expressione scripto in P·4.

Si nos trahe $C(m+n,m)$ de P·4, nos exprime numero de combinationes per integrale Euleriano, que habe sensu et pro m et n non integro.

·5 $S[x^m(1-x)^n|x,\Theta] = S[x^m/(1+x)^{m+n+2}|x,Q]$
[$x/(1+x)\,|\,x\ .\supset.$ P]

·6 $m,n \varepsilon \mathrm{N}_1 . m < n .\supset. \mathrm{S}[x^{m-1}/(1+x^n)|x, \mathrm{Q}] = \pi/[n \, \mathrm{s}(m\pi/n)]$
{ Euler BerolMisc. a.1743 t.7 p.151 }

·7 $a \varepsilon \theta .\supset. \mathrm{S}[x^{a-1}(1-x)^{-a}|x, \Theta] = \pi/\mathrm{s}(a\pi)$

·8 $a \varepsilon \theta . m,n \varepsilon \mathrm{Q}_0 .\supset. \mathrm{S}[x^{m+a}(1-x)^{n-a}|x, \Theta] =$
$\Pi(1 \cdots m+a) \; \Pi(1 \cdots n-a) \;/\; (m+n+1)! \; (\pi a)/\sin(\pi a)$

{ Euler BerolMisc. a.1743 t.7 p.181 }
Integrale præcedente pote es expresso per π, pro $m+n$ integro.

✻ 26. e S

·1 $x \varepsilon \mathrm{q} .\supset. \mathrm{S}(\mathrm{e}^x|x; 0,x) = \mathrm{e}^x - 1$ [$\mathrm{D}(\mathrm{e}^x|x, \mathrm{q})x = \mathrm{e}^x .\supset. \mathrm{P}$]

·2 $a \varepsilon \mathrm{Q} - \iota 1 . x \varepsilon \mathrm{q} .\supset. \mathrm{S}(a^x|x; 0,x) = (a^x - 1)/\log a$.

Dem. 1 $\mathrm{D}(a^x/\log a|x, \mathrm{q})x = a^x .\supset. \mathrm{P}$

Dem. 2 $\mathrm{S}(a^x|x; 0,x) = x \lim \Sigma[a \curlywedge (rx/n)|r, 0 \cdots (n-1)]/n \; n$
$= \lim\{(a^x - 1)/[a \curlywedge (x/n) - 1]\}|n$
$= (a^x - 1)/\lim\{[a \curlywedge (x/n) - 1]/(n/x)\}|n = \ldots$

·3 $\mathrm{S}(\mathrm{e}^{-x}|x, \mathrm{Q}) = 1$
[$\mathrm{S}(\mathrm{e}^{-x}|x, \mathrm{Q}) = \lim[\mathrm{S}(\mathrm{e}^{-x}|x, \theta x)|x, \mathrm{Q}, \infty]$
$= \lim(1 - \mathrm{e}^{-x})$ » $= 1$]

·4 $a \varepsilon \mathrm{Q} .\supset. \mathrm{S}(\mathrm{e}^{-ax}|x, \mathrm{Q}) = /a$

·5 $n \varepsilon \mathrm{Q} .\supset. \mathrm{S}(\mathrm{e}^{-x}x^n|x, \mathrm{Q}) = n\mathrm{S}(\mathrm{e}^{-x}x^{n-1}|x, \mathrm{Q})$
[$x \varepsilon \mathrm{Q} .\supset. \mathrm{D}(\mathrm{e}^{-x}x^n|x, \mathrm{Q})x = n\mathrm{e}^{-x}x^{n-1} - \mathrm{e}^{-x}x^n$ (1)
» . (1) $.\supset. \mathrm{e}^{-x}x^n = n\mathrm{S}(\mathrm{e}^{-x}x^{n-1}|x, \theta x) - \mathrm{S}(\mathrm{e}^{-x}x^n|x, \theta x)$ (2)
(2). Oper lim $.\supset.$ P]

·6 $n \varepsilon \mathrm{N}_0 .\supset. \mathrm{S}(\mathrm{e}^{-x}x^n|x, \mathrm{Q}) = n!$ [P·3·5 $.\supset.$ P]

« Integrale euleriano de secundo specie », quem Legendre *Exer.* t.2, p.4 indica per $\Gamma(n+1)$; illo exprime functio de n, que pro n integro coincide cum functione $n!$.

·7 $n \varepsilon \mathrm{N}_0 . a \varepsilon \mathrm{Q} .\supset. \mathrm{S}(\mathrm{e}^{-ax}x^n|x, \mathrm{Q}) = n!/a^{n+1}$ [$ax|x$ P·6 $.\supset.$ P]

·8 $n \varepsilon \mathrm{N}_0 . x \varepsilon \mathrm{Q}_0 .\supset. \mathrm{e}^x \mathrm{S}(\mathrm{e}^{-z}z^n|z, x+\mathrm{Q}) = n! \, \Sigma(x^r/r!|r, 0 \cdots n)$

·9 $m,n \varepsilon \mathrm{Q} - 1 .\supset. \mathrm{S}[x^m(1-x)^n|x, \Theta] =$
$\mathrm{S}(\mathrm{e}^{-x}x^m|x, \mathrm{Q}) \times \mathrm{S}(\mathrm{e}^{-x}x^n|x, \mathrm{Q}) \;/\; \mathrm{S}(\mathrm{e}^{-x}x^{m+n+1}|x, \mathrm{Q})$

Expressione de integrale Euleriano de primo specie per integrale Euleriano de secundo specie.

✱ 27. $n\varepsilon N_0 . x\varepsilon Q_0 . \supset .$

·1 $x e^x S(e^{-z}/z \,|\, z, x+Q) = \Sigma[(-1)^r r! x^{-r} \,|\, r, 0 \cdots (n-1)]$
$+(-1)^n x e^x S(e^{-z}/z^{n+1} \,|\, z, x+Q)$

·2 $x e^x S(e^{-z}/z \,|\, z, x+Q) - \Sigma[(-1)^r r! x^{-r} \,|\, r, 0 \cdots (n-1)] \;\varepsilon$
$\theta(-1)^n n! x^{-n}$

} LAGUERRE BsF. a.1879 t.7 p.72; Oeuvres t.1 p.4283 {

« Logarithmo - integrale » ; (Soldner a. 1809), « hyperlogarithmo » (Mascheroni), « Logologarithmo (Caluso) ». Indicare per « li e^{-x} ».

Terminos de serie $1-1/x+2!/x^2-3!/x^3+\ldots$ es reciproco de terminos de serie exponentiale $e^{-x} = 1-x/1+x^2/2!-\ldots$ Ergo serie considerato es divergente pro omni valore positivo de x. Sed differentia inter summa de suo primos n termine et logarithmo - integrale, habe expressione simplice, et que pote es utile in calculo numerico.

Resto minue, dum $n<x$; postea cresce sine limite. Serie de typo præcedente vocare « semi-convergente ».

Exemplo : pro $x = 100$:
$100 e^{100} S(e^{-z}/z \; z, 100+Q) = 1 - 0{\cdot}01 + 0{\cdot}0002 - 0{\cdot}000006 + 0{\cdot}00000024$
$= 0{\cdot}99019424$
cum errore negativo, minore de $10\uparrow -8$.

✱ 28. sin cos S

·0 $S(s, \Theta\pi/2) = S(c, \Theta\pi/2) = 1$

·1 $a\varepsilon q . \supset . S(s; 0,a) = 1 - ca$ } KEPLER a.1605, t.3 p.105; a.1609 t.3 p.391: « ... pro summis rectorum utimur simplicibus sinibus versis» {

Versione : integrale de sinu (recto) vale sinu verso, vel 1—cosinu.

Vide id. a.1618, t.6 p.407.

Dem. 1 $S(s; 0,a) = S(D-c; 0,a) = -ca + c0 = 1 - ca$

Dem. 2 $S(s; 0,a) = \lim (a/n) \Sigma[s(ra/n) \,|\, r, 1 \cdots n] \,|\, n$
$= s(a/2) \lim (a/n) s[(n+1)a/(2n)] / s[a/(2n)] \,|\, n$
$= 2s(a/2) \lim s[(a/2)(1+/n)] \; a/(2n) / s[a/(2n)] \,|\, n$
$= 2[s(a/2)]^2 = 1 - ca$]

·2 $m\varepsilon n - \iota 0 . \supset . S[e^{mix} \,|\, x, 2\Theta\pi] = 0$
[$S[\ldots] = \Delta[e\uparrow(mix)/(mi) \,|\, x; 0,2\pi] = 0$]

·3 $m,n\varepsilon n . m^2 -= n^2 . \supset .$
$S[s(mx)\, s(nx) \,|\, x, \Theta\pi] = S[c(mx)\, c(nx) \,|\, x, \Theta\pi] = 0$

·31 $m,n\varepsilon n . \supset . S[s(mx)\, c(nx) \,|\, x, \Theta\pi] = 0$
[P·2 . Df sin . Df cos $. \supset .$ P·3·31]

·4 $m \varepsilon N_1 . \supset . S[c(mx)^2|x, \Theta\pi] = S[s(mx)^2|x, \Theta\pi] = \pi/2$

·5 $m \varepsilon Q . \supset . S[(sx)^m|x, \Theta\pi/2]) = S[(cx)^m|x, \Theta\pi/2]$

·51 $S[(sx)^2 |x, \Theta\pi/2] = S[(cx)^2 |x, \Theta\pi/2] = \pi/4$

·6 $n \varepsilon 2+Q_0 . \supset . S[(cx)^n|x, \Theta\pi/2] = (n-1)/n \; S[(cx)^{n-2}|x, \Theta\pi/2]$

* 29·1 $m,n \varepsilon Q_0 . \supset .$
$S[x^m(1-x)^n |x, \Theta] = 2S[(sx)^{2m+1}(cx)^{2n+1} |x, \Theta\pi/2]$

·2 $n \varepsilon N_1 . \supset . S[(sx)^{2n} |x, \Theta\pi/2] = S[(cx)^{2n} |x, \Theta\pi/2] =$
$\Pi[2(1 \cdots n)-1]/\Pi[2(1 \cdots n)] \times \pi/2 = C(2n,n)/2^{2n} \times \pi/2$

·3 $n \varepsilon N_0 . \supset . S[(sx)^{2n+1}|x, \Theta\pi/2] = S[(cx)^{2n+1}|x, \Theta\pi/2] =$
$\Pi[2(1 \cdots n)]/\Pi[2(1 \cdots n)+1] = 2^{2n}/[(2n+1) \; C(2n,n)]$
[P28·5·6 $. \supset .$ P·2·3]

$x \varepsilon q . \supset :$

·4 $n \varepsilon N_1 . \supset . S[(sx)^{2n}|x; 0,x] =$
$\Sigma\{(-1)^{n-r} \; C(2n,r) \sin[(2n-2r)x]/(2n-2r)|x, 0 \cdots (n-1)\}/2^{2n-1}$
$+ x \; C(2n,n)/2^{2n}$

·5 $n \varepsilon N_1 . \supset . S[(cx)^{2n}|x; 0,x] =$
$\Sigma\{C(2n,r) \sin[(2n-2r)x]/(2n-2r)|r, 0 \cdots (n-1)\}/2^{2n-1} +$
$x \; C(2n,n)/2^{2n}$
[§e P15·3 $. \supset .$ P·4·5]

·6 $n \varepsilon N_0 . \supset . S[(sx)^{2n+1}|x; 0,x] = \Sigma\{(-1)^{n-r+1} C(2n+1,r) \times$
$[\cos[(2n-2r+1)x]-1]/(2n-2r+1)|r, 0 \cdots n\}/2^{2n}$

·7 $n \varepsilon N_0 . \supset . S[(cx)^{2n+1}|x; 0,x) =$
$\Sigma\{C(2n+1,r) \sin[(2n-2r+1)x]/(2n-2r+1)|r, 0 \cdots n\}/2^{2n}$
[§e P15·4 $. \supset .$ P·6·7]

·8 $n \varepsilon N_0 . \supset . S[(sx)^{2n+1}|x, \Theta\pi/2] = S[(cx)^{2n+1}|x, \Theta\pi/2] =$
$\Sigma[(-1)^{n-r} \; C(2n+1,r)/(2n-2r+1)|r, 0 \cdots n]/2^{2n}$
[P·6·7 $. \supset .$ P]

* 30·1 $a \varepsilon \theta\pi/2 . \supset . S\{(tx)^2 |x, \Theta a\} = ta - a$

·2 $a \varepsilon \theta\pi/2 . \supset . S(t, \Theta a) = -\log ca$

·3 $a \varepsilon \theta\pi/2 . \supset . S(/c, \Theta a) = \log\{ta + (ca)^{-1}\} = \log t(\pi/4 + a/2)$
{ ·1-·3 COTES a.1722 p.78-81 }

·4 $a\varepsilon\,\theta\pi/2 \,.\supset.\, \mathrm{S}[/(\mathrm{s}x\mathrm{c}x)|x, \Theta a) = \log \mathrm{t}a$

·5 $a\varepsilon\,\theta\pi \,.\supset.\, \mathrm{S}(/\mathrm{s}x|x, \Theta a) = \log \mathrm{t}(a/2)$

·6 $a\varepsilon\,\theta\pi/2 \,.\supset.\, \mathrm{S}(/\mathrm{t}x|x, \Theta a) = \log \mathrm{c}a$

✻ 31·1 $a,b\varepsilon \mathrm{Q} \,.\supset.\, \mathrm{S}\{/[(a\,\mathrm{s}x)^2+(b\,\mathrm{c}x)^2]\,|x, \Theta\pi/2\} = \pi/(2ab)$

·2 $a\varepsilon\mathrm{Q} \,.\, b\varepsilon\mathrm{q} \,.\, a^2>b^2 \,.\supset.\, \mathrm{S}[/(a+b\,\mathrm{c}x)\,|x, \Theta\pi] = \pi/\surd(a^2-b^2)$

·21 $z\varepsilon\,\theta\pi \,.\supset.\, \mathrm{S}[\,/(1+\mathrm{c}z\;\mathrm{c}x\,)|x, \Theta\pi\,] = \pi/\mathrm{s}z$

[$(1,\mathrm{c}z)|(a,b)\mathrm{P}{\cdot}2 \,.\supset.\, \mathrm{P}$]

·3 $a\varepsilon\mathrm{Q} \,.\, b,c\varepsilon\mathrm{q} \,.\, a^2>b^2+c^2 \,.\supset.$
$\mathrm{S}[/(a+b\,\mathrm{c}x+c\,\mathrm{s}x)\,|x, 2\pi\Theta] = 2\pi/\surd(a^2-b^2-c^2)$

·4 $z\varepsilon\,\theta\pi \,.\supset.\, \mathrm{S}[/(1-2x\,\mathrm{c}z+x^2)\,|x, \Theta] = (\pi-z)/(2\mathrm{s}z)$
{ EULER *Calc. Integr.* a.1794 t.4 p.288 (PetrNC a.1774 t.19) }

✻ 32·1 $\mathrm{S}\{(\mathrm{s}\,x)/x\,|x, \mathrm{Q}\} = \pi/2$
{ EULER (a.1781) *Calc. Integr.* a.1794 t.4 p.345 }

·2 $\mathrm{S}\{(\mathrm{s}x\,/x)^2\,|x, \mathrm{Q}\} = \pi/2$ { EULER ib. }

[$\mathrm{D}[(\sin x)^2/x\;|x, x] = 2\,\sin x\,\cos x\,/x - (\sin x)^2/x^2$ (1)
(1) $.\supset.\, (\sin b)^2/b - (\sin a)^2/a = \mathrm{S}[(\sin 2x)/x|x, a\frown b] - \mathrm{S}(\sin x/x)^2|x, a\frown b]$
$\mathrm{S}[(\sin x/x)^2|x, a\frown b] = -(\sin b)^2/b + (\sin a)^2/a + \mathrm{S}(\sin 2x\,/x\,|x, a\frown b)$
Oper $\lim(|a,\mathrm{Q},0) \,.\supset.$
$\mathrm{S}[(\sin x/x)^2|x, 0\frown b] = -(\sin b)^2/b + \mathrm{S}(\sin 2x\,/x|x, 0\frown b)$
$\text{»} \quad = \quad \text{»} \quad + \mathrm{S}(\sin z\,/z\;|z, 0\frown 2b)$
Oper $(\lim|b,\infty) \,.\supset.\, \mathrm{S}[(\sin x/x)^2|x, \mathrm{Q}] = \pi/2$]

✻ 33·1 $y\varepsilon\mathrm{Q} \,.\supset.\, \mathrm{S}(\mathrm{s}^{-1}y|y, \Theta y) = y\mathrm{s}^{-1}y+\surd(1-y^2)-1$

·2 $y\varepsilon\mathrm{Q} \,.\supset.\, \mathrm{S}(\mathrm{t}^{-1}, \Theta y) = y\mathrm{t}^{-1}y-[\log(1+y^2)]/2$

✻ 34. $a\varepsilon\mathrm{q} \,.\supset.$

·1 $\mathrm{sgn}a = 2/\pi\;\mathrm{S}\{\mathrm{s}(ax)\,/x\,|x, \mathrm{Q}\}$

·2 $\mathrm{mod}a = 2/\pi\;\mathrm{S}\{\mathrm{s}(ax)^2/x^2\,|x, \mathrm{Q}\}$

·3 $a,b\varepsilon\mathrm{q} \,.\supset.\, 2\mathrm{S}[\mathrm{s}(ax)\,\mathrm{s}(bx)/x^2\,|x, \mathrm{Q}] = \pi\,\min(\iota a\cup\iota b)$

✻ 35·1 $\mathrm{S}[x/(\mathrm{e}^x-1)\,|x, \mathrm{Q}] = \pi^2/6$

·8 $n\varepsilon\mathrm{N}_1 \,.\supset.\, \mathrm{S}[x^{2n-1}/(\mathrm{e}^{2\pi x}-1)\,|x, \mathrm{Q}] = \mathrm{B}_n/(4n)$
{ EULER PetrNC. a.1769 t.14 I p.151 }

✻ 36·1 $a \varepsilon \mathrm{Q} \,.\, b \varepsilon \mathrm{q} \,.\supset.\, \mathrm{S}[\mathrm{e}^{-ax}\mathrm{c}(bx) \,|x, \mathrm{Q}] = a/(a^2+b^2)$
$.\, \mathrm{S}[\mathrm{e}^{-ax}\mathrm{s}(bx) \,|x, \mathrm{Q}] = b/(a^2+b^2)$

[$\mathrm{S}(\mathrm{e}^{(-a+\mathrm{i}b)x}|x,\mathrm{Q}) = /(a-\mathrm{i}b) = (a+\mathrm{i}b)/(a^2+b^2)$. Oper real . Oper imag $.\supset.$ P]

·2 $p,q \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{S}\}\mathrm{e}^{-px}[\mathrm{c}(qx)-1]/x|x, \mathrm{Q}\{ = \log[p/\surd(p^2+q^2)]$.
$\mathrm{S}[\mathrm{e}^{-px}\mathrm{s}(qx)/x \,|x, \mathrm{Q}] = \mathrm{t}^{-1}(q/p)$

} Euler, a.1781 *Calc. Integr.*, a.1794 t.4 p.345 {

[$\mathrm{S}[\mathrm{e}^{(-p+q\mathrm{i})x}|x, \Theta x] = [1-\mathrm{e}^{(-p+q\mathrm{i})x}]/(p-q\mathrm{i}) \,.\supset.\, \mathrm{S}|\mathrm{S}[\mathrm{e}^{(-p+q\mathrm{i})x}|x, \Theta x]|q, \Theta q\{$
$= \mathrm{S}[/(p-q\mathrm{i})|q, \Theta q] - \mathrm{e}^{-px}\mathrm{S}[\mathrm{e}^{q\mathrm{i}x}/(p-q\mathrm{i})|q, \Theta q] \,.\supset.$
$\mathrm{S}[\mathrm{e}^{-px}(\mathrm{e}^{q\mathrm{i}x}-1)/x|x, \Theta x] = \log[p/(p-q\mathrm{i})] - \mathrm{i}\mathrm{e}^{-px}\mathrm{S}[\mathrm{e}^{q\mathrm{i}x}/(p-q\mathrm{i})|q, \Theta q]$.
Oper $(\lim|x, \infty) \,.\supset.\, \mathrm{S}[\mathrm{e}^{-px}(\mathrm{e}^{q\mathrm{i}x}-1)/x|x, \mathrm{Q}] = \log[p/(p-q\mathrm{i})] =$
$\log[p(p+q\mathrm{i})/(p^2+q^2)]$. §e P13·5 . §s^{-1} P4·5 $.\supset.\, \mathrm{S}|\mathrm{e}^{-px}[\mathrm{c}(qx)-1]/x|x, \mathrm{Q}\{$
$+\mathrm{i}\mathrm{S}[\mathrm{e}^{-px}\,\mathrm{s}(qx)/x|x,\mathrm{Q}] = \log[p/\surd(p^2+q^2)]+\mathrm{i}\mathrm{t}^{-1}(q/p)$. §q′ P1·6 $.\supset.$ P]

·3 $n \varepsilon \mathrm{N}_1 \,.\, a \varepsilon \mathrm{q} \,.\supset.$
$\mathrm{S}[\mathrm{e}^{-ax}(\mathrm{s}x)^{2n+1}|x, 2\Theta\pi] = (2n+1)!(1-\mathrm{e}^{-2\pi a})/\Pi\}[a^2+(2r+1)^2]|r, 0\cdots n\{$
$\mathrm{S}[\mathrm{e}^{-ax}(\mathrm{s}x)^{2n}\,|x, 2\Theta\pi] = (2n)!\,(1-\mathrm{e}^{-2\pi a})/\}a\Pi[(a^2+4r^2)\,|r, 0\cdots n]\{$

} Cauchy ParisCR. a.1854 t.39 p.129; Œuvres s.1 t.12 p.175 {

·4 $\mathrm{S}[(\mathrm{c}x + x\,\mathrm{s}x)/(1+x^2) \,|x, \mathrm{Q}] = \pi/\mathrm{e}$

} Laplace (ParisM. a.1782, publ. a.1785) Oeuvres t.10 p.264

·5 $a \varepsilon \mathrm{Q} \,.\, b \varepsilon \mathrm{q}' \,.\, \mathrm{real}\, b\, \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{S}[\mathrm{c}(ax)/(b^2+x^2) \,|x, \mathrm{q}] = \pi/b\; \mathrm{e}^{-ab}$

} Laplace, Dirichlet t.1 p.112 {

·6 $a \varepsilon \mathrm{Q} \,.\, b \varepsilon \mathrm{q}' \,.\, \mathrm{real}\, b\, \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{S}[\mathrm{e}^{-ax\mathrm{i}}/(b^2+x^2) \,|x, \mathrm{q}] = \pi/b\; \mathrm{e}^{-ab}$

}Dirichlet t.1 p.113{

✻ 37·1 $\mathrm{S}(\log \mathrm{s}, \Theta\pi/2) = \mathrm{S}(\log \mathrm{c}, \Theta\pi/2) = -(\pi \log 2)/2$

} Euler PetrA. a.1777 t.1_2 p.7 {

[$\mathrm{S}(\log \sin x|x, \Theta\pi/2) = \mathrm{S}[\log(\sin x/x)|x, \Theta\pi/2] + \mathrm{S}(\log x|x, \Theta\pi/2)$ (1)
(1) $.\supset.\, \mathrm{S}(\log \sin x|x, \Theta\pi/2)\, \varepsilon\, \mathrm{q}$ (2)
(1) $.\supset.\, \mathrm{S}(\log \sin, \Theta\pi/2) = \mathrm{S}[\log 2 + \log\sin(x/2) + \log\cos(x/2)|x, \Theta\pi/2]$
$= \pi/2 \log 2 + \mathrm{S}[\log\sin(x/2)|x, \Theta\pi/2] + \mathrm{S}[\log\sin(x/2)|x, \pi/2 \frown \pi] =$
$\pi/2 \log 2 + \mathrm{S}[\log\sin(x/2)|x, \Theta\pi] = \pi/2 \log 2 + 2\mathrm{S}(\log\sin x|x, \Theta\pi/2)$ (3)
(2) . (3) $.\supset.$ P]

[$\mathrm{S}(\log \sin, \Theta\pi/2) = \lim\Sigma\}\pi/(2n)\log\sin[r\pi/(2n)]|r, 1\cdots n\{|n$
$= \pi/2 \lim \log \Pi\}\sin[r\pi/(2n)]|r, 1\cdots n\{/n|n$ (1)
(1) . §π P6·4 $.\supset.\, \mathrm{S}(\log\sin, \Theta\pi/2) = \pi/2 \lim \log(\surd n/2^{n-1})/n|n$
$= \pi/2 \lim[(\log n)/(2n) - (\log 2)(n-1)/n]|n = -(\pi\log 2)/2$]

·2 $\mathrm{S}(x\log\sin x \,|x, \Theta\pi) = -(\pi^2\log 2)/2$

·3 $\mathrm{S}\}x\sin x/[1+(\cos x)^2] \,|x, \Theta\pi\{ = \pi^2/4$

·4 $r\varepsilon q$. mod$r > 1$.⊃. S[log(1—2r cx+r^2) |x, $\Theta\pi$] = πlog r^2
[Hp .⊃. S[log(1—2r cx+r^2) |x, $\Theta\pi$] =
lim[(π/n)Σ{log[1—2r c($m\pi$/n)+r^2] |m, 1⋯n}] |n =
lim[(π/n)logΠ{[1—2r c($m\pi$/n)+r^2] |m, 1⋯(n—1)}] |n (1)
Hp . §π P6·2 .⊃. r^{2n} —1 = (r^2—1)Π{[1—2r c($m\pi$/n)+r^2] |m, 1⋯(n—1)} (2)
(1) . (2) .⊃. S[log(1—2r cx+r^2) |x, $\Theta\pi$] =
lim{(π/n)log[(r^{2n}—1)/(r^2—1)]} |n = lim[(π/n)log(r^{2n}—1)] |n =
π log[lim(r^{2n}—1)$\wedge$/n |n] .⊃. P]

·5 S{[log(1+x)]/(1+x^2) |x, Θ} = (π/8) log2
{ Bertrand JdM. t.8 a.1843 p.112 }
[S{[log(1+y)]/(1+y^2) |y,Θ} =
S{[log(1+y)]Dt^{-1}y |y,Θ}=S[log(1+tx) |x,$\Theta\pi$/4] .
P9·4 .⊃. S[log(1+tx) |x,$\Theta\pi$/4] = S{log[1+t(π/4—x)] |x,$\Theta\pi$/4}=
S{log[2/(1+tx) |x,$\Theta\pi$/4}=(π/4)log2 —S[log(1+tx) |x,$\Theta\pi$/4] .⊃.
S[log(1+tx) |x,$\Theta\pi$/4]=(π/8)log2]

·6 S{(t^{-1}y)/(1+y) |y, Θ} = (π/8) log2
[P31·3 .⊃. S[(t^{-1}y)/(1+y) |y,Θ]=S[(t^{-1}y)Dlog(1+y) |y,Θ]=
(π/4)log2—S[log(1+y) Dt^{-1}y |y,Θ] . P·5 .⊃. P]

·7 S{x sx / [1+(sx)^2] |x, $\Theta\pi$} = π[log($\surd$2+1)]/$\surd$2

✻ 38·1 S(e$^{-x^2}$ |x, q) = $\surd\pi$
[S(e$^{-x^2}$|x, q) = S{lim[/(1+x^2/m)^m|m]|x,q} = limS[/(1+x^2/m)^m|x, q]|m =
2 limS{/[1+(tang$y$)^2]$^{m-1}\surd m$|y, $\Theta\pi$/2}|m =
2 lim{$\surd m$|S[(cos$y$)$^{2m-2}$|y, $\Theta\pi$/2]}|m =
2 lim$\surd m$ Π[1—/(2r)|r, 1⋯(m—1)]× π/2 |m =
π lim$\surd${1.3.3...(2m—3)(2m—3)m/[2.2.4.4...(2m—2)(2m—2)]} |m =
π lim$\surd\Pi${2E(n/2)+1]/[2E[(n+1)/2]|n, N_1} × lim$\surd$[m/2(m—1)]|m (1)
(1). §π P10·1 .⊃. P]
{ Euler PetrC. a.1730 (edito a.1738) t.5 p.44 }

·2 $x\varepsilon$Q .⊃. S(e$^{-x^2}$|x,Θx) = Σ{(—1)$^n x^{2n+1}$/[(2n+1)n!]|n,N_0} =
xe$^{-x^2}${1+Σ[(2x^2)$^{2n+1}$/Π[2×0⋯(n—2)+1]|n,N_0]}

Ce integrale se præsenta in «Calculo de probabilitate». Laplace inveni illo in theoria de refractione in astronomia, a. 1805 (Traité de Mécanique Céleste) Oeuvres, t. 4, pag.254.

Existe plure tabula de valores de S(e$^{-x^2}$|x, Θx):
Bertrand, Calcul des probabilités, Paris, 1889, p.329;
Kämpfe Bruno, Tafel des Integrals Φ, Leipzig 1893.

TABULA DE $\Phi x = (2/\sqrt{\pi})S(e^{-x^2}|x, \Theta x)$

	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
0	·000	·112	·222	·328	·428	·520	·603	·677	·742	·796
1	·842	·880	·910	·934	·952	·966	·976	·983	·989	·992
2	·995	·997	·998	·998	·999	.....				

Exemplo : Pro calcula $\Phi(1{\cdot}85)$:

Lege in horizontale 1, in verticale ·8 $\Phi(1{\cdot}8) = 0{\cdot}989$

Interpola $0{\cdot}05 \times [\Phi(1{\cdot}9) - \Phi(1{\cdot}8)]/0{\cdot}1 = 0{\cdot}001$

Summa $\Phi(1{\cdot}85) = 0{\cdot}990$

que differ de valore dato per tabula plus amplo, per 1 unitate de ultimo ordine.

* 39·1 $a \varepsilon q'$. real$a > 0$.⊃. $S(e-ax^2|x, q) = \sqrt{(\pi/a)}$

·2 $a \varepsilon Q$.⊃. $S[e\wedge -(x^2+a^2/x^2) \mid x, Q] = \sqrt{\pi}\, e^{-a}/2$

} LAPLACE (ParisM. a.1810, edito a.1811) Oeuvres t.12 p.368 {

·3 $S[e\wedge(ix^2) \mid x, q] = (1+i)\sqrt{(\pi/2)}$

·4 $S[s(x^2) \mid x, Q] = S[c(x^2) \mid x, Q] = \sqrt{(\pi/2)}/2$ [= P·3]

} EULER a.1781, *Calc. Integr.* a.1794 t.4 p.339 {

Euler inveni ce integrales in theoria de curva elastico, a.1744 p.276; et in 1781 determina valore de illos.

Fresnel, a.1818 t.1 p.178, inveni illos in theoria de diffractione de luce.

Vide et : Cauchy, Œuvres s.1 t.7 p.151.

Dirichlet, Œuvres t.1 p.245, 264 da novo demonstratione.

Fresnel calcula tabula de ce integrale inter limites variabile.

·5 $a,b,c \varepsilon q'$. real $a > 0$.⊃.
$S[e\wedge -(ax^2+bx+c) \mid x, q] = \sqrt{(\pi/a)}\, e\wedge[-c+b^2/(4a)]$

} CAUCHY a.1827 s.2 t.7 p.280 {

* 40. INTEGRATIONE PER SERIE.

·1 $u \varepsilon$ Intv . $f \varepsilon$ qf$(u{:}N_0)$. $\Sigma\{$l'mod$[f(x,r) \mid x' u] \mid r, N_0\} \varepsilon Q$:
$r \varepsilon N_0$.⊃$_r$. $S[f(x,r) \mid x, u] \varepsilon q$:⊃.
$S\{\Sigma[f(x,r) \mid r, N_0] \mid x, u\} = \Sigma\{S[f(x,r) \mid x, u] \mid r, N_0\}$ Comm(S,Σ)

f indica quantitate functione de duo variabile, uno in intervallo u, et altero sumente valores 0, 1, 2, ...

Id es, $f(x,0) + f(x,1) + f(x,2) + ...$ es serie ; et omni suo termine es functione de variabile x in intervallo u.

Si serie formato per limites supero de valores absoluto de termines de serie dato, ubi x varia in intervallo u, es convergente; et si omni termine de serie dato es integrabile in intervallo u; tunc integrale de serie vale serie de integrales.

[$x\varepsilon u \,.\, n\varepsilon N_1 \,.\, R_n = \Sigma\{l'\,\mathrm{mod}[f(x,r)|x`u]|r, n+N_1\} \;.\supset.$

$\Sigma[f(x,r)|r, N_0] = \Sigma\, f(x,r)|r, 0\cdots n] + \Sigma[f(x,r)|r, n+N_1] \,.$

$\mathrm{mod}\Sigma[f(x,r)|r, n+N_1] \leqq R_n$ (1)

$n\varepsilon N_1 \,.\, (1) \,.\, P9{\cdot}11 \;.\supset.$

$S'\{\Sigma[f(x,r)|r, N_0]|x, u\} = \Sigma\{S[f(x,r)|x, u]|r, 0\cdots n\} + S'\{\Sigma[f(x,r)|r, n+N_1]|x, u\} \,.$

$S_{,}\{$ » » » » » $+ S_{,}\{$ » » $\}$.

$\mathrm{mod}\, S'\{\Sigma[f(x,r)|r, n+N_1]|x,u\} \leqq \mathrm{Long}u \times R_n$

$\mathrm{mod}\, S_{,}\{$ » » $\} \leqq$ » » (2)

$\mathrm{Hp} \;.\supset.\; \lim R_n|n = 0$ (3)

$(2) \,.\, (3) \;.\supset.\; S'\{\Sigma[f(x,r)|r, N_0]|x,u\} = \Sigma\{S[f(x,r)|x,u]|r, 0\cdots n\} \,.$

$S_{,}\{$ » » $=$ » » $.\supset.\; P$]

In vero summa de serie vale summa de termines de 0 ad n, plus resto, que es, in valore absoluto, minore de resto in serie de limites supero, quem me voca R_n. Tunc integrale supero de summa de serie vale summa de integrales de primos termine, plus integrale supero de resto, que es minore de longitudo de intervallo de integrale, per R_n. Idem pro integrale infero. Si nune n verge ad ∞, R_n ve rge ad 0: tunc integrale supero de serie, et suo integrale infero, es expresso per serie de integrales.

Nos substitue ad conditione de convergentia uniforme que occurre in plure tractato, alio plus simplo. Vide pag. 233, 295.

✻ 41·1 $S(x^x|x, \Theta) = 1-2^{-2}+3^{-3}-\ldots = {\cdot}7834305\ldots$

{ Joh. BERNOULLI, a.1694 t.1 p.185 }

·2 $n\varepsilon q \;.\supset.\; S(x^{-nx}|x, \Theta) = \Sigma[n^r(r+1)^{-(r+1)}|r, N_0]$

{ EULER a.1768 t.1 p.144: « Quæ ob concinnitatem terminorum omnino est notatu digna. » }

✻ 42. $n,p\varepsilon q \,.\, \mathrm{mod}n < 1 \;.\supset.\; S[(1+2n\mathrm{c}x+n^2)^p|x, \Theta\pi] =$
$\pi\Sigma\{[C(p,r)n^r]^2|r, N_0\}$

[$x,n,p\varepsilon q.\, \mathrm{mod}n<1 \;.\supset.\; (1+2n\mathrm{c}x+n^2)^p = (1+n\mathrm{e}^{ix})^p\, (1+n\mathrm{e}^{-ix})^p =$

$[1+pn\mathrm{e}^{ix} + p(p-1)/2n^2\mathrm{e}^{2ix} + \cdots][1+pn\mathrm{e}^{-ix} + p(p-1)/2n^2\mathrm{e}^{-ix}+\ldots] =$

$\Sigma[C(p,r)n^r]^2|r, N_0]+$

$\Sigma[C(p,r)C(p,s)n^{r+s}\,\mathrm{e}^{(r-s)ix}|(r,s), (N_0;N_0)\frown(r,s)\ni(r-=s)]$ (1)

(1) . OperS . P 28·2 $.\supset.$ P]

✻ 43. $a,b\varepsilon\text{q} \,.\, n\varepsilon\text{N}_1 \,.\, f, \text{D}^{n+1}f\ \varepsilon\ (\text{qF}a\frown b)\text{cont} \,.\supset.$

$$fb - \Sigma[(b-a)^r/r!\ \text{D}^r fa \mid r, 0\cdots n] = /n!\ \text{S}[(b-x)^n\ \text{D}^{n+1}fx \mid x; a,b]$$
$$= (b-a)^{n+1}/n!\ \text{S}\{(1-t)^n \text{D}^{n+1}f[a+t(b-a)] \mid t, \Theta\}$$

[$\text{Hp} \,.\, g = \{fb - \Sigma[(b-x)^r/r!\ \text{D}^r fx \mid r, 0\cdots n] \mid x, a\frown b\} \,.\supset.\, gb=0$ (1)

$\text{D}g = \{\Sigma[(b-x)^{r-1}(r-1)!\ \text{D}^r fx \mid r, 1\cdots n]$
$\quad - \Sigma[(b-x)^r/r!\text{D}^{r+1}fx \mid r, 0\cdots n]\} \mid x, a\frown b$
$\quad = [-(b-x)^n/n!\ \text{D}^{n+1}fx \mid x, a\frown b]$ (2)

$gb - ga = \text{S}(\text{D}g, a\frown b)$ (3)

$(1) \,.\, (2) \,.\, (3) \,.\supset.\, \text{P}$]

{ LAGRANGE a.1797 ; Œuvres t.9 p.73 }

Expressione, sub forma de integrale, de differentia inter valore fb et summa de primos $n+1$ termine in formula de Taylor.

✻ 44. FORMULAS DE QUADRATURA.

·1 $f\varepsilon\ (\text{qFq})\text{integ} \,.\, \text{grad}f=1 \,.\, a,b\,\varepsilon\text{q} \,.\, a<b \,.\supset.$

$$\text{S}(f, a\frown b) = (b-a)(fa+fb)/2$$

[$p,q\varepsilon\,\text{q} \,.\, f=[(p+qx) \mid x, \text{q}] \,.\supset.\, fa=p+qa \,.\, fb=p+qb \,.$
$\text{S}(f, a\frown b) = p(b-a)+q(b^2-a^2)/2 = (b-a)[p+q(a+b)/2] \,.\supset.\, \text{P}$]

Integrale de functione integro de gradu uno, extenso ad aliquo intervallo, vale amplitudo de intervallo per valore medio arithmetico inter primo et ultimo valore de functione.
Integrale considerato mensura area de trapezio.

·2 $a,b\varepsilon\text{q} \,.\, a<b \,.\, f,\text{D}^2f\ \varepsilon\ \text{qF}a\frown b \,.\supset.$

$$\text{S}(f,a\frown b) - (b-a)(fa+fb)/2\ \varepsilon\ -(b-a)^3(\text{D}^2f\text{‘}a\frown b)/12$$

[$g=[fa+(x-a)(fb-fa)/(b-a)] \mid x \,.\, x\varepsilon\, a\frown b \,.\, \S\text{D P18·1} \,.\supset.$
$fx-gx\ \varepsilon\ -(x-a)(b-x)(\text{D}^2f\text{‘}a\frown b)/2 \,.\, \text{S}(g,a\frown b) = (b-a)(fa+fb)/2 \,.$
$\text{S}[(x-a)(b-x) \mid x, a\frown b] = (b-a)^3/6 \,.\supset.\, \text{P}$]

Si in intervallo de a ad b, functio f habe derivata de ordine 2, tunc differentia inter integrale de f, de a ad b, et area de trapezio constructo super ordinatas fa et fb, vel errore que resulta si ad integrale nos substitue trapezio, es expresso per formula scripto.

In vero, si nos voca gx functio de gradu 1, que pro $x=a$ et $x=b$ coincide cum fx, tunc $fx-gx$ es exprimibile ope derivata de f, de ordine 2; integrale de g vale trapezio; ...

Exemplo. Si nos pone $fx = 1/x$, vel $f = (1/x)|x = /$, $a=1$, $b=2$, seque $\mathrm{D}fx = -1/x^2$, $\mathrm{D}^2fx = 2/x^3$. Ergo $\mathrm{S}(/, 1 \urcorner 2)$, $(= \log 2$, vide P19) vale $(1+1/2)/2$, cum errore de forma $-1/(6x^3)$, ubi x es valore incognito) inter 1 et 2. Ce errore es negativo, et minore in valore absoluto de 1/6; und $0{\cdot}75 > \log 2 > 0{\cdot}75 - 1/6 = 0{\cdot}59$.

·3 $\mathrm{Hp}{\cdot}2 \,.\, n \varepsilon \mathrm{N}_1+1 \,.\supset.\, \mathrm{S}(f, a \urcorner b) -$
$(b-a)[fa+fb+2\Sigma\{f[a+r(b-a)/n]|r, 1\cdots(n-1)\}]/(2n) \; \varepsilon$
$-(b-a)^3(\mathrm{D}^2f\text{‘}a \urcorner b)/(12n^2)$

[$\mathrm{P}{\cdot}2 \,.\supset.\, \mathrm{S}(f, a \urcorner b) -$
$\Sigma[\{f[a+r(b-a)/n]+f[a+(r+1)(b-a)/n]\}(b-a)/n \,|r, 0\cdots(n-1)]$
$\varepsilon \; -(b-a)^3/(12n^3)\Sigma\{\mathrm{D}^2f\text{‘}[a+(r+\theta)(b-a)/n] \,|r, 0\cdots(n-1)\}$
$\varepsilon \; -(b-a)^3/(12n^3)\times n\mathrm{D}^2f\text{‘}a \urcorner b \,.\supset.\, \mathrm{P}$]

Si functio f habe derivata de ordine 2 in toto intervallo de a ad b, tunc nos divide intervallo de integratione in n parte aequale, et nos substitue ad omni integrale partiale, trapezio correspondente. Summa de trapezios es valore approximato de integrale; et differentia vale errore praecedente (P·2) diviso per quadrato de numero de partes.

In vero, errore totale es summa de errores partiale, dato per formula ·2. Ce summa contine factore commune $-(b-a)^3/(12n^3)$, multiplicato per summa de n valores de D^2f, que vale n multiplicato per valore medio inter valores de D^2f.

Exemplo. Nos considera integrale praecedente

$$\log 2 = \mathrm{S}(/, 1 \urcorner 2).$$

Nos divide intervallo de 1 ad 2 in 10 parte, cum valores 1, 1·1, 1·2, ... 1·9, 2; valore approximato de integrale es

$$(1/1+2/1{\cdot}1+2/1{\cdot}2+\ldots+2/1{\cdot}9+1/2)/20\,;$$

errore vale praecedente diviso per 100; ergo es negativo, et minore in valore absoluto de 1/600.

·4 $\mathrm{Hp}{\cdot}1 \,.\supset.\, \mathrm{S}(f, a \urcorner b) = (b-a)f[(a+b)/2]$

·5 $\mathrm{Hp}{\cdot}2 \,.\supset.\, \mathrm{S}(f, a \urcorner b)-(b-a)\, f[(a+b)/2] \;\varepsilon\; (b-a)^3(\mathrm{D}^2f\text{‘}a \urcorner b)/24$

[$g = \{f[(a+b)/2] + [x-(a+b)/2]\, \mathrm{D}f[(a+b)/2]\}|x \,.\, x \varepsilon a \urcorner b \,.\supset.$
$fx - gx \;\varepsilon\; [x-(a+b)/2]^2\, (\mathrm{D}^2f\text{‘}a \urcorner b)/2 \,.$
$\mathrm{S}(g, a \urcorner b) = (b-a)\, f[(a+b)/2] \,.$
$\mathrm{S}\{[x-(a+b)/2]^2|x, a \urcorner b\} = (b-a)^3/12 \,.\supset.\, \mathrm{P}$]

·6 $\mathrm{Hp}\; \mathrm{P}{\cdot}2 \,.\supset.\, \mathrm{S}(f, a \urcorner b) = (b-a)(fa+fb)/2$
$-\mathrm{S}[(x-a)(b-x)\, \mathrm{D}^2fx|x, a \urcorner b]/2$

$[\ \mathrm{D}[(2x-a-b)fx-(x-a)(x-b)\mathrm{D}fx]|x=[2fx-(x-a)(x-b)\mathrm{D}^2fx]|x$ (1)

$\mathrm{Increm}\{[(2x-a-b)fx-(x-a)(x-b)\mathrm{D}fx]|x,\ a\frown b\}=(b-a)(fb+fa)$ (2)

$(1)\ .\ (2)\ .\ \mathrm{Oper\ S}\ .\supset.\ \mathrm{P}\ ]$

Exprime differentia inter valore exacto de integrale, et valore approximato dato per formula de trapezio, sub forma de integrale.

✻ 45.

·1 $f\varepsilon(\mathrm{qFq})\mathrm{integ}\ .\ \mathrm{grad}f\leqq 3\ .\ a,b\varepsilon\mathrm{q}\ .\ a<b\ .\supset.$

$\mathrm{S}(f,a\frown b)=(b-a)\{fa+fb+4f[(a+b)/2]\}/6$

$[\ g=\{f[(a+b)/2+z(b-a)/2]\mid z,\mathrm{q}]\ .\supset.\ g\varepsilon(\mathrm{qFq})\mathrm{integ}\ .\ \mathrm{grad}g=\mathrm{grad}f\ .$

$g(-1)=fa\ .\ g0=f[(a+b)/2]\ .\ g1=fb\ .$

$\mathrm{S}(f,a\frown b)=(b-a)/2\times\mathrm{S}[(g,(-1)\frown 1)]$ (1)

$p,q,r,s\ \varepsilon\mathrm{q}\ .\ g=[(p+qz+rz^2+sz^3)\mid z,\mathrm{q}]\ .$

$g(-1)=p-q+r-s\ .\ g0=p\ .\ g1=p+q+r+s\ .$

$\mathrm{S}[g,(-1)\frown 1]=2p+2r/3=[g(-1)+4g0+g1]/3$ (2)

$(1)\ .\ (2)\ .\supset.\ \mathrm{P}\ ]$

Si f es functio integro, de gradu non superiore ad tres, tunc integrale de f, extenso ad intervallo de a ad b, vale amplitudo $b-a$ de intervallo, multiplicato per valore medio arithmetico inter fa, fb et $f[(a+b)/2]$ cum pondo 1,1,4.

Cavalieri a.1639 p.446:

« Per havere la capacità della botte... moltiplicaremo la terza parte della lunghezza della botte..., in due cerchi maggiori ed uno dei minori... ».

Gregory, a.1668; Cotes, *opuscula* a.1722 p.33.

Simpson, a.1743 p.109. Plure Auctore voca formula praecedente « formula de Simpson ».

·2 $a,b\varepsilon\mathrm{q}\ .\ a<b\ .\ f,\mathrm{D}^4f\ \varepsilon\ \mathrm{qF}a\frown b\ .\supset.\ \mathrm{S}(f,a\frown b)-$

$(b-a)\{fa+fb+4f[(a+b)/2]\}/6\ \varepsilon\ -(b-a)^5(\mathrm{D}^4f'a\frown b)/(4!\ 5!)$

$[\ g=\{f[(a+b)/2+x(b-a)/2]|x,\ -1\frown 1\}\ .\supset.$

$\mathrm{S}(f,a\frown b)=(b-a)/2\times\mathrm{S}(g,-1\frown 1)$ (1)

$\mathrm{Hp}(1)\ .\ h=[(1-x^2)g0+x(1-x^2)\,\mathrm{D}g0+x^2(1+x)/2\,g1+x^2(1-x)/2\,g(-1)]|x\ .\supset.$

$h\varepsilon\,(\mathrm{qfq})\mathrm{integ}\ .\ \mathrm{grad}h\leqq 3\ .\ h(-1)=g(-1)=fa\ .\ h0=g0=f[(a+b)/2]\ .$

$\mathrm{D}h0=\mathrm{D}g0\ .\ h1=g1=fb$ (2)

$\mathrm{Hp}(2)\ .\ x\varepsilon -1\frown 1\ .\supset.\ gx-hx\ \varepsilon\ x^2(x^2-1)\mathrm{D}^4g'(-1\frown 1)/4!\ .$

$\mathrm{D}^4gx=(b-a)^4/16\ \mathrm{D}^4f[(a+b)/2+x(b-a)/2]$ (3)

$\mathrm{Hp}(2)\ .\ (3)\ .\supset.\ \mathrm{S}(g,-1\frown 1)-\mathrm{S}(h,-1\frown 1)$

$\varepsilon\ \mathrm{D}^4g'(-1\frown 1)/4!\ \mathrm{S}[x^2(x^2-1)\mid x,-1\frown 1]$ (4)

$\mathrm{Hp}(2).\ \mathrm{P}{\cdot}1\ .\supset.\ \mathrm{S}(h,-1\frown 1)=[h(-1)+4h0+h1]/3$

$=\{fa+4f[(a+b)/2]+fb\}/3$ (5)

$\mathrm{S}[x^2(x^2-1)\mid x,-1\frown 1]=-4/15$ (6)

$(1)\ .\ (4)\ .\ (5)\ .\ (6)\ .\supset.\ \mathrm{P}\ ]$

Formula de P·1, exacto pro functiones de gradu 2 aut 3, es approximato pro functione arbitrario. Errore in isto approximatione vale minus amplitudo de intervallo de integratione, ad potestate 5, multiplicato pro uno ex valores de derivata de ordine 4, de functione f in intervallo, toto diviso per 4! 5!.

Si nos applica ce regula ad exemplo de P44·2, $S(/, 1 \frown 2) = \log 2$, nos habe valore approximato

$$(1/1 + 4/1{\cdot}5 + 1/2)/6 = 0{\cdot}694 \ldots$$

cum errore de forma $-1/(120x^5)$, ubi $x \varepsilon 1 \frown 2$; ergo $0{\cdot}68 < \log 2 < 0{\cdot}694 \ldots$

·3 $\mathrm{Hp}{\cdot}2 \,.\, n \varepsilon N_1 \,.\supset.\, S(f, a \frown b)$
$\varepsilon\, (b-a)[fa+fb+2\Sigma\{f[a+2r(b-a)/n] \mid r, 1 \cdots (n-1)\} +$
$4\Sigma\{f[a+(2r+1)(b-a)/(2n)] \mid r, 0 \cdots (n-1)\}]/(6n) -$
$(b-a)^5(D^4 f \,\text{'}\, a \frown b)/(4!\, 5!\, n^4)$

Si nos divide intervallo de a ad b in n partes, et nos applica ad omni parte formula præcedente, nos obtine valore magis approximato de integrale. Differentia decresce ut numero de partes ad potestate 4.

Si, in exemplo præcedente, nos pone $n=5$, resulta:

Valore approximato
$= [1/1+1/2+$
$+ 2(1/1{\cdot}2+1/1{\cdot}4+1/1{\cdot}6+1/1{\cdot}8)$
$+ 4(1/1{\cdot}1+1/1{\cdot}3+1/1{\cdot}5+1/1{\cdot}7+1/1{\cdot}9)]/30$
$= (1{\cdot}5+2\times 2{\cdot}72818+4\times 3{\cdot}45955\ldots)/30$
$= 20{\cdot}79456/30 = 0{\cdot}693152$

Errore vale errore in calculo præcedente, diviso per $5^4 = 625$; ergo errore es negativo, et minore in valore absoluto de 1/75000. Seque:

$$0{\cdot}69314 < \log 2 < 0{\cdot}693152.$$

Calculo numerico cum formula ·3 es pauco plus complexo que calculo cum formula P44·3, et approximatione fi 100 vice majore.

Me da expressiones de restos P·2·3 in «*Applicazioni geometriche* a.1887, p.206».

F. Rimondini, *Sul calcolo approssimato degli integrali doppi*, Torino A. a.1904-05, extende formula P·2 ad integrale duplo.

·4 $\mathrm{Hp}{\cdot}1 \,.\supset.$
$S(f, a \frown b) = (b-a)\{fa+3f[(2a+b)/3]+3f[(a+2b)/3]+fb\}$

{ Newton, a.1711, Opuscula t.1 p.281 }

Vide formulas successivo in Formul. t.4, p.187.

❋ 46. Arcu

$u \varepsilon \mathrm{Intv} \,.\, p \varepsilon \mathrm{p\,f}\, u \;.\supset:$

·0 $\mathrm{Chorda}(p,u) = \mathrm{mod}(p\mathrm{l}'u - p\mathrm{l}_{\prime}u)$ Def

·1 $h \varepsilon \mathrm{Cls}'u \,.\, \mathrm{Num}h \,\varepsilon\, \mathrm{N}_1 \,.\, \mathrm{l}_{\prime}u, \mathrm{l}'u \,\varepsilon h \;.\supset.\; \mathrm{polyg}(p,u,h) =$
$\Sigma[\mathrm{Chorda}(p, \mathrm{min}_r h \,\bar{}\, \mathrm{min}_{r+1} h) \,|\, r, 1 \cdots (\mathrm{Num}h - 1)]$ Def

·2 $\mathrm{Arcu}(p,u) =$
$\mathrm{l}'[\mathrm{polyg}(p,u,h) \,|\, h \text{‘} \mathrm{Cls}'u \frown h \ni (\mathrm{Num}h \,\varepsilon\, \mathrm{N}_1 \,.\, \mathrm{l}_{\prime}u, \mathrm{l}'u \,\varepsilon h)$ Def

Si u es intervallo, et p es puncto mobile functione de variabile (tempore) in intervallo u, tunc Chorda(p,u) vale distantia de positiones extremo de puncto.

Et, si h es classe de valores in u in numero finito, continente extremos de u, vel si h es divisione de intervallo u in partes (vide pag. 340), tunc polyg(p,u,h), lege « longitudo de linea polygonale inscripto in trajectoria de p in intervallo u diviso per valores h » indica summa de chordas.

Arcu(p,u) « arcu descripto per puncto p, quando variabile varia in u » es limite supero de lineas polygonale inscripto, respondente ad omni divisione h de intervallo u in partes.

Vide pag. 117.

arcu : A.F. arc, H.I. arco, D. (raro) arcus.
F. arche, A. arch, R. arca arcada (in Architectura).

·3 $\mathrm{D}p \,\varepsilon\, (\mathrm{vF}u)\mathrm{cont} \;.\supset.\; \mathrm{Arc}(p, u) = \mathrm{S}(\mathrm{mod\,D}p, u)$

Si puncto mobile p habe derivata continuo in toto intervallo u, tunc arcu vale integrale de valore absoluto de derivata (velocitate) de p.

Dem.

$x,y \varepsilon u \,.\, x < y \,.\, \S\mathrm{D\,P44} \;.\supset.\; py - px \,\varepsilon\, (y-x)\, \mathrm{Med\,D}p\text{‘}x \,\bar{}\, y$ (1)

Sume duo valore x et y in u. Theorema de valore medio (pag. 312) dice que vectore differentia de duo puncto vale incremento de variabile multiplicato per valore medio de derivata in intervallo.

$\mathrm{Hyp}(1) \;.\supset.\; \mathrm{Chorda}(p, x \,\bar{}\, y) \leqq (y-x)\, \mathrm{l}'\mathrm{modD}p\text{‘}x \,\bar{}\,\bar{}\, y$ (2)

Ergo chorda que uni punctos px et py es minore de incremento de tempore multiplicato per limite supero de valores absoluto de velocitate in idem intervallo.

$h \varepsilon \text{Cls}'u \,.\, \text{Num}h \varepsilon N_1 \,.\, \iota_, u, \iota' u \varepsilon h \,.\, (2) \,.\supset.$
$\text{polyg}(p,u,h) \leqq s'(\text{modD}p, u,h)$ (3)

Et si h es divisione de intervallo u in partes, ex definitione de numero s′ (pag. 346) resulta que linea polygonale inscripto in trajectoria de p, in intervallo u, diviso per valores h, es minore de summa s′ respondente ad functione mod Dp, in intervallo u, diviso per valores h.

(3) . Df Arcu $.\supset.$ $\text{Arcu}(p,u) \leqq S'(\text{modD}p, u)$ (4)

Unde, ex definitione de arcu, resulta que arcu es minore de integrale supero de valore absoluto de velocitate.

Hyp(1) $.\supset.$ $\text{Arc}(p, x\frown y) \geqq \text{Chorda}(p, x\frown y)$ (5)
» $.\supset.$ $\text{Arc}(p, x\frown y)/(y-x) \geqq \text{mod}[(py-px)/(y-x)]$ (6)

Si x et y es duo valore, ut in Prop. (1), nos habe que arcu supera chorda; unde arcu diviso per intervallo de tempore supera chorda diviso per idem intervallo.

$x\varepsilon u \,.\, (4) \,.\, (6) \,.\supset.\, D[\text{Arcu}(p, \iota_, u\frown x)|x, u]x = \text{modD}px$ (7)
(7) $.\supset.$ P

Ergo, si x es valore in intervallo dato, derivata de arcu descripto per p, ab extremo infero de intervallo, ad x, facto pro x, vale valore absoluto de velocitate. Integratione duce ad formula.

·4 $\text{D}p \,\varepsilon\, (\text{vF}u)\text{cont} \,.\, x\varepsilon u \,.\, \text{D}px -= 0 \,.\supset.$
$\lim[\text{Arcu}(p, x\frown y) / \text{Chorda}(p, x\frown y)|y, u, x] = 1$

[P·3 $.\supset.$ $\lim[\text{Arc}(p, x\frown y)/\text{mod}(y-x)|y, u, x] = \text{modD}px$ (1)
Df D $.\supset.$ $\lim[\text{d}(px, py)/\text{mod}(y-x)|y, u, x] =$
$\lim\{\text{mod}[(py-px)/(y-x)]|y, u, x\} = \text{modD}px$ (2)
(1) . (2) $.\supset.$ P]

Si puncto p habe derivata continuo, et pro valore x, differente de 0, tunc limite de arcu descripto in intervallo ab x ed y, diviso per chorda respondente, ubi y verge ad x, vale 1.

·5 $\text{D}p \,\varepsilon\, \text{vF}u \,.\, \iota' \text{mod D}p`u \,\varepsilon \text{Q} \,.\supset.$
$S_,(\text{modD}p, u) \leqq \text{Arc}(p, u) \leqq S'(\text{modD}p, u)$

Casu de derivata non continuo, sed limitato.

✻ 47. (Cx|p) P 46.

Nos extende definitione de arcu ad numero complexo de ordine arbitrario, in loco de puncto.

Interessante es casu de functione reale de variabile reale.

$u\varepsilon$ Intv . $f\varepsilon$ qf$u$ $.\supset.$ Arcu(f,u) = « variatione de functione f in intervallo u » secundo Jordan a. 1893. Vide Scheffer, ActaM. t. 5. H. Lebesgue a.1904 p. 62.

❊ 48. Long

$u \varepsilon \mathrm{Cls'q} \;.\supset.$

$\mathrm{Long}_{,} u = \mathrm{l}' \{ h \times \mathrm{Num}\; \mathrm{n} \frown p \ni [(p+\Theta)h \supset u] \,|\, h`\mathrm{Q} \}$ Df

$\mathrm{Long}' u = \mathrm{l}_{,} \{ h \times \mathrm{Num}\; \mathrm{n} \frown p \ni [\exists (p+\Theta)h \frown u] \,|\, h`\mathrm{Q} \}$ Df

$\mathrm{Long}\, u = \iota(\iota \mathrm{Long}_{,} u \frown \iota \mathrm{Long}' u)$ Df

Si u es classe de quantitates reale, tunc nos sume aliquo quantitate positivo h, et considera segmentos inter duo valores successivo de

$$\ldots\; -2h,\; -h,\; 0,\; h,\; 2h,\; \ldots$$

vel de ph, ubi p es n. Numero de segmentos parte de figura u, multiplicato per h, longore commune de illos, si varia h, habe limite supero, quem nos indica per $\mathrm{Long}_{,}u$, lege « longore infero de classe u ».

Numero de segmentos que habe aliquo puncto commune cum u, multiplicato per h, quando h sume omni valore positivo, habe limite infero, indicato per $\mathrm{Long}'u$, lege « longore supero de u ».

$\mathrm{Long}u$, lege « longore de u », es valore commune de longore supero et de longore infero; habe sensu, si ce duo longore coincide.

Definitiones præcedente es analogo ad Df de S′, $\mathrm{S}_{,}$, S.

Stolz, MA. a.1884 t.23 p.152 et Cantor AM. a.1884 t.4 p.388 considera Long′. $\mathrm{Long}_{,}$ et Long occurre in meo libro a. 1887 p. 155, et in Jordan a. 1893 p. 28.

L. ***longo:*** F. long, H. luengo, I. lungo, Port. longo.
$\supset$ long-i-metria A.D.F.H.I.R.
$\subset$ E. longho: A. long, D. lang.

L. ***longitudo,*** longitudine; A.F. longitude, H. longitud, I. longitudine (in Geographia).
$\supset$ A.D.F.H. longitudin-al.
$=$ ***longore*** (non L.), F. long-ueur.
$=$ long-itia (non L.), I lunghezza.
$=$ H. long-ura.
$=$ A. length (‖ long-itia), D. Länge (‖ long-io-ne)

·1 $u \varepsilon \mathrm{Intv} \;.\supset.\; \mathrm{Long}'u = \mathrm{Long}_{,}u = \mathrm{Long}u$

Si classe u es intervallo, suo longitudo supero et infero et proprio vale differentia de extremos, ut es definito in pag. 142.

·2 $v \varepsilon \mathrm{Cls'Intv} \,.\, \mathrm{Num}v \;\varepsilon\; \mathrm{N}_0 \;.\supset.$
$\mathrm{Long}' \bigcup v = \mathrm{Long}_{,} \bigcup v = \mathrm{Long} \bigcup v$

Si v es systema de plure intervallo in numero finito, tunc figura $\bigcup v$, reunione de intervallos v, summa, in sensu logico, de intervallos v, habe longore supero æquale ad infero.

·3 $\text{Long}'u - \text{Long}_{,}u = \text{Long}'\text{am}u$

Differentia inter longores supero et infero de u vale longore supero de campo am$u$ (p.142).

·4 $\text{Long}'\eta = 1 \quad . \quad \text{Long}_{,}\eta = 0$

Campo η (p. 104) de numeros rationale minore de 1, habe 1 pro longore supero, et 0 pro longore infero.

·5 $\text{l}'\text{mod}u \,\varepsilon \text{Q} \,.\, \text{Num}\delta u \,\varepsilon\, \text{N}_0 \cup \iota\text{NumN}_0 \,.\supset.$
$\text{Long}'u = \text{Long}_{,}u = \text{Long}u = 0$
{ G. Cantor MA. a.1883 t.21 p.54 }

Si campo u es limitato, es si habe numero finito de elementos (tunc gruppo derivato $=\wedge$), vel numero infinito, sed gruppo derivato habe numero finito de elementos, vel numero infinito sed numerabile, tunc longore supero (et infero et proprio) de u es nullo.

·6 $u = \Sigma[(\iota 0 \cup \iota 2)/3^n \,|\, n, \text{N}_1] \,.\supset.\, \delta u = u \,.\, \text{Long}'u = 0$

Classe de numeros, que es expresso in fractione, analogo ad decimale sed in base 3, pro solo cifras 0 et 2, es perfecto, et habe longore supero nullo.

Classe perfecto habe potestate de continuo, ut es scripto in pag. 141, Prop. 5·3. Ergo existe classe perfecto, de longore nullo.

Et existe classe perfecto, condensato in nullo intervallo, id es δu contine nullo intervallo, et longore supero de classe non es nullo.

Vide Du Bois-Reymond, *Functionentheorie* a. 1882 p. 189.

❋ 49. $u\varepsilon \,\text{Cls}'\text{p} \,.\, \exists\, \text{p}_2 \cap a\ni(u\supset a) \,.\supset.$

·1 $\text{Long}_{,}u = \text{l}'\, x\ni[\exists(o,i,h)\ni\{o\varepsilon\text{p} \,.\, i\varepsilon\text{v} \,.\, i^2{=}1 \,.\, u\supset \text{recta}(o,i) \,.$
$h\varepsilon\text{Q} \,.\, x{=}\, h\times \text{Num}\, p\ni[p\varepsilon\text{n} \,.\, o+(p+\Theta)hi\supset u]\}]$ Df

·2 $\text{Long}'u = \text{l}_{,}x\ni[$ » » » »
» $.\, x{=}\, h\times \text{Num}\, p\ni[p\varepsilon\text{n} \,.\, \exists[o+(p+\Theta)hi] \cap u\}]$ Df

·3 $\text{Long}u = \imath(\iota\text{Long}'u \cap \iota\text{Long}_{,}u)$ Df

u es classe de punctos, vel figura, et existe recta a que contine u; id es u es classe de punctos super recta. Tunc sume puncto o et vectore unitario i, in modo que u jace super recta (o,i); sume quantitate positivo h. Divide recta in segmentos de longore h, cum punctos

.... $o-2hi$, $o-hi$, o, $o+hi$, $o-2hi$,...

Segmento inter duo puncto consecutivo habe expressione $o+(p+\Theta)hi$, ubi p es numero integro positivo aut negativo aut nullo.

Numera segmentos que continere in u:

$$\mathrm{Num}\, p \ni [p \varepsilon \mathrm{n} \,.\, o + (p+\Theta)hi \supset u];$$

et multiplica illo per longore commune de segmentos h; resulta quantitate x que depende de o, i, h. Sume limite supero de valores de x, pro omni triade o, i, h, compatibile cum conditiones scripto. Ce limite vocare « longore infero de u », et indicare per $\mathrm{Long}_{\prime} u$.

Si nos numera segmentos que habe aliquo puncto commune cum u, et multiplica per h, et sume limite infero, resulta « longore supero de u », indicato per $\mathrm{Long}' u$.

Si longore infero æqua supero, valore commune vocare « longore proprio de u », et indicare per $\mathrm{Long}\, u$.

·4 $a,b \varepsilon \mathrm{p} \,.\supset.\, \mathrm{Long}(a^{-}b) = \mathrm{d}(a,b)$

Longore de segmento vale distantia de suo extremos.

* 50·0 $u \varepsilon \mathrm{Cls'q} \,.\, \mathrm{l'mod} u \,\varepsilon \mathrm{Q} \,.\, f \varepsilon \mathrm{qF}u \,.\supset.$
$\mathrm{S}f = \mathrm{S}\,\iota\, (\mathrm{qFq}) \frown g \ni (x \varepsilon u \,.\supset_x.\, gx = fx : x \varepsilon \mathrm{q}\text{-}u \,.\supset_x.\, gx = 0)$ Df

·01 $(\mathrm{S}'|\mathrm{S})\mathrm{P}{\cdot}0$ ·02 $(\mathrm{S}_{\prime}|\mathrm{S})\mathrm{P}{\cdot}0$ Df

Si u es classe de numeros reale finito; et f es functione reale definito in campo u, id es, considerato simul cum campo de variabilitate, tunc $\mathrm{S}f$ « integrale de f » es integrale de illo functione definito per omni valore reale de variabile, que pro omni x in campo u, redde $gx = fx$, et pro omni valore non in u, vale 0. Idem pro integrale supero et infero.

·1 $u \varepsilon \mathrm{Cls'q} \,.\, \mathrm{l'mod} u \,\varepsilon \mathrm{Q} \,.\supset.$
$\mathrm{Long}' u = \mathrm{S}'(\iota 1 : u) \,.\, \mathrm{Long}_{\prime} u = \mathrm{S}_{\prime}(\iota 1 : u)$

Si u es classe limitato, tunc longitudo supero de u vale integrale supero de functione que habe valore constante 1 in campo u (et 0 ex campo u). Longitudo infero vale integrale infero.

Vide Poussin, *Cours d'Analyse*, a. 1903 t. 1 p. 221.

* 51. NOVO CONDITIONE DE INTEGRABILITATE.

$u \varepsilon \mathrm{Cls'q} \,.\, f \varepsilon \mathrm{qF}u \,.\, x \varepsilon \delta u \,.\supset.\, \mathrm{l}'fx = \max \mathrm{Lm}(f,u,x)$ Def
$\mathrm{l}_{\prime}fx = \min \mathrm{Lm}(f,u,x)$ —
$\mathrm{O}fx = \mathrm{l}'fx - \mathrm{l}_{\prime}fx$ —

Si u es classe de quantitates, et f es quantitate functione definito in campo u, et x es elemento de classe derivata de u, tunc $\mathrm{l}'fx$, lege « limite supero de functione f in x », vel « extremo oscillatorio supero de

functione f in x » indica maximo valore de classe « limes » (pag. 230) de functione f, ubi variabile sume valores in campo de variabilitate de f, et verge ad x.

Symbolo $\mathrm{l}'fx$ vale $(\mathrm{l}'f)x$, vel $\mathrm{l}'(f,x)$; non habe sensu $\mathrm{l}'(fx)$. Nam ente definito depende de f et de x, et non de numero fx. Vide observatione analogo pro derivata, in pag. 276.

Idem pro $\mathrm{l}_{,}fx$ « limite infero, vel extremo oscillatorio infero de functione f in x ».

$\mathrm{O}fx$ vocare « oscillatione de functione f in puncto x ».

·1 $u \varepsilon \mathrm{Intv} \,.\, f \varepsilon \mathrm{qF}u \,.\, \mathrm{l}'f\text{‘}u, \mathrm{l}_{,}f\text{‘}u \;\varepsilon \mathrm{q} \,.\supset .$
$\mathrm{S}'(f,u) = \mathrm{S}'(\mathrm{l}'f, u) \,.\, \mathrm{S}_{,}(f,u) = \mathrm{S}_{,}(\mathrm{l}_{,}f, u) \,.\, \mathrm{S}'(f,u) - \mathrm{S}_{,}(f,u) = \mathrm{S}'(\mathrm{O}f, u)$

Integrales supero et infero de functione f in intervallo u æqua integrales de extremos oscillatorio de f.

Differentia de integrales de f vale integrale supero de oscillatione de f in intervallo dato. Vide Poussin, *Cours d'Analyse* a. 1903 p. 221; Lebesgue a. 1904 p. 35.

Ex definitione de integrale, seque (pag. 342): « ut integrale existe, es necesse et suffice que integrale supero æqua infero ».

Ergo « ut functione f es integrabile in u, es necesse et suffice que integrale supero de oscillatione de functione in u es nullo ».

Ante consideratione de integrales supero et infero (a. 1875, vide pag. 343), Riemann (a. 1854; publicato a. 1867; Werke pag. 226) exprime conditione de integrabilitate sub forma, que Du Bois Reymond a. 1882 p. 189, habe reducto ad:

·2 $\mathrm{S}(f,u) \varepsilon \mathrm{q} \;.=:\; h \varepsilon \mathrm{Q} \,.\supset_h .\, \mathrm{Long}'\, u \frown x \ni (\mathrm{O}fx > h) = 0$

Ut functione f es integrabile, es necesse et suffice que, si h es quantitate positivo, parvo ad arbitrio, gruppo de u, ubi oscillatione de functione supera h, habe longore supero nullo.

Ce conditione sume novo forma, post introductione de novo idea de longore, indicato per « long » et definito ut seque.

✻ 52. $u \varepsilon \mathrm{Cls'q} \,.\supset .$

$\mathrm{long}\,u = \mathrm{l}_{,}x \ni [\exists\, \mathrm{Cls'Intv} \frown v \ni (u \supset \bigcup v \,.\, x = \mathrm{Long}_{,} \bigcup v)$ Df

Dato u, classe de quantitates, sume classe v de intervallos (in numero infinito), tale que omni u pertine ad aliquo v, vel classe u continere in summa, in sensu logico, de v. De ce systema de intervallos $\bigcup v$ sume longore infero x; id es, inscribe in $\bigcup v$ classe de intervallos in numero finito, et sume longore supero de longore proprio de ce classe de intervallos. Limite infero de x, ubi varia classe v de intervallos, es indicato per « long u », que me voca « longore medio de u ».

Ergo «longore medio» es limite infero de classe de quantitates, que es limite supero de alio quantitates.

·1 $\text{Long}'u \geqq \text{long}u \geqq \text{Long}_{\prime}u$

·2 $\text{Num}u = \text{NumN}_0 \;.\supset.\; \text{long}u = 0$

[$x\varepsilon\,(u\text{FN}_0)\text{rep} \;.\; h\varepsilon\text{Q} \;.\; v = (x_n \pm \Theta h/2^n)|n`\text{N}_0 \;.\supset.$
$v\varepsilon\,\text{Cls}'\text{Intv} \;.\; u \supset \bigcup v \;.\; \text{Long}_{\prime}v = 4h \;.\supset.\; \text{long}u < 4h \;.\supset.\; \text{long}u = 0$]

Si u es classe numerabile de punctos, long$u$ vale 0.

In vero nos pote pone numeros u in correspondentia reciproco cum numeros naturale x_0 x_1 ... Nunc si nos comprehende omni elemento x_n de u in intervallo de centro x_n, et de radio $h/2^n$, longore interno de ce intervallos, vel limite supero de longores de numero finito de intervallos in illos, vale $4h$, quantitate parvo ad arbitrio.

·3 $\text{long}\eta = 0$

[§Num P10·7 (pag. 137) $.\supset.$ $\text{Num}\eta = \text{NumN}_0$. P·2 $.\supset.$ P]

·4 $u\varepsilon\,\text{Intv} \;.\; f\varepsilon\,\text{qf}u \;.\; \text{l}'f`u, \text{l}_{\prime}f`u\varepsilon\text{q} \;.\supset:$
$\text{S}'(f,u) = \text{S}_{\prime}(f,u) \;.=.\; \text{long}\{u \cap x \ni [\lim(f,u,x) -= fx]\} = 0$

Si f es functio reale definito in intervallo u, limitato supra et infra, tunc conditione necessario et sufficiente pro integrabilitate de functione, es que longore medio de classe de punctos in u, ubi functione f non es continuo, es nullo.

Mirabile transformatione de criterio de integrabilitate.

Conditione de integrabilitate es expresso per conditione de classe de punctos ubi functione es discontinuo.

Ce theorema es invento in idem tempore, per Lebesgue et Vitali, in plure publicatione, et in modo explicito:

Lebesgue, *Leçons sur l'intégration*, Paris a.1904, p.29. Vitali, *Sulla integrabilità delle funzioni*, Rendiconti Ist. Lomb. a.1904, p.73.

·5 $\text{Hp}·4 \;.\; v\varepsilon\,\text{Cls}'u \;.\; \text{Num}v\;\varepsilon\;\text{N}_0 \cup \iota\text{NumN}_0 \;.\; f\varepsilon\,[\text{qf}(u - v)]\text{cont} \;.\supset.$
$\text{S}(f,u)\varepsilon\text{q}$ [P·1·3 $\supset$ P

Si functione f es continuo, excepto punctos in numero finito, aut infinito numerabile, tunc functione es integrabile. Seque de Prop. præcedente.

·6 $u\varepsilon\text{Intv} \;.\; f,g\varepsilon\,\text{qf}u \;.\; \text{S}(f,u), \text{S}(g,u)\;\varepsilon\text{q} \;.\; h\varepsilon\;\text{qf}(f`u \,;\, g`u)\text{cont} \;.\supset.$
$\text{S}[h(fx, gx)\;|x,\; u]\;\varepsilon\text{q}$

Functione h continuo de plure functione f,g integrabile, es integrabile.

✻ 53. Area

$u \varepsilon \mathrm{Cls}'\mathrm{p} \,.\, \exists\, \mathrm{p}_3 \frown a \ni (u \supset a) \,.\supset.$

·1 $\mathrm{Area}_{\iota} u = \mathrm{l}'x \ni [\exists (o,i,j,h) \ni \{ o \varepsilon \mathrm{p} \,.\, i,j \varepsilon \mathrm{v} \,.\, i^2 = j^2 = 1 \,.\, i \times j = 0 \,.$
$u \supset \mathrm{plan}(o,i,j) \,.\, h \varepsilon \mathrm{Q} \,.\, x = h^2 \times \mathrm{Num}(p,q) \ni [p,q \varepsilon \mathrm{n} \,.$
$o + (p+\Theta) hi + (q+\Theta) hj \supset u] \}]$ Df

·2 $\mathrm{Area}' u = \mathrm{l}_{\iota} x \ni [$ » » » »
$\exists [o + (p+\Theta) hi + (q+\Theta) hj] \frown u] \}]$ Df

·3 $\mathrm{Area}\, u = \imath (\iota \mathrm{Area}' u \frown \iota \mathrm{Area}_{\iota} u)$ Df

Si u es classe de punctos, vel figura, in aliquo plano a, tunc sume puncto o et duo vectore i, j unitario et orthogonale, in plano de u, et numero positivo h. Divide toto plano in quadratos, de vertices $o+phi+qhj$, ubi p et q es numero integro, positivo aut negativo. Numero de quadratos interno ad figura u, multiplicato per h^2, area de uno quadrato, da area de figura ex quadratos in u. Numero de quadrato que habe aliquo puncto commune cum u, da area de figura ex quadratos que contine u. Limite supero de area de figuras interno ad u, vocare $\mathrm{Area}_{\iota} u$ « area infero de u », et limite infero de area de figuras continente u vocare $\mathrm{Area}' u$ « area supero de u ».

Si duo area es æquale, nos voca « area proprio », et indica per $\mathrm{Area}\, u$, valore commune de illos.

L. ***area:*** A.D.H.I. area, F. aire.
$\supset$ F. are: A. are, D. ar, I. ara, R. ar' = 100 m^2.

✻ 54. $o \varepsilon \mathrm{p} \,.\, i,j \varepsilon \mathrm{v} \,.\, i^2 = j^2 = 1 \,.\, i \times j = 0 \,.$

$u \varepsilon \mathrm{Cls}'\mathrm{plan}(o,i,j) \,.\, \mathrm{l}'\mathrm{mod}(u - u)\, \varepsilon \mathrm{Q} \,.\supset\therefore$

·1 $x \varepsilon \mathrm{q} \,.\supset_x.\, \mathrm{Long}' \{ \lambda u \frown \lambda [\mathrm{plan}(o,i,j) - u] \frown (o + ix + j\mathrm{q}) \} = 0 \,:\supset.$
$\mathrm{Area}\, u = \mathrm{S} \{ \mathrm{Long} [u \frown (o + ix + j\mathrm{q})] | x, \mathrm{q} \}$

Si o es puncto, et i, j es vectore unitario et orthogono, et si u es classe de elementos (puncto) in plano o, i, j, et limite supero de distantia de duo puncto de figura es finito, tunc per puncto $o+xi$, de abscissa x, duce recta parallelo ad j: $o+ix+j\mathrm{q}$. Me voca, in modo provisorio, fine (perimetro) de figura u punctos pertinente ad classe limite de u, et ad classe limite

de punctos de plano que non es u; id es « fine u » $= \lambda u \frown \lambda[\mathrm{plan}(o,i,j) - u]$. Si recta $o+ix+j\mathrm{q}$ seca campo fine de u secundo figura rectilineo que habe longore supero nullo pro omni valore de x, tunc area de figura u vale integrale de longore de sectione de recta $o+ix+j\mathrm{q}$ cum figura u, ubi x varia et sume omni valore reale. In hypothesi scripto, ce figura rectilineo habe longore proprio, functione continuo de x, ergo integrabile, et figura u habe area proprio.

Demonstratione es simile ad Dem. de P 57·1, infra.

·2 $\mathrm{Area}'u \leqq \mathrm{S}_{,}\{\mathrm{Long}'[\lambda u \frown (o+ix+j\mathrm{q})] | x,\mathrm{q}\}$

·3 $\mathrm{Area}_{,}u \geqq \mathrm{S}'\{\mathrm{Long}_{,}[u - \lambda[\mathrm{plan}(o,i,j) - u] \frown (o+ix+j\mathrm{q})] | x,\mathrm{q}\}$

In omni casu, nos habe limitationes pro area supero et infero.

·4 $k \varepsilon \mathrm{Intv} \,.\, f \varepsilon \mathrm{Qf}k \ \mathrm{l}'f`k, \mathrm{l}_{,}f`k \,\varepsilon\, \mathrm{q} \ .\supset.$
$\mathrm{Area}' \bigcup [o+xi+(\Theta fx)j \,|\, x, k] = \mathrm{S}'(f,u)$
$\mathrm{Area}_{,}$ ——————————— $\mathrm{S}_{,}$ ——

Si in intervallo k es dato functione f positivo, limitato supra et infra, tunc $o+xi+(\Theta fx)j$ = « segmento rectilineo inter punctos $o+xi$ et $o+xi-(fx)j$ ». Si varia x et sume valores in classe k, resulta classe de segmentos. Summa, in sensu logico, de ce segmentos, id es figura descripto per ce segmento variabile, habe area supero et infero æquale ad integrale supero et infero de f in k. Es interpretatione de integrale, in pag.339 et sequentes.

✻ 55·1 $o \varepsilon \mathrm{p} \,.\, u,v \varepsilon \mathrm{v} \ .\supset.\ \mathrm{Area}(o+\theta u+\theta v) = \mathrm{mod}(u \alpha v)$

Area de parallelogrammo.

·2 $a,b,c \varepsilon \mathrm{p} \,.\, a -= b \ .\supset.\ \mathrm{Area}(a \bar{} b \bar{} c) = \mathrm{d}(a,b) \times \mathrm{d}[c, \mathrm{recta}(a,b)]/2$
$= \mathrm{mod}[(b-a)\alpha(c-a)]/2$
$= \sqrt{}\{[\mathrm{d}(a,b)+\mathrm{d}(b,c)+\mathrm{d}(c,a)][\mathrm{d}(a,b)+\mathrm{d}(b,c)-\mathrm{d}(c,a)][\mathrm{d}(a,b)-\mathrm{d}(b,c)+\mathrm{d}(c,a)][-\mathrm{d}(a,b)+\mathrm{d}(b,c)+\mathrm{d}(c,a)]\}/4$

Area de triangulo. Suo expressione per lateres es de Herone. Vide pag.265 Prop.2·6.

·3 $o \varepsilon \mathrm{p} \,.\, a,b \varepsilon \mathrm{v} \,.\, a^2 = b^2 = 1 \,.\, a \times b = 0 \,.\, i = b/a \,.\, k \varepsilon \mathrm{Intv} \,.$
$r \varepsilon (\mathrm{qF}k)\mathrm{cont} \ .\supset.\ \mathrm{Area}[o+z(rt)\mathrm{e}^{it}a] | (z,t)`(\Theta,k) = \mathrm{S}[(rt)^2 | t, k]/2$

Puncto de coordinatas polare $z(rt)$ et t, si z varia inter 0 e 1, et t in intervallo k, describe area expresso per integrale.

✻ 56. Volum

$u \varepsilon$ Cls'p .⊃.

·1 $\text{Volum}_{,} u =$
$\text{l}' x \ni [\exists\, (o,i,j,k,h) \ni \{ o \varepsilon \text{p} \,.\, i,j,k \varepsilon \text{v} \,.\, i^2 = j^2 = k^2 = 1 \,.\, i \times j = j \times k = k \times i = 0 \,.\, h \varepsilon \text{Q} \,.\, x = h^3 \times \text{Num}(p,q,r) \ni [\, p,q,r \varepsilon \text{n} \,.\, o + (p+\Theta)hi + (q+\Theta)hj + (r+\Theta)hk \supset u]\}]$ Df

·2 $\text{Volum}' u =$
$\text{l}_{,} x \ni [$ » » $.\, x = h^3 \times \text{Num}(p,q,r) \ni [\, p,q,r \varepsilon \text{n} \,.\, \exists [o + (p+\Theta)hi + (q+\Theta)hj + (r+\Theta)hk] \frown u]\}]$ Df

·3 $\text{Volum}\, u = \imath(\iota \text{Volum}_{,} u \frown \iota \text{Volum}' u)$ Df

u es classe de punctos, vel figura. Nos sume punto o, tres vectore unitario orthogonale i, j, k, et numero positivo h. Si p, q, r es numeros integro, $o + (p+\Theta)hi + (q+\Theta)hj + (r+\Theta)hk$ indica cubo de que uno vertice es $o + phi + qhj + rhk$, et latere, directo secundo axi coordinato, habe longore h. Si varia p, q, r, nos divide toto spatio in cubos. Numero de cubos que continere in figura u, multiplicato per h^3, volumen de uno cubo, da volumen de uno figura composito de cubos juxtaposito, et interno ad figura data. Ce volumen depende de axis coordinato, et de latere h. Limite supero de ce volumen, quando varia axis et h, es dicto « volumen infero de u ».

L. ***Volumen*** volumine: A.F.I. volume, D.H. volumen.
⊃ volumin-oso A.D.F.H.I. ⊂ volv(e) + -men.
L. volve: A. volve, D. volv-ieren, I. volge-re, H. volve-r.
⊃ L.(A.D.F.H.I.R.) e-volu-tione, re-volu-tione, ...
⊂ E velu ⊃ G. ely-tro hel-ice, A. wallow, D. walzen.
E. velna = lana: A. wolle, R. volna, L. vello villo, H.I. vello.

✻ 57·0 (p | q) § in ex am (pag. 142)

Definitiones dato in pag. 142, de campo in (interno), ex (externo), am (circa, fine, frontière) de campo dato, subsiste si in loco de campo de quantitates nos considera campo de punctos.

$u \varepsilon \text{Cls'p} \,.\, \text{l}'\text{mod}(u-u)\, \varepsilon \text{Q} \,.\, o \varepsilon \text{p} \,.\, i \varepsilon \text{v} \,.\, \text{mod} i = 1$.⊃.

·1 $x \varepsilon \text{q} \,.\supset_x.\, \text{Area}'[\text{am} u \frown \text{plan}(o+xi, \text{I}i)] = 0$:⊃.
$\text{Volum}\, u = \text{S}\{\text{Area}[u \frown \text{plan}(o+xi, \text{I}i)] | x, \text{q}\}$

Dato u classe de punctos, vel figura, tale que limite supero

de distantia de duo puncto de figura es finito, sume puncto o et vectore unitario i. Si pro omni abscissa x, plano per puncto $o+xi$, et normale ad i [plano de tripuncto $(o+xi)a\mathrm{I}i$, vide pag. 198, 199] seca campo fine de u secundo figura de area supero nullo, tunc volumen de u vale integrale de area de sectione de u cum plano per puncto $o+xi$, et normale ad i, ubi varia x, et sume omni valore reale.

Dem.

$j,k\varepsilon \mathrm{v} \,.\, j^2=k^2=1 \,.\, i\times j=j\times k=k\times i=0$.

Sume alio duo vectore j et k, unitario, et orthogonale inter se et cum i.

$x\varepsilon \mathrm{q} \,.\, g_{,}x = \mathrm{Volum}_{,}\, u \frown [o+(x-\mathrm{Q})i+\mathrm{q}j+\mathrm{q}k]$.

$g'x = \mathrm{Volum}'$ » » » .

Sume quantitate x, voca $g_{,}x$ et $g'x$ volumen infero et supero de parte de figura u que es in semispatio de punctos cum abscissa minore de x.

$h\varepsilon \mathrm{Q} \,.\, a = (\mathrm{n}{:}\mathrm{n}) \frown (p,q)\ni[o+xi+(p+\Theta)hj+(q+\Theta)hk \supset \mathrm{in}\ u]$.

$b =$ » » » » $\frown \mathrm{am}\ u \mathrel{-}= \Lambda]$.

$c =$ » » » » $\supset \mathrm{ex}\ u]$.

Sume quantitate positivo h; divide toto plano $o+xi+\mathrm{q}j+\mathrm{q}k$ in quadratos de latere h; et voca a dyades de numeros integro p et q, tale que quadrato correspondente $o+xi+(p+\Theta)hj+(q+\Theta)hk$ es toto interno ad u.

Voca b dyades de quadratos que habe aliquo puncto commune cum fine de u.

Voca c dyades de quadratos externo ad u.

$l = \mathrm{dist}\{[o+xi+(p+\Theta)hj+(q+\Theta)hk] \mid (p,q)`(a\cup c),\ \mathrm{am}\ u\} \,.\supset.\, l\varepsilon \mathrm{Q}$

Voca l distantia de figura composito de quadratos a et c, ab campo fine de u. Distantia de duo figura (pag.176 Prop.22·3) es limite infero de distantias de punctos de uno figura ab punctos de altero. Si ambo figura es clauso, vale distantia minimo. In nostro casu, figuras es clauso, et habe nullo puncto commune; ergo l es quantitate positivo non nullo.

$y\varepsilon\, x+\theta l \,.\supset.\, [o+x\urcorner yi+(p+\Theta)hj+(q+\Theta)hk] \mid (p,q)`a \supset u \frown (o+xi+\mathrm{q}j+\mathrm{q}k)$
$\supset$ » » » » $`(a\cup b)$

Si y es abscissa inter x et $x+l$, tunc figura composito de parallelepipedo, de basi quadratos a, et de altitudine vectore $(y-x)i$, es parte de strato de figura u, inter planos normale ad i, per punctos $o+xi$ et $o+yj$; ce strato es parte de parallelepipedos de basi quadratos a et b, et de idem altitudine.

Volumen de figura inscripto $=(y-x)\times h^2(\mathrm{Num}a)$;

Volumen infero de strato $= g_{,}y-g_{,}x$;

Volumen supero $= g'y-g'x$;

Volumen de figura circumscripto $= (y-x)\times h^2(\mathrm{Num}a+\mathrm{Num}b)$.

Divide per $y-x$:

$y\varepsilon\, x\mp\Theta l \,.\supset.\, h^2(\mathrm{Num}a) \leqq \mathrm{D}(g_{,};x,y) \leqq \mathrm{D}(g';x,y) \leqq h^2(\mathrm{Num}a+\mathrm{Num}b)$

nam formula subsiste et pro $y<x$.

Nunc si y verge ad x, et derivata de g, et g' non es supposito:

$\text{Lm}[\text{D}g_,;x,y)|y,\text{q},x] \supset h^2(\text{Num}a + \Theta\text{Num}b)$

» g' » » »

Varia h in omni modo, et sume limites supero et infero de secundo membro:

$\text{Lm}[\text{D}(g_,;x,y)|y,\text{q},x] \supset \text{Area}_,[\text{in}\, u \frown (o+xi+\text{q}j+\text{q}k)] +$
$\Theta\, \text{Area}'[\text{am}\, u \frown (o+xi+\text{q}j+\text{q}k)]$

Classe limes de ratione incrementale de g, continere in area infero de sectione de campo interno ad u cum plano $(o+xi,\text{I}i)$, plus fractione de area sectione cum campo fine.

Per hypothesi, ultimo area es nullo; ergo area de sectione cum campo in$u$, λu et u es æquale. Classe Lm consta ex uno solo individuo:

$\text{D}g_,x = \text{D}g'x = \text{Area}[u \frown \text{plan}(o+xi, \text{I}i)]$

Integra ab $-\infty$ ad x; $g_,x$ et $g'x$ es nullo pro valore de x minore de abscissa de omni puncto de figura. Ergo:

$g_,x = g'x = \text{S}[\text{Area}\, u \frown \text{plan}(o+xi, \text{I}i)|x, x-\text{Q}]$

Et si x verge ad $+\infty$, vel nos tribue ad x valore superiore ad abscissa de omni puncto, seque theorema.

Regula pro calculo de area (Prop.54) et de volumen (Prop.56) occurre in Kepler et in Cavalieri a.1635 libro 2 Prop.3:

« Figurae planae habent inter se eandem rationem, quam eorum omnes lineae iuxta quamuis regulam assumptae; Et figurae solidae, quam eorum omnia plana iuxta quamuis regulam assumpta ».

Enunciatione de hypothesi es recente.

Vide meo libro: *Applicazioni geometriche* a.1887 p. 178 221.

·2 $\text{Volum}'u \leqq \text{S}_,\{\text{Area}'[\lambda u \frown \text{plan}(o+xi, \text{I}i)]|x, \text{q}\}$

·3 $\text{Volum}_,u \geqq \text{S}'\{\text{Area}_,[\text{in}u \frown \text{plan}(o+xi, \text{I}i)]|x, \text{q}\}$

✻ 58.

·5 $o\varepsilon\text{p} \,.\, u,v,w\varepsilon\text{v} \,.\supset.\, \text{Volum}(o+\theta u+\theta v+\theta w) = \text{mod}(uavaw/\psi)$

Volumen de parallelepipedo super vectores u,v,w æqua valore absoluto de ratione de trivectore $uavaw$ ad trivectore unitario ψ. Vide p.188, 198, 199 P 22·4.

·6 $a\varepsilon\text{p} \,.\, b\varepsilon\, \text{p}-\iota a \,.\, c\varepsilon\, \text{p}-\text{recta}(a,b) \,.\, d\varepsilon\, \text{p}-\text{plan}(a,b,c) \,.\supset.$
$\text{Volum}(a^-b^-c^-d) = \text{Area}(a^-b^-c) \times \text{d}[d, \text{plan}(a,b,c)]/3$

Tetrahedro.

✻ 59. Volumen de cylindro et de cono

$a \varepsilon \mathrm{p}_3 \,.\, u \varepsilon \mathrm{Cls}'a \,.\, \mathrm{Area}u \,\varepsilon \mathrm{Q} \,.\supset:$

·1 $i \varepsilon \mathrm{v}{-}\iota 0 \,.\supset.\, \mathrm{Volum}(u + \theta i) = \mathrm{Area}u \times \mathrm{mod}i \times \sin(i,a)$

Si a es plano, et u es figura in plano a, cum area determinato, tunc, si i es vectore non nullo, volumen de cylindro aut prisma que resulta si ad omni puncto de u nos adde fractione de vectore i, vale area de basi u, per modulo de i, per sinu de angulo de i cum plano a.

·2 $o \varepsilon \mathrm{p} \,.\supset.\, \mathrm{Volum}(o \frown u) = \mathrm{Area}u \times \mathrm{d}(o,a)/3$

Et volumen de cono aut pyramide, que resulta si nos junge puncto o cum omni puncto de u vale area de basi u per distantia de o ab plano a, diviso 3.

[$\mathrm{Hp} \,.\, i \varepsilon \mathrm{v} \,.\, i^2 = 1 \,.\, p \varepsilon a \,.\, a = \mathrm{plan}(p, \mathrm{I}i) \,.\, h = \mathrm{d}(o,a) \,.\supset.$
$\mathrm{Volum}(o \frown u) = \mathrm{S}[\mathrm{Area}(o \frown u) \cap \mathrm{plan}(o + xi, \mathrm{I}i) | x, \Theta h]$
$= \mathrm{S}[\mathrm{Area}u \times (x/h)^2 | x, \Theta h] = \mathrm{Area}u \times h/3$]

In vero, sume vectore i unitario, et normale ad plano a. Voca h distantia de o ab a. Volumen quæsito vale integrale de area de sectione in cono de plano per puncto $o+xi$, et normale ad i, ubi varia x ab 0 ad h. Cono partiale de altitudine x es homothetico ad cono dato; ergo areas es ut quadratos de altitudines. Ergo area sectione in puncto $o+xi$ vale area $u \times (x/h)^2$. Post integratione cum regula de potestates, pag.350, seque formula.

Propositione occurre in Euclide L. XII Prop.7, pro pyramide, Prop.10 pro cono de revolutione, et in Cavalieri a.1635 l.7 Prop.8 pro cono generale:

« Quilibet cylindricus triplus est conici in eadem basi, & altitudine cum eo existentis ».

✻ 60. Volumen de sphæra.

$o \varepsilon \mathrm{p} \,.\, r \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{Volum}\{\mathrm{p} \cap x \ni [\mathrm{d}(x,o) < r]\} = 4\pi r^3/3$

Si o es puncto, et r es quantitate positivo, tunc volumen de solido formato per punctos que dista de o minus que r, id es, volumen de sphæra de centro o et de radio r vale $4\pi r^3/3$.

[$\mathrm{Hp} \,.\, i \varepsilon \mathrm{v} \,.\, i^2 = 1 \,.\supset.\, \mathrm{Volum}\{\mathrm{p} \cap x \ni [\mathrm{d}(x,o) < r]\} =$
$\mathrm{S}\{\mathrm{Area}\, \mathrm{plan}(o + zi, \mathrm{I}i) \cap x \ni [\mathrm{d}(x, o+zi) < \sqrt{(r^2 - z^2)}] | z, (-r) \vdash r\} =$
$\mathrm{S}[\pi(r^2 - z^2) | z, (-r) \vdash r] = 2\pi r^3 \mathrm{S}[(1 - t^2) | t, \Theta] = 4\pi r^3/3$]

Invero, nos sume vectore unitario ad arbitrio i. Tunc volumen quæsito vale integrale de area de sectione in sphaera cum plano passante per puncto $o+zi$, et normale ad i, que es circulo de centro $o+zi$, et de radio $\sqrt{(r^2-z^2)}$, ubi integrale es extenso de $-r$ ad $+r$. Ce area vale $\pi(r^2-z^2)$, et post integratione, nos habe valore scripto.

¦ ARCHIMEDE t.1 p.40: *Πᾶσα σφαῖρα τετραπλασία ἐστὶ κώνου τοῦ βάσιν μὲν ἔχοντος ἴσην τῷ μεγίστῳ κύκλῳ τῶν ἐν τῇ σφαίρᾳ, ὕψος δὲ τήν ἐκ τοῦ κέντρου τῆς σφαίρας.* ¦

* 61. $a,b \varepsilon \mathrm{p}_2 \,.\, (\mathrm{Sym}\, a)b = b \,.\, r \varepsilon \mathrm{Q} \,.\supset.$

$$\mathrm{Volum}\ \mathrm{p} \cap x \ni [\mathrm{d}(x,a) < r \,.\, \mathrm{d}(x,b) < r] = 2r^3/3$$

Parte commune ad cylindros, que habe pro axi duo recta a et b, que se seca ad angulo recto, et radios æquale.

* 62. VOLUMEN IN COORDINATAS CURVILINEO.

$x_0, x_1 \varepsilon \mathrm{q} \,.\, y_0, y_1 \varepsilon (\mathrm{qF}x_0 \urcorner x_1)\mathrm{cont} \,.\, z_0, z_1 \varepsilon [\mathrm{qF}\,(x,y) \ni (x \varepsilon x_0 \urcorner x_1 \,.\, y \varepsilon y_0 x \urcorner y_1 x)]\mathrm{cont} \,.\, c = (x,y,z) \ni [x \varepsilon x_0 \urcorner x_1 \,.\, y \varepsilon y_0 x \urcorner y_1 x \,.\, z \varepsilon z_0(x,y) \urcorner z_1(x,y)] \,.\, p \varepsilon (\mathrm{pF}c)\mathrm{sim} \,.\, \mathrm{D}_1 p \,\alpha\, \mathrm{D}_2 p \,\alpha\, \mathrm{D}_3 p / \psi \;\varepsilon\; (\mathrm{QF}c)\mathrm{cont} \;:\supset.\; \mathrm{Volum}\, p`c = \mathrm{S}[\mathrm{S}\{\mathrm{S}[\mathrm{D}_1 p(x,y,z) \,\alpha\, \mathrm{D}_2 p(x,y,z) \,\alpha\, \mathrm{D}_3 p(x,y,z) / \psi \,|z, z_0(x,y) \urcorner z_1(x,y)] \,|y, y_0 x \urcorner y_1 x\} \,|x, x_0 \urcorner x_1]$

Es dato duo quantitate x_0 et x_1, et duo functione $y_0\, y_1$ definito in intervallo $x_0 \urcorner x_1$, et continuo, et duo functione z_0 et z_1 definito de dyades (x,y) tale que x es in $x_0 \urcorner x_1$, et y es in $y_0 x \urcorner y_1 x$. Nos voca c campo de triade (x,y,z) tale que x es in $x_0 \urcorner x_1$, y in $y_0 x \urcorner y_1 x$, z in $z_0(x,y) \urcorner z_1(x,y)$; p es puncto functione de triades de systema c, et ad valores differente de x,y,z responde punctos differente, et ratione de trivectore super tres derivata de $p(x,y,z)$ pro x,y,z ad trivectore unitario ψ es functione positivo et continuo in campo c. Tunc volumen loco de puncto p, pro triades de campo c, resulta ut seque:

In $\mathrm{D}_1 p(x,z,z) \,\alpha\, \mathrm{D}_2 p(x,y,z) \,\alpha\, \mathrm{D}_3 p(x,y,z)/\psi$ varia z inter $z_0(x,y)$ et $z_1(x,y)$ et integra. Postea varia y inter $y_0 x$ et $y_1 x$, et integra. In fine varia x inter x_0 et x_1, et integra. Integrale triplo mete volumen quæsito.

Regula præcedente es sæpe exposito ut seque.

Parallelepipedo descripto per puncto $p(x,y,z)$ si x,y,z varia in intervallos infinitesimo de amplitudine $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, vale

$$\mathrm{D}_1 p(x,y,z) \,\alpha\, \mathrm{D}_2 p(x,y,z) \,\alpha\, \mathrm{D}_3 p(x,y,z)/\psi\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

vel $m\,dx\,dy\,dz$, ci nos voca m coefficiente de $dx\,dy\,dz$.

Varia z inter z_0 et z_1, et summa parallelepipedos:

$$(\int m\,dz)\,dx\,dy$$

mete volumen de « filo » infinitesimo, descripto per puncto $p(x,y,z)$, ubi x et y varia in intervallos infinitesimo dx et dy, et z in intervallo finito $z_0(x,y)\text{⌐}z_1(x,y)$.

Varia y inter y_0x et y_1x, et summa filos:

$$[\int(\int m\,dz)\,dy]\,dx$$

mete « strato » descripto per puncto, ubi x varia in intervallo infinitesimo dx, et y et z intervallos finito.

Varia x inter $x_0\,x_1$ et summa stratos:

$$\int\{\int(\int m\,dz)\,dy]\,dx$$

mete volumen.

Variabiles x,y,z unde depende puncto p vocare suo « coordinatas curvilineo ».

Propositione es considerato in plure libro ut evidente. Non existe demonstratione elementare. Vide infra, integrales multiplo.

❋ 63. Area de superficie non in plano.

$u \varepsilon$ Cls'p . Volum$u = 0$. $\supset$.
  area$u = \lim\{\text{Volum}\{p \frown x\ni[d(x,u) < h]\}/(2h) \mid h, Q, 0\}$ Df

Si u es figura, et suo volumen es nullo, vel u es superficie curvo, considera solido formato de punctos que dista de u minus que h, quantitate positivo dato; divide volumen de isto strato per $2h$, suo spissore. Limite de ratione, si h verge ad 0, vocare « area de figura u ».

Si figura u es in plano, « area u » nunc definito = Area'u, de Prop. 53.

Ce definitione de area es dato per Borchardt a.1854, JfM. t.19 p.369; et per Minkowski a.1901.

Super differentes definitione dato per area de superficie curvo, vide Formulario, t. 4, p.300-301,

Lebesgue, Intégrale, Longueur, Aire; Ann. di Mat. a.1902.
Fréquet, Sur une généralisation des notions d'aire, NouvAdM. t.4 a.1904.
Sibirani, Periodico di Matematica a.1905.

✻ 64.

$x_0, x_1 \varepsilon \mathrm{q} \,.\, y_0, y_1 \,\varepsilon\, (\mathrm{qF} x_0 \frown x_1)\mathrm{cont} \,.\, c = (x,y) \ni (x \varepsilon x_0 \frown x_1 \,.\, y \varepsilon\, y_0 x \frown y_1 x)$
$. \, p \varepsilon\, (\mathrm{pF}c)\mathrm{sim} \,.\, \mathrm{D}_1 p, \mathrm{D}_2 p \,\varepsilon\, (\mathrm{vF}c)\mathrm{cont} \,.\supset.\, \mathrm{area}\, p\text{‘}c =$
$\mathrm{S}\{\mathrm{S}[\mathrm{mod}\, \mathrm{D}_1 p(x,y) \,\alpha\, \mathrm{D}_2 p(x,y) \,|y, y_0 x \frown y_1 x] \,|x, x_0 \frown x_1\}$

Es dato duo quantitate x_0 et x_1. et duo functione $y_0\, y_1$ de x in intervallo de x_0 ad x_1, continuo. Nos voca c campo de dyades (x,y) tale que x es inter x_0 et x_1, et y inter $y_0 x$ et $y_1 x$. p es puncto functione in campo c; ad valores differente de variabiles responde positiones distincto de p. Ce puncto habe derivatas partiale pro x et y continuo. Tunc area de loco de puncto p, si variabiles varia in c, resulta ut seque.

Calcula modulo de bivectore super derivatas partiale de p; varia y inter y_0 et y_1, et integra. Postea varia x inter x_0 et x_1, et integra.

Vel, area es decomposito in parallelogrammos infinitesimo

$$\mathrm{D}_1 p(x,y) \,\alpha\, \mathrm{D}_2 p(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

Summa pro y, inter limites dato; nos calcula area de « zona ».
Summa pro x; et habe area quæsito.

✻ 65. $o \varepsilon \mathrm{p} \,.\, r \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{area}\, \{\mathrm{p} \frown x \ni [\mathrm{d}(x,o) = r]\} = 4\pi r^2$

Superficie de sphæra de centro o et de radio r vale $4\pi r^2$.

[$\mathrm{area}\{\mathrm{p} \frown x \ni [\mathrm{d}(x,o) = r]\} = \lim\{\mathrm{Volum}\; \mathrm{p} \frown x \ni [\mathrm{d}(x,o) < r+h] -$
$\mathrm{Volum}\; \mathrm{p} \frown x \ni [\mathrm{d}(x,o) < r-h]/(2h) | h, \mathrm{Q}, 0\} = \mathrm{D}[4\pi r^3/3 | r, \mathrm{Q}, r] = 4\pi r^2$]

In vero, per definitione, area quæsito vale limite de volumen de solido formato per punctos distante de superficie minus que h, diviso per $2h$, quando h tende ad 0. Ce volumen es differentia inter sphæras de centro o et de radio $r+h$ et $r-h$; et limite quæsito es derivata de volumen de sphæra, ubi varia radio r.

{ ARCHIMEDE t.1 p.136 : *Πάσης σφαίρας ἡ ἐπιφάνεια τετραπλασία ἐστὶ τοῦ μεγίστου κύκλου τῶν ἐν αὐτῇ.* }

·1 $o \varepsilon \mathrm{p} \,.\, i \varepsilon \mathrm{v} \,.\, i^2 = 1 \,.\, h,k,r \varepsilon \mathrm{Q} \,.\, h < k < r \,.\supset.$
$\mathrm{area}\{\mathrm{p} \frown x \ni [\mathrm{d}(x,o) = r \,.\, h < (x-o) \times i < k]\} = 2\pi r(k-h) \,.$
$\mathrm{Volum}\{\mathrm{p} \frown x \ni [\mathrm{d}(x,o) < r \,.$ » $\} = \pi r^2(k-h) - \pi(k^3-h^3)/3$

Segmento de sphæra.

·2 $o \varepsilon \mathrm{p} \,.\, i \varepsilon\, \mathrm{v}\text{-}\iota 0 \,.\, r \varepsilon \mathrm{Q} \,.\, a \varepsilon\, \theta\pi/2 \,.\supset.$
$\mathrm{Volum}\{\mathrm{p} \frown x \ni [\mathrm{d}(x,o) < r \,.\, \mathrm{ang}(x-o,i) < a]\} = 2\pi r^3 (1-\cos a)/3 \,.$
$\mathrm{Area}\{\mathrm{p} \frown x \ni [\mathrm{d}(x,o) = r \,.\, \mathrm{ang}(x-o,i) < a]\} = 2\pi r^2(1-\cos a)$

Sectore sphærico de radio r et de angulo ad vertice $2a$.

✻ 66. $u \varepsilon$ Cls'p . area$u = 0$.⊃.
arc$u$ = lim{Volum[p⌒xꓱ[1,mod$(x-u) < h$]/(πh^2)] | h, Q, 0} Df

Si u es classe de punctos, que habe extensione superficiale nullo, tunc « arc u » indica limite de volumen de filo formato de punctos que dista de u minus que h, diviso per area de sectione πh^2, ubi h verge ad 0.

Si p es puncto mobile, functione de variabile (tempore) in intervallo k, si p es functione continuo, et si nullo arcu es descripto duo vice, Arcu(p,k), arcu descripto per p in intervallo k, ut es definito in pag. 370, = arc$(p'k)$, id es mensura lineare de figura loco de punctos p in intervallo k.

·1 $o \varepsilon$p . $i \varepsilon$ v-ι0 . $r \varepsilon$Q .⊃.
Long{p⌒xꓱ[d$(x,o) = r$. $(x-o) \times i = 0$]} = $2\pi r$

·2 $o \varepsilon$p . $r \varepsilon$Q . $u \varepsilon$ Cls'p ⌒ xꓱ[d$(x,o) = r$] . area$u$ εQ .⊃.
Volum$(o^- u) = r \times$ area$u$ /3

Volumen de sectore de sphæra, de radio r, et de basi figura u in superficie de sphæra, vale radio per area basi, diviso per 3.

·3 $o \varepsilon$p . $r \varepsilon$Q . $u \varepsilon$ Cls'p ⌒ xꓱ[d$(x,o) = r$] . arc u εQ .⊃.
area $(o^- u) = r \times$(arc u)/2

·4 $i \varepsilon$ v-ι0 . $o \varepsilon$p . $u \varepsilon$ Cls'plan$(o$, I$i)$. arc$u$ εQ .⊃.
area$(u + \theta i)$ = arc$u$ × mod$i$

Area de cylindro.

·5 $o \varepsilon$p . $i \varepsilon$ v-ι0 . $a \varepsilon$ $\theta\pi/2$.⊃.
area{p⌒xꓱ[ang$(x-o,i) = a$. $(x-o) \times i$ ε Θi^2]} = πi^2sin$a$ /(cos$a$)^2

Cono de revolutione; angulo ad vertice $2a$, altitudo mod$i$.

·6 $o \varepsilon$p . $i \varepsilon$ v-ι0 . $a \varepsilon$ $\theta\pi/2$. $u \varepsilon$ Cls' p⌒xꓱ[ang$(x-o,i) = a$] .
area u εQ .⊃. area[proj plan$(o$, I$i)$]'u = (area u)cos$a$

·7 $a \varepsilon$p$_3$. $o \varepsilon a$. $r \varepsilon$Q . $h \varepsilon$ Θr .⊃.
Volum p⌒xꓱ{d[x,a⌒yꓱ[d$(y,o) = r$]] < h} = $2\pi^2 r^2 h$.
area p⌒xꓱ{d[x,a⌒yꓱ[d$(y,o) = r$]] = h} = $4\pi^2 r h$

{ Kepler a.1615 t.4 p.582:

« Omnis annulus sectionis circularis... est aequalis cylindro, cujus altitudo aequat longitudinem circumferentiae, quam centrum figurae circumductae descripsit, basis vero eadem est cum sectione annuli ». }

VIII

APPLICATIONES AD GEOMETRIA ET COMPLEMENTO

VIII. APPLICATIONES AD GEOMETRIA
ET COMPLEMENTO

§ 1-27 es reproductione de *Theoria de Curvas* per Dr. G. PAGLIERO assistente de Analysi infinitesimale in Universitate de Torino, publicato in anno 1905-06.

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§1 Parabola

$o\varepsilon\mathrm{p} \,.\, a,b\varepsilon\mathrm{v} \,.\, a^2 = b^2 = 1 \,.\, a\times b = 0 \,.\, i = b/a \,.\supset\therefore$

$p = [(o+2xia+x^2a)|x, \mathrm{q}] \,.\, f = o+a \,.\, d = \mathrm{recta}(o-a, ia) \,.\, x\varepsilon\mathrm{q} \,.\supset$

Si o es puncto, a et b es vectore unitario et orthogonale, et si i indica rotatione que porta a in b, et si p habe expressione scripto, tunc puncto px, si x varia, et sume omni valore reale, genera curva dicto « parabola ». o es « vertice », f es « foco ». d es « directrice », recta(o,a) es « axi ».

✻ 1·1 $px = f+(x+i)^2a$

·2 $\mathrm{d}(px, f) = \mathrm{d}(px, d) = x^2+1$ { PAPPO l.VII p.1012 }

Omni puncto de parabola æquidista ab foco et ab directrice.

[$\mathrm{d}(px, o+a) = \mathrm{mod}[(x+i)^2a] = \mathrm{mod}[(x+i)^2] = [\mathrm{mod}(x+i)]^2 = x^2+1$ $= \mathrm{d}(o+2xia+x^2a, o+2xia-a) = \mathrm{d}[px, \mathrm{recta}(o-a, ia)]$]

·3 $[\mathrm{Sym\ recta}(o,a)]\, px = p(-x)$

Symmetrico pro recta (o,a) de puncto px es puncto $p(-x)$. Ergo recta (o,a) es « axi de symmetria » de parabola.

·4 $X,Y\varepsilon\mathrm{q} \,.\supset: o+Xa+Yia \;\varepsilon\; p`\mathrm{q} \,.=.\, Y^2 = 4X$

Conditione necessario et sufficiente ut punto de coordinatas numeros reale X, Y es super parabola, es æquatione scripto, que vocare « æquatione de parabola ».

✻ 2·0 $\mathrm{D}px = 2(x+i)a$. $\mathrm{D}^2px = 2a$

·1 $\mathrm{rectaT}px = [o+(2x+z)ia+(x^2+zx)a]\,|\,z`\mathrm{q}$

[$\mathrm{rectaT}\,px = \mathrm{recta}(px, \mathrm{D}px) = \mathrm{recta}(o+2xia+x^2a, ia+xa) = \ldots$

·2 $o+iax,\; o-ax^2 \;\varepsilon\; \mathrm{rectaT}px$

Intersectione de tangente cum axi (o,a), (o,ia).

·3 $x=0$.$\supset$. rectaTpx = recta[px, U($px-f$)+a]

·4 rectaNpx = recta[px, U($px-f$)−a]

Recta tangente et normale ad parabola es bisectrice de vectores a et $px-f$ (radio focale).

·5 proj (rectaT px) $f = o+xia$

Loco de projectiones de puncto m super tangentes ad curva c vocare « podaria de curva c relativo ad puncto m ». Ergo podaria de parabola relativo ad foco es recta tangente in vertice.

·6 Cc $px = o+(2+3x^2-2x^3i)a$

[Cc $px = px-$Dpx/Imag(D^2px/Dpx)
$= o+a+(x+i)^2a-2(x+i)a$/Imag{$2a$/[$2(x+i)a$]}
= » » » » /Imag $1/(x+i)$
= » » » » /Imag $(x-i)/(x^2+1)$
= » » » » /[$-i/(x^2+1)$]
= » » $-2(x+i)(x^2+1)ia$ = ...]

Centro de curvatura describe parabola de ordine 3/2. Vide §4.

·7 Rc px = 2×d(px, rectaN$px \frown d$) = $2(1+x^2)$↑(3/2)

Radio de curvatura es duplo de normale limitato ad directrice.

* 3·1 $x\varepsilon$Q .$\supset$. Area $\bigcup$[($o+2xia+\Theta x^2a$)|x‘ Θx] = $4x^3/3$

[... = S($4x^2|x$, Θx) = ...]

{ 2·2 3·1 ARCHIMEDE, *Τετραγωνίσμος παραβολῆς*, P2 P17 }

·2 $x\varepsilon$Q .$\supset$. Arc($p,\Theta x$) = $x\surd(1+x^2)$ + log[$x+\surd(1+x^2)$]

[Arc(p, Θx) = S(modDp, Θx) = S[2 mod($x+i$) |x, Θx] =
2S[$\surd(1+x^2)|x$, Θx] = ...]

{ HUYGENS a.1657, vide LoriaB. a.1903 p.4. }

·3 Arc(p, $-1^{-}1$)/Chorda(p, $-1^{-}1$) = 1·1477935...

Ratione de arcu de parabola ad chorda normale ad axi in foco.

* 4. $o\varepsilon$p . $a,b\varepsilon$v . $m,n\varepsilon$q .$b=(m+in)a$. $x\varepsilon$q . $x+m/2 = ny/2$.
$\supset$. $o+xb+x^2a = (o-mnia-m^2a/4)+(2iy+y^2)n^2a/4$

Puncto functione de secundo gradu de x describe semper parabola, que nos reduce ad forma præcedente. Vide motu de puncto grave, pag.324.

Parabola A.H.I.R., F. parabole, D. parabel, G. παραβολή = comparatione, similitudo. $\subset$ para (p.204) + bola.

G. bol- balle = pone, jecta. $\supset$ bol-ide sym-bol-o pro-ble-ma ball-istica.

Apollonio (tomo I, Prop. 11, 12, 13) voca «parabola, ellipsi, hyperbola» sectiones de cono « rectangulo, acutangulo, obtusangulo » secundo Archimede. Sectione de cono habe aequatione de forma $y^2 = 2px + qx^2$, et es « parabola » si $q=0$, « ellipsi » si $q<0$, « hyperbola » si $q>0$, id es, si y^2 « æqua, aut defice, aut supera » $2px$.

§2 Ellipsi

Hp §1 . $m,n\varepsilon$Q . $m>n$. $p=[(o+ ma\cos x + nb\sin x)|x, \mathrm{q}]$.
$f=o+a\surd(m^2-n^2)$. $d=$ recta$[o+m^2a/\surd(m^2-n^2), ia]$.
$f'=o-$ » . $d'=$ » $-$ » . $x\varepsilon$q .⊃:

x = anomalia de eccentro (Kepler, Astronomia nova, a.1609, Opera t.3 p.408). Punctos f,f' es vocato « focos » (Kepler), rectas d, d' es « directrices » (De l'Hospital).

❋ 1·0 $X,Y\varepsilon$q .⊃: $o+Xa+Yia\ \varepsilon p$'q .=. $X^2/m^2+Y^2/n^2=1$

·01 $r,u\varepsilon$q .⊃: $f+re^{iu}a\ \varepsilon\ p$'q .=.
$r=\surd(m^2-n^2)/[1+(/m)\surd(m^2-n^2)\cos u]$

·1 $px=o+[(m+n)\mathrm{e}^{ix}+(m-n)\mathrm{e}^{-ix})]a/2$
[§ e P13·4 .⊃. P]

Alio expressione de puncto de ellipsi ; resulta de relatione inter functiones trigonometrico et exponentiale p.251.

·2 [Sym recta(o,a)]$px=p(-x)$.
[Sym recta(o,ia)]$px=p(\pi-x)$. (Sym o)$px=p(\pi+x)$

Ellipsi habe pro axi de symmetria recta (o,a)et recta(o,ia), et pro centro o.

·3 d(px,f) + d(px,f') = $2m$ ¦ Apollonio l. 3 § 52 et 51¦

·4 d$(px,f)=m-\surd(m^2-n^2)\cos x$ = $\surd(m^2-n^2)/m$ d(px, d)
·5 d$(px,f')=m+$ » = » d(px, d')
¦ Pappo vii P 235 et 238¦

❋ 2·1 D$px=(-m\mathrm{s}x+in\mathrm{c}x)a=[(m+n)\mathrm{e}^{ix}-(m-n)\mathrm{e}^{-ix}]ia/2$
$=p(x+\pi/2)-o$

Vectores $px-o$, $p(x+\pi/2)-o$ es « hemi-diametros coniugato ».

·2 $(px-o)^2+[p(x+\pi/2)-o]^2=m^2+n^2$

·3 d ¦proj (rectaT px) f, o¦ = m

Projectione de uno foco super tangente in uno puncto de ellipsi dista ab centro de quantitate constante m. Vel, podaria de ellipsi, relativo ad foco, es circulo de centro puncto o, et de radio m.

·4 x -ε nπ .⊃. rectaTpx = recta[px, U($px-f$)—U($px-f'$)]

·5 rectaNpx = recta[px, U($px-f$) + U($px-f'$)]

·6 $o+(m+n)\mathrm{e}^{ix}a\varepsilon$ rectaNpx

·7 $\mathrm{d}(f, \mathrm{rectaT}\, px) \times \mathrm{d}(f', \mathrm{rectaT}\, px) = n^2$

} Keill LondonT. a. 1709 {

·8 $\mathrm{d}(px, f) \times \mathrm{d}(px, f') = (\mathrm{D}px)^2$

·9 $\mathrm{d}[f, (\mathrm{Sym\ rectaT}\, px) f'] = 2m$

·91 $(px - o)\, \alpha\, [p(x + \pi/2) - o] = mn(\alpha ab)$

❋ 3·0 $\mathrm{D}^2 px = -(px - o)$

·1 $\mathrm{Cc}\, px = o + [n(\cos x)^3 - mi(\sin x)^3] a (m^2 - n^2)/(mn)$

·2 $\mathrm{Rc}\, px = (\mathrm{D}px)^2 / \mathrm{d}(o, \mathrm{rectaT} px) = (\mathrm{modD}px)^3 / (mn)$

❋ 4·1 $\mathrm{Area}\ o^{-} p^{\prime} 2\Theta\pi = \pi mn$

} Archimede *Περὶ κωνοειδέων* P5 {

·1 $x \varepsilon 2\Theta\pi$.⊃. $\mathrm{Area}(o^{-} p^{\prime} \Theta x) = mnx/2$

·2 $\mathrm{Arc}(p, 2\Theta\pi) = 4\mathrm{S}\rbrace\sqrt{[(m \sin x)^2 + (n \cos x)^2]}\ |x, \Theta\pi/2\lbrace$

$= 2\pi m \rbrace 1 - \Sigma[\Pi\rbrace[1 - /(2r)]^2 |r, 1 \cdots s\lbrace (1 - n^2/m^2)^s/(2s-1)\ |s, \mathrm{N}_1]\lbrace$

$= \pi(m+n)\, \Sigma\rbrace [\mathrm{C}(/2, r)]^2 [(m-n)/(m+n)]^{2r}\ |r, \mathrm{N}_0\lbrace$

$= \pi(m+n)\rbrace 1 + [(m-n)/(m+n)]^2/4 + [(m-n)/(m+n)]^4/64 + \ldots$

·3 $\mathrm{Arc}(p, 2\Theta\pi) > \pi(m+n)$

} Keplero a.1609 t.3 p.401:

« Tota elliptica circumferentia est proxime medium arithmeticum inter circulum diametri longioris et circulum diametri brevioris ». {

·4 $\mathrm{Arc}(p, 2\Theta\pi) < \pi(m+n) + \pi(\sqrt{m} - \sqrt{n})^2/2$

ParisCR. a.1889 p.360.

·5 $\mathrm{Arc}(p, 2\Theta\pi) < \pi\rbrace m + n + \sqrt{[2(m^2+n^2)]}\lbrace/2$

} Hartmann HoffmannZ. t.30 p.256 {

Ellipsi. A. ellipsis, D.F. ellipse, H. elipsi, I. ellissi, R. elipsis'
⊂ G. ἔλλειψι-ς. = (in origine) defectu.
⊂ en (p.311) + lip(e) (‖ L. linque) + -si.

§3. Hyperbola

Hp §1 . $m, n \varepsilon \mathrm{Q}$. $p = \rbrace[o + (m + in)ax + (m - in)a/x]\ |x, \mathrm{q}\text{-}\iota 0\lbrace$

$f = o + 2a\sqrt{(m^2+n^2)}$. $d = \mathrm{recta}[o + 2m^2/\sqrt{(m^2 + n^2)}\, a, ia]$.

$f' = o -$ » . $d' =$ » $-$ » .

$x \varepsilon \mathrm{q}\text{-}\iota 0$.⊃.

❋ 1·0 $x,y\varepsilon q$.⊃: $o+xa+yia\ \varepsilon p'q$.=. $x^2/m^2-y^2/n^2=4$

·1 [Sym recta(o,a)] $px = p(/x)$

·2 [Sym recta(o,ia)] $px = p(-/x)$. (Sm o)$px = p(-x)$

o es « centro »; recta(o,a) et recta(o,ia) es « axi »; $o\pm 2ma$ es « vertice ». Punctos f,f' es « focos », rectas d, d' es « directrices ».

❋ 2·0 $x\mathrm{D}px = (m+in)ax-(m-in)a/x$

$qx=o+2(m+in)ax$. $rx=o+2(m-in)a/x$.⊃.

·1 qx,rx ε rectaT px . $px=(qx+rx)/2$

·2 $oa(rx)a(qx) = 4mn(aab)$

recta$[o, (m\pm in)a]$ es « asymptotos »; $px-o$, $px-rx$ es « hemi-diametros coniugato ».

❋ 3·1 mod[d(px, f)—d(px, f')] $=4m$

·2 d(px, f) = mod$[(x+/x)\surd(m^2+n^2)-2m]$ =
$\surd(m^2+n^2)/m\times$d(px, d) = d{px, d ⌒ recta$[px, (m\pm in)a]$}

·3 d [proj (rectaT px) f, o] $= 2m$

·4 d [(Sym rectaT px) f, f'] $= 4m$

·5 d (f, rectaT px) ×d(f', rectaT px) $= 4n^2$

❋ 4·1 $x-=\pm 1$.⊃. rectaT px =
recta$[px$, U$(px-f)$ + U$(px-f')]$

·2 rectaN px = recta$[px$, U$(px-f)$ —U$(px-f')]$

❋ 5·1 Cc $px = o+ [n(x+/x)^3-m(x-/x)^3i]a\ (m^2+n^2)/(4mn)$

$x\varepsilon$ 1+Q .⊃.

·2 Area $o^{-}p'$ $1^{-}x = 2mn$ log$x$
{ ·2 GREGORIUS a S. Vinc. a.1647 p.594 }

·3 Arc$(p, 1^{-}x)$ =
$\surd(m^2+n^2)$S{$\surd[1+2(n^2-m^2)/[(n^2+m^2)x^2]+/x^4]|x, 1^{-}x$}

Hyperbola A., D. hyperbel, F. hyperbole, H. hiperbola, I. iperbola R. giperbola. ⊂ G. ὑπερβολή. ⊂ hyper (|| L. super) + bola (v. para-bola)

Asymptoto, ADF asymptote, R asimptota.
⊂ G. ἀσύμπτωτο-ς (Eutocio) = non coincidente.
⊂ a- (⊃ A.D.F.H.I.R. a-chromatico a-mnestia a-tomo ... || L. in- D. un-) +syn- (p.20) + pto- (⊃ A.—R. sym-ptoma || L. pete) + -to.

§4. Parabola de vario ordine.

Hp §1 . $m,n \,\varepsilon\, \mathrm{q}-\iota 0$. $k = \mathrm{q} \frown x \ni (x^n, x^m \,\varepsilon\, \mathrm{q})$.
$p = [(o+x^n a+x^m ia) \,|\, x, k]$. $x \,\varepsilon\, k$.⊃.

·0 $x,y \varepsilon \mathrm{q}$.⊃: $o+xa+yia \,\varepsilon\, p`k$.=. $x^m = y^n$

·1 $o+(m-n)x^n a/m \,\varepsilon$ rectaT px

·2 $m+n-2 -= 0$.⊃. Cc $px =$
$px+(n^2x^{2n-2}+m^2x^{2m-2})\, \mathrm{D}px/[mn(m+n-2)x^{m+n-3}]$

·3 $m+n \,\varepsilon \mathrm{Q}$.⊃. Area $\bigcup[(o+x^n a)\frown px \,|\, x, \Theta x] = nx^{m+n}/(m+n)$

·4 $y,z \,\varepsilon \mathrm{Q}$. $y<z$.⊃. $\mathrm{Arc}(p, y\frown z) =$
$\mathrm{S}\{\sqrt{[n^2x^{2n-2}+m^2x^{2m-2}]} \,|\, x, y\frown z\}$

Parabola de ordine m/n. Pro $m/n = -1$ curva es hyperbola.

§5 Linea exponentiale

❋ 1. Hp§1. $p = \{[o+(x+ie^x)a] \,|\, x, \mathrm{q}\}$. $x \varepsilon \mathrm{q}$.⊃.

·1 $o+(x-1)a \;\varepsilon$ rectaT px

·2 $\lim \{ \mathrm{d}\,[\, px, \mathrm{recta}\,(o, a)] \,|\, x, \mathrm{q}, -\infty \} = 0$

·3 Cc $px = o + [(x-1-e^{2x}) + i(2e^x+e^{-x})]a$
$= px + (e^x+e^{-x})(i-e^x)a$

·4 $\mathrm{Rc}px = e^{-x}(1+e^{2x}) \uparrow (3/2)$

·5 $\mathrm{Rc}p(-\log 2/2) = 3\sqrt{3}/2 = \min(\mathrm{Rc}p, \mathrm{q})$

$y,z \,\varepsilon \mathrm{q}$. $y<z$.⊃.

·6 $\mathrm{Arc}(p, y\frown z) = \mathrm{S}[\sqrt{(1+e^{2x})} \,|\, x, y\frown z] =$
$\Delta\{\sqrt{(1+e^{2x})}+\log[\sqrt{(1+e^{2x})}-1]-x \,|\, x; y,z\}$

·7 Area$\bigcup[\, px\frown(o+xa) \,|\, x`\, y\frown z] = \mathrm{S}(e^x|x, y\frown z) = e^z - e^y$
{ ·1 ·7 Torricelli a.1644 }

·8 Volum$\bigcup\{\mu(2\pi\, oaa)[(o+xa)\frown px] \,|\, x`y\frown z\}$
$= \pi \mathrm{S}(e^{2x}|x, y\frown z) = \pi(e^{2z}-e^{2y})/2$

·9 area$\bigcup[\mu(2\pi\, oaa)px \,|\, x`y\frown z] = 2\pi \mathrm{S}[e^x \sqrt{(1+e^{2x})} \,|\, x, y\frown z]$
$= \pi \times \Delta\{e^x \sqrt{(1+e^{2x})}+\log[e^x + \sqrt{(1+e^{2x})}] \,|\, x; y,z\}$

Linea vocare et « curva logarithmica ».
Recta(o, a), de que in P·2, es « asymptoto ».

❋ 2. Hp§1. $m\varepsilon$Q-ι1 . p=$\{[o+(x+im^x)a]|x$, q$\}$. $x\varepsilon$q .⊃.

·1 $o+(x-/\log m)a$ ε rectaTpx

·2 lim $\{$ d $[px$, recta$(o, a)]$ $|x$, q, $-\infty$ $\} = 0$

·3 Cc$px = o + \{(x - /\log m - m^{2x}\log m) + i[2m^x + m^{-x}/(\log m)^2]\}a$
$= px + [m^x + m^{-x}/(\log m)^2](i - m^x \log m)a$

y, z εq .⊃.

·4 Arc$(p, y^-z) =$ S$\{\sqrt{[1+(m^x \log m)^2]}|x, y^-z\} = (/\log m)\times$
$\Delta\{\sqrt{[1+(m^x \log m)^2]}+\log[\sqrt{[1+(m^x \log m)^2]}-1]$
$-x\log m\, |x; y,z\}$

·5 area⋃$[px^-(o+xa)|x' y^-z]$ = S$(m^x\, |x, y^-z)$ =
$(m^z - m^y)/\log m$

§6 Catenaria, Tractória.

❋ 1. Hp§1. $p = \{[o + ax + ia(e^x + e^{-x})/2]\, |x$, q$\}$. $x\varepsilon$q .⊃.

·1 [Sym recta(o, ia)] $px = p(-x)$

·2 D$px = a + ia(e^x - e^{-x})/2$

·3 mod D$px = (e^x + e^{-x})/2$

·4 d$(o+ax$, rectaT$px) = 1$

$rx = o + ax + a(e^{2x} - e^{-2x})/4$.⊃.

·5 rx ε rectaNpx

·6 Cc $px = o + ax + (e^x + e^{-x})ia - (e^{2x} - e^{-2x})a/4 =$ (Sm $px)rx$

·7 Rc$px =$ d$(px, rx) = (e^x + e^{-x})^2/4$

·8 $x\varepsilon$Q .⊃. Arc$(p, \Theta x) =$ Area⋃$[(o + xa)^- px|x' \Theta x] =$
S$[(e^x + e^{-x})/2\, |x, \Theta x] = (e^x - e^{-x})/2$

·9 $x\varepsilon$Q .⊃. Volum⋃$\{\mu(2\pi\, oaa)[(o+xa)^- px]\,|x' \Theta x\}=$
(/2) area⋃$[\mu(2\pi\, oaa)px|x' \Theta x] = (e^{2x} - e^{-2x} + 4x)\pi/8$

Catenaria es positione de æquilibrio de catena grave homogeneo.

$\}$ LEIBNIZ a.1691 t.5 p.246 $\}$

P·7. Radio de curvatura vale normale usque ad axi (o,a).

* 2. HpP1. $q = \{[px - \mathrm{Arc}(p,\Theta x) \mathrm{U} \mathrm{D}px] \mid x, \mathrm{Q}_0\} \,.\, x \varepsilon \mathrm{Q}_0 \,.\supset.$

·0 $qx = o + [x - (\mathrm{e}^x - \mathrm{e}^{-x} - 2i)/(\mathrm{e}^x + \mathrm{e}^{-x})]a$

·01 $\lim\{\mathrm{d}[px, \mathrm{recta}(o,a)] \mid x, \mathrm{Q}_0, \infty\} = 0$

·1 $\mathrm{D}qx = (\mathrm{e}^x - \mathrm{e}^{-x} - 2i)(\mathrm{e}^x - \mathrm{e}^{-x})/(\mathrm{e}^x + \mathrm{e}^{-x})^2 a$

·2 $\mathrm{mod}\, \mathrm{D}qx = (\mathrm{e}^x - \mathrm{e}^{-x})/(\mathrm{e}^x + \mathrm{e}^{-x})$

·3 $\mathrm{rectaN}qx = \mathrm{recta}(px, qx)$

·4 $o + xa \,\varepsilon\, \mathrm{rectaT}qx$ ·5 $\mathrm{d}(qx, o + xa) = 1$

·6 $\mathrm{Cc}qx = px$

·7 $\mathrm{Rc}qx = \mathrm{d}(px, qx) = (\mathrm{e}^x - \mathrm{e}^{-x})/2$

·8 $\mathrm{Arc}(q, \Theta x) = \mathrm{S}[(\mathrm{e}^x - \mathrm{e}^{-x})/(\mathrm{e}^x + \mathrm{e}^{-x}) | x, \Theta x] = \log[(\mathrm{e}^x + \mathrm{e}^{-x})/2]$

·9 $\mathrm{area} \bigcup \{[qx \frown \mathrm{proj}[\mathrm{recta}(o,a)]qx] \mid x`\mathrm{Q}_0\} = \pi/4$

Puncto qx de P·0, si x varia et sume omni valore reale, genera « tractoria », evolvente de catenaria. Recta (o,a), de que in P·01, es « asymptoto ». Puncto $o+ia$ que es « vertice » de catenaria, es « cuspide » pro tractoria.

P·3·4 Si nos uni cum filo punto qx ad puncto $o+xa$, que describe recta (o, a), tunc filo es semper longo 1, et punto qx move se in directione de filo. Unde vocabulo ori.

Catenaria A.I. (Jac. Bernoulli Ac.Er. a.1690), F. chaînette, D. Kettenlinie. ⊂ catenari(o) + -a (indice de feminile; intellige « linea »).

catenario (L a. + 50 circa) = de catena. ⊂ catena + -rio.

catena A.I., A. chain, F. chaîne, H. cadena, D. kette.

Tractoria (Huygens, Joh. Bernoulli). ⊂ trah(e) + -tor + -ia

§7 Sinusoide.

Hp§1 . $p = [(o + ax + ia \sin x) | x, \mathrm{q}] \,.\, x \varepsilon \mathrm{q} \,.\supset.$

·1 $n \varepsilon \mathrm{n} \,.\supset.\, [\mathrm{Sm}(o + n\pi a)]\, px = p(2n\pi - x)$.

$\{\mathrm{Sym}\, [\mathrm{recta}(o + (2n+1)\pi a/2, ia)]\} px = p[(2n+1)\pi - x]$

·2 $\mathrm{Transl}(2\pi a)\, px = p(2\pi + x)$ . .

$\mathrm{Transl}(\pi a/2)\, (o + ax + ia \cos x) = p(\pi/2 + x)$

·3 $\mathrm{D}px = a + ia\, \mathrm{c}x$. $\mathrm{D}^2 px = -ia\, \mathrm{s}x$

·4 $\mathrm{Mod}\, \mathrm{D}px = \surd(1 + \mathrm{c}x^2)$

·5 $o + (x - \mathrm{t}x)\iota \,\varepsilon\, \mathrm{rectaT}px$

·6 $\mathrm{Cc}\, px = px - ia(1+i\mathrm{c}x)(1+\mathrm{c}x^2)/\mathrm{s}x$

·7 $\mathrm{Rc}px = (1+\mathrm{c}x^2)\uparrow(3/2)/\mathrm{s}x$

·8 $\mathrm{Arc}(p,\Theta\pi/2) = \mathrm{S}[\surd(1+\mathrm{c}x^2)|x,\Theta\pi/2] = \mathrm{S}\{\surd[\mathrm{s}x^2+2\mathrm{c}x^2]\,|x,\Theta\pi/2\}$

·9 $x\varepsilon\Theta\pi$.⊃. $\mathrm{Area}\bigcup[(o+xa)^{-}px|x`\,\Theta x] = \mathrm{S}(\mathrm{s},\,\Theta x) = 1-\mathrm{c}x$

·10 $\mathrm{Area\ Med}\; p`\,\Theta\pi = 2$

·11 $\mathrm{Volum}\bigcup\{\mu(2\pi\ oaa)[(o+xa)^{-}px]|x`\,\Theta\pi/2\} = \pi^2/4$

·12 $\mathrm{area}\bigcup[\mu(2\pi\ oaa)px|x`\,\Theta\pi/2] = \pi[\surd2+\log(1+\surd2)]$

« Linea sinuum » (Wallis), « Cycloidis socia » (Roberval).
Omni puncto $o+\mathrm{n}\pi a$ es centro de symmetria.
Sinusoide habe infinito centro, et infinito axi de symmetria.

Sinusoide, voce internationale. ⊂ sinus + -oide.

§8 Tangentoide

Hp§1 . $p = [(o+ax+ia\ \mathrm{tng}x)|x,\ -\pi/2^{-}\pi/2]$. $x\varepsilon\ {}^{-}\pi/2\ \ \pi/2$.⊃.

·0 $(\mathrm{Sym}\ o)px = p(-x)$

1\. $\mathrm{D}px = a+ia/\mathrm{c}x^2$

·2 $o+a+ia\ \varepsilon\ \mathrm{rectaT}p0$. $o+ax-a\mathrm{s}(2x)/2\ \varepsilon\ \mathrm{rectaT}px$

·3 $\mathrm{Cc}px = px+ia(1+i/\mathrm{c}x^2)(1+\mathrm{c}x^4)/\mathrm{s}(2x)$

·4 $\mathrm{Rc}px = (1+\mathrm{c}x^4)\uparrow(3/2)/[\mathrm{s}(2x)\mathrm{c}x^2]$

$x\varepsilon\theta\pi/2$.⊃.

·5 $\mathrm{Area}\bigcup[(o+xa)^{-}px|x`\,\Theta x] = \mathrm{S}(\mathrm{t}x\,|x,\,\Theta x) = -\log \mathrm{c}x$

·6 $\mathrm{Volum}\bigcup\{\mu(2\pi\ oaa)[(o+xa)^{-}px]|x`\,\Theta x\} = \pi(\mathrm{t}x-x)$

§9 Curva de luce

Hp§1 . $p = [(o+xa+ia\log\cos x)|x,\ 0^{-}\pi/2]$. $x\varepsilon\ 0^{-}\pi/2$.⊃.

·1 $\mathrm{D}px = a-ia\mathrm{t}x$. $\mathrm{D}^2px = -ia/\mathrm{c}x^2$

·2 $\mathrm{mod}\,\mathrm{D}px = /\mathrm{c}x$

·3 $\mathrm{Cc}px = px-ia-a\mathrm{t}x$

·4 $\mathrm{Rc}px = /\mathrm{c}x$

·5 $\text{Area} \bigcup [px \frown (o+xa) | x \text{ ‘ } 0 \frown \pi/2] = \mathrm{S}(\log \mathrm{c}, \Theta\pi/2) =$ $(\pi/2)\log 2$

·6 $x \varepsilon\, \Theta\pi/2 \,.\supset.\, \mathrm{Arc}(p, \Theta x) = \mathrm{S}(/\mathrm{c}, \Theta x) = \log \mathrm{t}(\pi/4 + x/2)$

Trajectoria de radio de luce in atmosphæra, si densitate varia secundo formula de barometro, temperatura constante.

Es «catenaria æquiresistente». Vide Cesàro a.1905, p.214.

§10 Spira mirabile

$\mathrm{Hp}\S 1 \,.\, h\varepsilon \mathrm{Q} \,.\, p = [\{o + a\mathrm{e}\curvearrowright[(h+i)x]\} | x, \mathrm{q}] \,.\, x\varepsilon \mathrm{q} \,.\supset.$

·0 $h > 1 \,.\supset.\, \lim(p, \mathrm{q}, -\infty) = o$

·1 $\mathrm{D}px = (h+i)(px - o) \quad . \quad \mathrm{D}^2 px = (h+i)^2(px - o)$

·2 $\mathrm{rectaT}px = \mathrm{recta}[px, (h+i)(px-o)]$

{ Descartes a.1638, *OEuvres*, éd. Tannery t.2 p.360 }

·3 $\mathrm{Cc}px = px + i\mathrm{D}px = o + hia\mathrm{e}\curvearrowright[(h+i)x] = o + hi(px - o)$

·4 $\mathrm{Rc}px = \mathrm{mod}\, \mathrm{D}px = \mathrm{e}^{hx}\sqrt{(1+h^2)}$

·5 $x\varepsilon\, 2\Theta\pi \,.\supset.\, \text{Area}\, o \frown p\text{‘}\Theta x = [\mathrm{e}\curvearrowright(2hx) - 1]/(4h)$

·6 $y, z\, \varepsilon \mathrm{q} \,.\, y < z \,.\supset.\, \mathrm{Arc}(p, y \frown z) = \sqrt{(1+h^2)}\mathrm{S}(\mathrm{e}^{hx} | x, y \frown z) =$ $\sqrt{(1+h^2)}\, (\mathrm{e}^{hz} - \mathrm{e}^{hy})/h$

·7 $\mathrm{Arc}(p, x - \mathrm{Q}) = \mathrm{e}^{hx}\sqrt{(1+h^2)}/h$

{ Torricelli, a.1640; vide Loria, LinceiR. s.5 t.6 a.1897 }

{ Jac. Bernoulli, *Acta Eruditorum*, a.1692, voca curva « Spira mirabilis... quoniam enim semper sibi similem eandem Spiram gignit, utcumque volvatur, evolvatur, radiet ... Libenter Spiram hanc tumulo meo juberem incidi cum epigrapho: *Eadem mutata resurgo* ».

« Spirale logarithmica » Joh. Bernoulli.

P·0. o es « puncto asymptotico ».

Spira L., G. *σπεῖρα*, A.D.F. spire, H. espira.

Spirale non L. classico, L. de Bernoulli, D.F.I., A spiral, H espiral, R. spiralı-naja. ⊂ spira + -le

Mirabile L., A.F.H. ad-mirable.

§11 Spirale de ordine m

Hp§1 . $m \varepsilon$ q$-\iota 0$. $p = [(o+x^m e^{ix}a)\,|x, Q]$. $x\varepsilon Q$.$\supset$.

·1 $o - x^{m+1}e^{ix}ia/m$ ε rectaTpx . $o+mx^{m-1}e^{ix}ia$ ε rectaNpx

·2 $m\varepsilon Q$. $x\varepsilon\, 2\Theta\pi$.$\supset$. Area $o\urcorner p‘\Theta x = x^{2m+1}/[2(2m+1)]$

·3 Cc$\,px = px + x^{m-1}(m^2+x^2)/(m^2+m+x^2) \times (m+ix)e^{ix}ia$

·4 $y\varepsilon\, x+Q$.$\supset$. Arc$(p, x\urcorner y) = S[x^{m-1}\surd(m^2+x^2)|x, x\urcorner y]$

§12 Spirale de Archimede

Hp§1 . $p = [(o+xe^{ix}a)\,|x, q]$. $x\varepsilon q$.$\supset$:

·1 D$px = (1+ix)e^{ix}a$. D$^2px = (2i-x)e^{ix}a$

·2 $o - x^2ie^{ix}a$ ε rectaTpx ·3 $o+ie^{ix}a$ ε rectaN$\,px$

·4 $x\varepsilon\, 2\Theta\pi$.$\supset$. Area $o\urcorner p‘\Theta x = x^3/6$

{ ·2-·4 Archimede *Περὶ Ἑλίκων* P20, P28 }

·5 $x\varepsilon Q$.$\supset$. Arc$(p, \Theta x) = S[\surd(1+x^2)|x, \Theta x] =$
$\{x\surd(1+x^2)+\log[x+\surd(1+x^2)]\}/2$

{ Roberval, a.1643; Wallis a.1655, vide LoriaB. a.1903 p.3 }

·6 Cc$\,px = px+i(1+x^2)$D$px/(2+x^2) =$
$o+[x+i(1+x^2)]/(2+x^2)]\, e^{ix}a$

·7 Rc$px = (1+x^2)\uparrow(3/2)/(2+x^2)$

§13 Spirale de ordine —1

Hp§1 . $p = [(o+e^{ix}a/x)|x, Q]$ $x\varepsilon Q$.$\supset$:

·0 lim[d[$\,px$, recta$(o+ai, a)$]$|x$, Q, 0$] = 0$

·01 lim$(p$, Q, $\infty) = o$

·1 D$px = (ix-1)e^{ix}a/x^2$. D$^2px = (2-x^2-2ix)e^{ix}a/x^3$

·2 $o+ie^{ix}a$ ε rectaT$\,px$

·3 Cc$\,px = px+i(1+x^2)$D$px/x^2 = o-[x^{-3}+i(x^{-2}+x^{-4})]e^{ix}a$

·4 Rc$px = (1+x^2)\uparrow(3/2)/x^4$

·5 Area $o\urcorner p‘y\urcorner z = (1/y-1/z)/2$

·6 $y, z \,\varepsilon\, \mathrm{Q} \,.\, y < z \,.\supset.$
$\mathrm{Arc}(p, y\frown z) = \mathrm{S}[\sqrt{(1+x^2)}/x^2 \,|\, x, y\frown z] =$
$\Delta\{[2\log[x+\sqrt{(1+x^2)}] - \sqrt{(1+x^2)}/x] \,|\, x\,;\, y,z\}$

« Spiralis hyperbolica », ita vocato per Joh. Bernoulli, a.1710 (Opera omnia t.1 p.480). Recta in P·0 es « asymptoto ».

Puncto o es « asymptotico ».

§14 Cochleoide

Hp§1 . $p = [(o+\mathrm{e}^{ix}\,\mathrm{s}x/x\, a) | x, \mathrm{q}-\iota 0] \,.\, x\varepsilon\, \mathrm{q}-\iota 0 \,.\supset.$

·0 $px = \mathrm{G}(o+\mathrm{e}^{iz}a \,|\, z, 2\Theta x)$

·1 $o+a\mathrm{e}^{2ix} \varepsilon$ rectaT px

·2 $\Sigma\{\mathrm{Area}[o\frown p`n\pi\frown(n+1)\pi] \,|\, n, \mathrm{N}_0\} = \pi/4$

Loco de barycentros de arcu de circulo $0\frown 2x$.

Cochlea L. ⊂ G κόχλο-ς. ⊃ I chiocciola, F coque.
Cochleoide ⊂ coclea — -a + -oide

§15 Sinus-spirale

Hp§1 . $m\varepsilon\mathrm{Q} \,.\, p = \{[o+a\mathrm{e}^{ix}(\sin mx)^{(1/m)}] \,|\, x, \Theta\pi/m\}$.
$x\varepsilon\, \Theta\pi/m \,.\supset.$

·1 $\mathrm{D}px = \mathrm{e}^{[(m+1)ix]}a\,(\sin mx)^{(1/m-1)}$
$= (px-0)\mathrm{e}^{(mix)}/\sin mx$

·2 $\mathrm{D}^2px = [\mathrm{e}^{(mix)} - m\mathrm{e}^{(-mix)}]\mathrm{D}px/\sin mx$

·3 $\mathrm{Cc}\, px = px + i(\mathrm{D}px)/(m+1)$

·4 $\mathrm{Rc}px = (\sin mx)^{(1/m-1)}/(m+1)$

·5 $\mathrm{Area}\; o\frown p`\Theta\pi/m = (2/m)\,\mathrm{S}[\sin z^{(2/m)} \,|\, z, \Theta\pi/2]$

·6 $\mathrm{Arc}(p, \Theta\pi/m) = (2/m)\,\mathrm{S}[\sin z^{(1/m-1)} \,|\, z, \Theta\pi/2]$

De la Goupillière, AnnN. a.1876 p.97.

·1 « Angulo de tangente cum radio vectore es mx ».

·2 « Radio de curvatura vale normale polare diviso per $m+1$ ».

Pro $m=1$ curva es circulo: pro $m=-1$ recta: pro $m=2$ « lemniscata de Jac. Bernoulli ». Pro $m=-2$ hyperbola. Pro $m=1/2$ cardioide. Pro $m=-1/2$ parabola.

§16 Cycloide

❋ 1. Hp§1 . $p = [(o+xa+\mathrm{e}^{ix}ia)\,|x, \mathrm{q}]$. $x\varepsilon\mathrm{q}$.$\supset$:

·0 $p(2\pi+x) = px+2\pi a$

·1 $n\varepsilon\mathrm{n}$.$\supset$. Sym recta$(o+n\pi a, ia)px = p(2n\pi-x)$

·2 $\mathrm{D}px = a(1-\mathrm{e}^{ix}) = -2\sin(x/2)ia\mathrm{e}\!\upharpoonleft\!(ix/2)$. $\mathrm{D}^2px = -\mathrm{e}^{ix}ia$

·3 $x\varepsilon\, 2\Theta\pi$.$\supset$. mod $\mathrm{D}px = 2\sin(x/2)$

·4 $o+xa-ia$ ε rectaT px . $o+xa+ia$ ε rectaN px

·5 Cc $px = px+2i\mathrm{D}px = (o+2ia)+(x-i\mathrm{e}^{ix})a$

$= p(\pi+x)+2ia-\pi a$

·6 $x\varepsilon\, 2\Theta\pi$.$\supset$. Rc$px = 4\sin(x/2)$

·7 Area Med$(p‘\, 2\Theta\pi) = 3\pi$

{ Roberval a.1634; vide Loria a.1902 p.461 }

·8 $x\varepsilon\, 2\Theta\pi$.$\supset$. Arc$(p, \Theta x) = 8\,[\sin(x/4)]^2$

{ Wren a.1650; vide Wallis, *Opera*, t.1 p.538. }

·9 Arc$(p, 2\Theta\pi) = 8$

$o+ia+2\mathrm{n}\pi a$ es cuspide, $o-ia+(2\mathrm{n}+1)\pi a$ es vertice. Omni recta$(o+\mathrm{n}\pi a, ia)$ es axi.

❋ 2. Hp§1 . $h\varepsilon\mathrm{Q}$. $p = [(o+xa+h\mathrm{e}^{ix}ia)\,|x, \mathrm{q}]$.$\supset$.

·1 $\mathrm{D}px = a(1-h\mathrm{e}^{ix})$. $\mathrm{D}^2px = -h\mathrm{e}^{ix}ia$

·2 mod $\mathrm{D}px = 1+h^2-2h\cos x$

·3 $o+xa+ia$ ε rectaN px

·4 Area$[(o+hia)\overline{\ }p‘2\Theta\pi] = \pi h^2+2\pi h$

·5 Cc$px = px+(1+h^2-2h\cos x)i\mathrm{D}px/(h^2-h\cos x)$

·6 Arc$(p, 2\Theta\pi) =$

$\mathrm{S}\{\sqrt{[(1-h)^2\mathrm{c}x^2 + (1+h)^2\mathrm{s}x^2]}\,|x, 2\Theta\pi\}$ { Pascal t.3 p.441 }

Cycloide cum punctos duplo si $h>1$, et punctos de flexu si $h<1$.

cycloide, nomen dato per Galileo a. 1599. F. cycloïde, A. cycloid, I. cicloide, R. tsicloida. $\subset$ cyclo + -ide.

cyclo G. κύκλο-ς, A.F. cycle, D. cyclus, R. tsiel', H.I. ciclo. = circulo $\subset$ E. qeqlo, A. wheel, S. c'acra = rota.

§17 Evolvente de circulo

Hp§1 . $p = [(o + a\mathrm{e}^{ix} - xai\mathrm{e}^{ix}) \mid x, \mathrm{q}] \,.\, x \varepsilon \mathrm{q} \,.\supset.$

·0 $\mathrm{D}px = xa\mathrm{e}^{ix}$

·1 $\mathrm{Cc}\,px = o + a\mathrm{e}^{ix}$

·2 $x\varepsilon\mathrm{Q} \,.\supset.\, \mathrm{Arc}(p, \Theta x) = x^2/2$

·3 $\mathrm{Area}\bigcup[(o + a\mathrm{e}^{ix})\overline{\ }px \mid x`\Theta x] = x^3/6$

·4 $\mathrm{proj}(\mathrm{rectaT}\,px)o = o - xia\mathrm{e}^{ix}$

Podaria de evolvente de circulo relato ad centro es spirale de Archimede.

Evolvente D.I., A. evolvent. $\subset$ e (=ex) + volve (p.379) + -nte.

§18 Asteroide

Hp§1 . $p = [o + (3\mathrm{e}^{ix} + \mathrm{e}^{-3ix})a \mid x, \mathrm{q}] \,.\supset.$

·0 $p(2\pi + x) = px$ ·01 $px = o + 4[(\mathrm{c}x)^3 + i(\mathrm{s}x)^3]a$

·1 $\mathrm{D}px = 3i(\mathrm{e}^{ix} - \mathrm{e}^{-3ix})a = -6a\mathrm{e}^{-ix}\sin 2x$

·2 $x\varepsilon\, \Theta\pi/2 \,.\supset.\, \mathrm{mod}\,\mathrm{D}px = 6\sin 2x$

·3 $\mathrm{rectaT}px = [o + 4(\mathrm{c}x)^2(\mathrm{c}x - 3z\mathrm{s}x)a + 4(\mathrm{s}x)^2(\mathrm{s}x + 3z\mathrm{c}x)ia]z`\mathrm{q}$

$qx = o + 4a\mathrm{c}x \,.\, rx = o + 4ia\mathrm{s}x \,.\supset.$

·4 $qx, rx\ \varepsilon\ \mathrm{rectaT}\,px$

·5 $m\varepsilon\mathrm{q} \,.\supset.\, mqx + (1-m)rx = o + 4[ma\mathrm{c}x + (1-m)ia\mathrm{s}x]$

·6 $\mathrm{d}(qx, rx) = 4$

·7 $\mathrm{Cc}px = px - i\mathrm{D}px = o + 2(3\mathrm{e}^{ix} - \mathrm{e}^{-3ix})a$
$= o + 2\mathrm{e}\upharpoonright(i\pi/4)\,p(x - \pi/4)$

·71 $x\varepsilon\, \Theta\pi/2 \,.\supset.\, \mathrm{Rc}px = 6\sin 2x$

·8 $x\varepsilon\, \Theta\pi/2 \,.\supset.\, \mathrm{Arc}(p, \Theta x) = 6\mathrm{S}(\sin 2x|x, \Theta x) = 3(1 - \cos 2x)$

·81 $\mathrm{Arc}(p, \Theta\pi/2) = 6$

·9 $\mathrm{Area}\ o\overline{\ }p`\Theta x = 3[x - (1/4)\sin 4x]$

·91 $\mathrm{Area}\ o\overline{\ }p`\Theta\pi/2 = 3\pi/2$

·10 Volum $\bigcup\{\mu(2\pi\, oaa)[[o+4(\mathrm{c}x)^3a]^{-}px]|x, \Theta\pi/2\} =$
$2^{10}\pi/(3\times5\times7)$

·11 area $\bigcup[\mu(2\pi\, oaa)px|x, \Theta\pi/2] = 96\pi/5$

$n\varepsilon 0\cdots3 .\supset. p(n\pi/2)$ es cuspide . recta$[(o, a\mathrm{e}\nwarrow(in\pi/4))]$ es axi.
Puncto considerato in P·5 describe ellipsi, de semi-axi $4ma$, $4(1\text{-}m)ia$.
P·7 « Evoluta de asteroide es asteroide ».

Asteroide A.D.F.H.I.R., ⊂ G. ἀστεροειδή-ς. ⊂ aster + -o- + -ide.
Aster G. ἀστήρ, ἀστέρ-ι. ⊃ astr-o-nom-ia A.D.F.H.I.R. ...
⊂ E. ster ⊃ L. stella (⊂ ster-ula), A. star, D. ster-n.

§19 Epicycloide

✻ 1. Hp§1 . $m\varepsilon\, \mathrm{N}_1+1$. $p = [(o+m\mathrm{e}^{ix}a+\mathrm{e}^{mix}a)\,|x, \mathrm{q}]$. $x\varepsilon \mathrm{q} .\supset.$

·0 $k\varepsilon\, 0\cdots(m-2) .\supset.$
Sym recta$\{o, a\mathrm{e}\nwarrow[k\pi i/(m-1)]\}\; px = p[2k\pi/(m-1) - x]$

·1 $\mathrm{D}px = mia(\mathrm{e}^{ix}+\mathrm{e}^{mix}) = 2mia\,\mathrm{e}\nwarrow[(m+1)ix/2]\cos[(m-1)x/2]$

·11 $x\varepsilon\, \Theta\pi/(m-1) .\supset.$ mod $\mathrm{D}px = 2m\cos[(m-1)x/2]$

·2 $o+(m-1)\mathrm{e}^{ix}a\ \varepsilon$ rectaNpx . $o+(m+1)\mathrm{e}^{ix}a\ \varepsilon$ rectaTpx

·3 proj (rectaTpx) $o = o + (m+1)(\mathrm{e}^{ix}+\mathrm{e}^{mix})a/2$

·4 $x\varepsilon\, \Theta\pi/(m-1) .\supset.$ Arc$(p, \Theta x) = [4m/(m-1)]\sin[(m-1)x/2]$

·5 Arc$[p, \Theta\pi/(m-1)] = 4m/(m-1)$

·6 Arc$(p, 2\Theta\pi) = 8m$

·7 $x\varepsilon\, \Theta\pi/(m-1) .\supset.$ Area$(o^{-}p\text{‘}\Theta x) =$
$[m(m+1)/2]\{x+[/(m-1)]\sin[(m-1)x]\}$

·8 Area $o^{-}p\text{‘}\Theta\pi/(m-1) = m(m+1)\pi/[2(m-1)]$

·9 Area $o^{-}p\text{‘}2\Theta\pi = \pi m\,(m+1)$

·10 Cc$px = px + 2i\mathrm{D}px/(m+1)$
$= o+[(m-1)/(m+1)](ma\mathrm{e}^{ix}-\mathrm{e}^{mix}a)$
$= o+[(m-1)/(m+1)]\mathrm{e}\nwarrow[-\pi i/(m-1)]\ \{p[x+\pi/(m-1)]-o\}$

Curva generato per puncto de circumferentia de radio 1 que se evolve supra alio circumferentia fixo de radio $m-1$.

Pro $m=2$ « cardioide ».

Pro $m=3$ « caustica per reflexione de radios parallelo in circumferentia ».
Pro $m=-1$, « recta ». Pro $m=-3$ « asteroide ».

❋ 2. $\text{Hp}\S1 . h,k\varepsilon \text{q} . m\varepsilon \text{Q} . n\varepsilon m-\text{Q} .$
$p=[(o+h\text{e}^{mix}a+k\text{e}^{nix}a)|x, \text{q}] \mathbin{:} x\varepsilon\text{q} .\supset.$

·1 $\text{rectaN}\, px = [o+h(1+mz)\text{e}^{mix}a+k(1+nz)\text{e}^{nix}a]|z\text{‘q}$

·2 $o+k(1-n/m)\text{e}^{nix}a,\ o+h(1-m/n)\text{e}^{mix}a\ \varepsilon\ \text{rectaN}\, px$

·3 $h=k .\supset. px = o+2h\cos[(m-n)x/2]\text{e}\!\upharpoonright\![(m+n)ix/2]\, a$

·31 $h=-k .\supset. px = o+2h\sin[(m-n)x/2]\text{e}\!\upharpoonright\![(m+n)ix/2]\, ia$

·4 $mh=\pm\, nk .\supset. \text{Cc}px = px+2i\text{D}px/(m+n)$

$x\varepsilon\Theta\pi/(m-n) .\supset.$

·5 $\text{Arc}\,[p, \Theta\pi/(m-n)] =$
$/(m-n)\,\text{S}\{\surd[(mh+nk)^2\,\text{c}x^2+(mh-nk)^2\,\text{s}x^2]|x, \Theta\pi\}$

·6 $mh = nk .\supset. \text{Arc}(p, \Theta x) = 4hm\sin[(m-n)x/2]/(m-n)$

·61 $mh=-nk .\supset. \text{Arc}(p, \Theta x) =$
$4hm\{1-\cos[(m-n)x/2]\}/(m-n)$

·7 $\text{Area}(o\bar{\ }p\text{‘}\Theta x) =$
$(mh^2+nk^2)x/2 + [hk(m+n)\sin(mx-nx)]/(2m-2n)$

Pertine ad genere « epicycloidale ». Si $h=\pm k$, vocare « rosa », D rhodonee.
Si $mh=\pm nk$ es « epicycloide proprio », secundo P1.
Si $m=-n=1$, ellipsi. $m=1$, $n=2$, limace de Pascal.

§20 Limace de Pascal

$\text{Hp}\S1 . m\varepsilon\text{Q} . p=[o+(\text{c}x+m)\text{e}^{ix}a\,|\,x, \text{q}] . x\varepsilon\,\text{q} .\supset.$

·0 $p(2\pi+x) = px$

·1 $px = o + a/2 + (m\text{e}^{ix}+\text{e}^{2ix}/2)a$

·2 $[\text{Sym recta}(o,a)]px = p(2\pi-x)$

·3 $\text{D}px = (\text{e}^{ix}+m)\text{e}^{ix}ia . \text{D}^2px = -(2\text{e}^{ix}+m)\text{e}^{ix}a$

·4 $\text{rectaN}px = [o+(px-o)(1+z)+zi\text{s}x\,\text{e}^{ix}a|z\text{‘q}]$

·5 $o-i\text{s}x\text{e}^{ix}a\ \varepsilon\ \text{rectaN}px$

·6 $\text{Cc}px = o+[1+m^2+2m\,\text{c}x+m\text{e}^{ix}(\text{s}x)^2]a/(2+3m\text{c}x+m^2)$

·7 $\text{Area}\, o\bar{\ }p\text{‘}2\Theta\pi = \pi/2+\pi m^2$

·8 $\text{Arc}(p, 2\Theta\pi) = 4\text{S}\{\surd\{[(1+m)\text{s}x]^2+[(1-m)\text{c}x]^2\}|x, \Theta\pi/2\}$

Puncto p describe conchoide de circulo de centro puncto $o+a/2$, et de radio $/2$, relato ad o, et ad segmento m. Es podaria de circulo de centro $o+a$ et de radio m. Es epicycloide. Pro $m=1$, curva es cardioide.

Limace L.F., I. lumaca, A.L. limax, F. limaçon. $\subset$ lim(o) + -ace.
Nomen dato ad curva per Roberval, CR. a. 1708 p. 78.
limo H.I., F. limon. $\subset$ E. limo (p.161) $\supset$ A. loam lime, D. lehm leim.

§21 Cardioide

Hp§1 . $p=[o+(\mathrm{c}x+1)\mathrm{e}^{ix}a|x, \mathrm{q}]$. $x\varepsilon\mathrm{q}$.$\supset$.

·0 $p(2\pi+x)=px$

·1 $px=o+2[\mathrm{c}(x/2)]^2\mathrm{e}^{ix}a$

·2 $qx=o+2\mathrm{c}x\,\mathrm{e}^{ix}a$.$\supset$. $\mathrm{Proj}(\mathrm{recta}\mathrm{T}qx)o=p(2x)$

·3 $\mathrm{Cc}px=o+2a/3+2[\mathrm{s}(x/2)]^2\mathrm{e}^{ix}a/3$
$\qquad =o+2a/3-[p(x+\pi)-o]/3$

·4 $\mathrm{Area}\,o^{-}p\text{‘}\Theta x=[3x+\mathrm{s}x(4+\mathrm{c}x)]/4$

·5 $\mathrm{Area}\,o^{-}p\text{‘}2\Theta\pi=3\pi/2$

·6 $x\varepsilon\,\Theta\pi$.$\supset$. $\mathrm{Arc}(p,\Theta x)=4\mathrm{s}(x/2)$

·7 $\mathrm{Arc}(p, 2\Theta\pi)=8$

Puncto $p\pi=o$ es cuspide.
P·3 « Evoluta de cardioide es cardioide ».

Cardioide es « epicycloide », « sinus-spirale », « conchoide de circulo », « podaria de circulo », « caustica per reflexione in circulo », « inverso de parabola relato ad foco ».

Cardioide (Castiglioni, Phil.T. a.1741) $\subset$ cardia — -a + -oide.
cardia G καρδία || L corde. $\supset$ L cor, F coeur, H cor-azon, I cuore. || A heart, D herz, R serd-tse.

§22 Cissoide de Diocle

Hp§1 . $p=[o+(\mathrm{s}x)^2\mathrm{e}^{ix}a/\mathrm{c}x\,|x, -\pi/2^{-}\pi/2]$. $x\varepsilon -\pi/2^{-}\pi/2$.$\supset$.

·1 $[\mathrm{Sm}\,\mathrm{recta}(o,a)]px=p(-x)$

·2 $\mathrm{Lim}\{\mathrm{d}[px, \mathrm{recta}(o+a, ia)]\,|x, -\pi/2^{-}\pi/2, \pi/2\}=0$

·3 $\mathrm{D}px=[\;/(\mathrm{c}x)^2-\mathrm{e}^{2ix}]ia$ ·31 $\mathrm{D}^2px=2i\mathrm{D}px+2\mathrm{e}^{ix}a/(\mathrm{c}x)^3$

·4 $o+\mathrm{s}x\,\mathrm{e}^{ix}[1+/(\mathrm{c}x)^2]ia\ \varepsilon\,\mathrm{rectaN}px$

·5 $\mathrm{Cc}px = px + [3(\mathrm{c}x)^2+1]i\mathrm{D}px/[6(\mathrm{c}x)^2]$

·6 $\mathrm{Area}\,(o+a)^{-}p'(-\pi/2^{-}\pi/2) = 3\pi/4$ { WALLIS t.1 p.545 }

L.D.F.I. ***Cissoide***, A. cissoid, R. tsissoida.
G κισσο-ειδή-ς = hederi-forme. ⊂ cisso (=|| L. hedera) + -ide.

§23 Podaria

$o\varepsilon\mathrm{p}\,.\,k\varepsilon\,\mathrm{Intv}\,.\,p\varepsilon\,\mathrm{pF}k\,.\,q = [\mathrm{proj}\,(\mathrm{rectaT}px)\,o\,|x,k]\,.\,x\varepsilon k\,.$
$\mathrm{D}px\ \varepsilon\ \mathrm{v}{-}\iota 0\,.\,\mathrm{D}^2px\ \varepsilon\mathrm{v}\ .\supset.$

·1 $qx = px + [(o-px)\times\mathrm{UD}px]\mathrm{UD}px$

·2 $\mathrm{D}qx = \mathrm{Imag}(\mathrm{D}^2px/\mathrm{D}px)[qx - \mathrm{proj}\,(\mathrm{rectaN}px)\,o]$

·3 $(o+px)/2,\ \mathrm{proj}\,(\mathrm{rectaN}px)\,o,\ \mathrm{proj}\,(\mathrm{planN}px)\,o\ \varepsilon\ \mathrm{planN}qx$

Dato puncto fixo o, et puncto mobile p, functione de varabile in intervallo k, nos voca qx projectione super tangente ad p in x de o.
Puncto q describe « podaria, relato ad o, de linea p ».
Podarias de epicycloides es rosas.

Podaria, F. podaire (Laurent, non in vocabulario).
⊂ G. pod- (=|| L. pede) + L. -aria. (Vide caten-aria).
Aliquo scriptore non ama vocabulo hybrido « podaria » composito ex duo elemento de lingua differente, et ute forma minus diffuso in Mathematica sed multo in Musica:
L.H.I. ***Pedale***, F pédale, A.D. pedal, R. pedalı̣. ⊂ ped(e) + -ale.
L. pede, F. pied, H. pié, I. piede. ⊃ A.D.F.H.I.R. ped-ale ex-ped-itione
G. pod- ποδ-ί ⊃ A.D.F.H.I.R. anti-pode poly-po pod.agra.
E. pod, S pad, A foot, D fuss. (R. pada-tı̣ = cade).

§24 Conchoide

$o\varepsilon\mathrm{p}\,.\,k\varepsilon\,\mathrm{Intv}\,.\,p\varepsilon\,(\mathrm{p}{-}\iota o)\mathrm{F}k\,.\,l\varepsilon\mathrm{Q}\,.\,q = \{[px + l\,\mathrm{U}(px-o)]\,|x,k\}$
$x\varepsilon k\ .\supset.\ \mathrm{plan}[o,\mathrm{I}(px-o)] \frown \mathrm{planN}\,px \supset \mathrm{planN}\,qx$
{ DESCARTES a.1637 Œuvres, t.6, p.323 }

q describe « conchoide de p, cum « polo » in o, et « regula » $p'k$.

Conchoide L.scientifico, A conchoid, F conchoïde, I concoide.
⊂ concha — -a + -oide.
concha L ⊂ G κόγχη, H concha, F conque, coqu-ille, I conch-iglia.

§25 Conchoide de Nicomede

Hp§1 . $m\varepsilon$Q . $p = \{[o+(/\mathrm{c}x+m)\mathrm{e}^{ix}a] | x, (-\pi/2)\overline{}(\pi/2)\}$.
$x\varepsilon$ Variab$p$.⊃:

·0 [Sym recta(o,a)]$px = p(-x)$

·1 Dpx = $[/(\mathrm{c}x)^2+m\mathrm{e}^{ix}]ia$

·2 $o+\mathrm{e}^{ix}\mathrm{s}x/(\mathrm{c}x)^2ia$ ε rectaNpx

·3 lim$\{$d[px, recta$(o+a, ia)$]$|x$, Variab$p$, $\pm\pi/2\}=0$

·4 $x\varepsilon$ $\Theta\pi/2$.⊃.
Area $o\overline{}p$'Θx = t $x/2+m\log[\mathrm{t}(\pi/4+x/2)]+m^2x/2$

Puncto p describe conchoide de recta $(o+a, ia)$, relato ad o.

§26 Helice

$o\varepsilon$p . $a,b,c\varepsilon$v . $a^2=b^2=c^2=1$. $a\times b = b\times c = c\times a = 0$.
$r,h\varepsilon$Q . $u = b/a$. $p = [(o+r\mathrm{e}^{ux}a+hxc)|x$, q] . $x\varepsilon$q .⊃.

·1 $o+r\mathrm{e}^{ux}a-rxu\mathrm{e}^{ux}a$ ε rectaT px

·2 $o+hxc$ ε rectaN px

·3 Cc$px = px - \mathrm{e}^{ux}(h^2+r^2)a/r = o-\mathrm{e}^{ux}h^2a/r+hxc$

·4 $y\varepsilon$ x+Q .⊃. Arc$(p, x\overline{}y) = (y-x)\surd(r^2+h^2)$

r vocare « radio de helice », $2\pi h$ « passu », h « passu reducto ».

L. ***Helice***, G. ἕλικ-ι, F.H. hélice, L.A.D. helix, I. elice elica.

§27 Inversione

$k\varepsilon$ Cls'q . k⊃ δk . $p\varepsilon$ pFk . $o\varepsilon$ p-p'k .
$q = \{[o+(px-o)/(px-o)^2]|x, k\}$. $x\varepsilon k$.⊃:

·1 D$qx = \{$D$px-2$[U$(px-o)\times$Dpx]U$(px-o)\}/(px-o)^2$

·2 Dpx ε v-ι0 .⊃. (UDpx+UDqx)$\times(px-o) = 0$

·3 Ccpx ε p-ιo .⊃. Ccqx ε recta$(o$, Cc$px)$

Nota. Vide tractatione diffuso de curvas præcedente, et de alio, in *G. Pagliero, Applicationes de Calculo infinitesimale*, Torino, Paravia a.1907.

§28 γ

·0 $\gamma = \lim\{\Sigma/(1\cdots n) - \log n\}\,|n$ Df

·1 $n \varepsilon \mathrm{N}_1 \,.\supset.\, \Sigma/(1\cdots n) - \log n > \gamma > \Sigma/[1\cdots(n+1)] - \log n$

·2 $\gamma = 0\cdot$

57721 56649 01532 86060 65120 90082 40243 10421 59335 93992
35988 05767 23488 48677 26777 66467 09369 47063 29174 67495
14631 44724 98070 82480 96050 40144 86542 83622 41739 97644
92353 62535 00333 74293 73377 37673 94279 25952 58247 09491
60087 35203 94816 56708 53233 15177 66115 28621 19950 15079
84793 74508 569 ...

Pro n vergente ad ∞, summa de primos n termine de serie harmonico (pag. 223 P23·1) verge ad ∞, ut $\log n$. Differentia verge ad limite finito, que vocare γ.

Numero γ habe in analysi maximo importantia, post numeros π et e. Vocare « constante de Eulero ».

Euler, PetrC., a.1734-35 t.7 p.156 indica illo per C et per O; Mascheroni per A. Signo γ, (que non occurre in Euler, et ne in Mascheroni) es adoptato per *Encyclopädie*, et plure Auctore.

Euler, ibid. calcula $\mathrm{E}(10^6\gamma)$, post $\mathrm{E}(10^{10}\gamma)$ in a.1744 CorrM. t.1 p.283; et $\mathrm{E}(10^{15}\gamma)$ in PetrNC. a.1769 t.14 I p.154.

Mascheroni, a.1790	calcula	E(10↾19 γ)	
Gauss, a.1812 *Werke*, t.3 p.154	»	»	23 »
Nicolai, » »	»	»	45 »
Glaisher, LondonP. a.1871 t.19 p.54	»	»	100 »
Adams, » » a.1878 t.28 p.88	»	»	263 »

·3 $\gamma = \Sigma\mathrm{N}_1^{-2}/2 - \Sigma\mathrm{N}_1^{-3}/3 + \ldots = \Sigma\{(-1)^n(\Sigma\mathrm{N}_1^{-n}/n)\,|n, \mathrm{N}_1+1\}$

·4 $\gamma = \Sigma\{\,\Sigma(\mathrm{N}_1+1)^{-n}(n-1)/n\,|n, \mathrm{N}_1+1\}$

{ ·3·4 Euler PetrNC. a.1769 p.154 }

·5 $1-\gamma = \Sigma\{\,\Sigma(\mathrm{N}_1+1)^{-n}/n\,|n, \mathrm{N}_1+1\}$

{ Euler PetrA. a.1781 t.5 II p.45 }

·6 $\mathrm{e}\upharpoonleft\gamma = \Pi[(\mathrm{e}\upharpoonleft/n)/(1+/n)\,|n, \mathrm{N}_1]$

·7 $\gamma = -\mathrm{S}(\mathrm{e}^{-x}\log x\,|x, \mathrm{Q})$

·8 $a\varepsilon\mathrm{N}_1 \,.\, n\varepsilon\mathrm{N}_1+1 \,.\supset.\, \Sigma/(1\cdots a)\ \varepsilon\ \log a + \gamma + /(2a) +$
$\Sigma[(-1)^r\mathrm{B}_r/(2ra^{2r})\,|r, 1\cdots(n-1)] + \theta(-1)^n\mathrm{B}_n/(2na^{2n})$

{ Euler PetrNC. a.1769 t.14 I p.153 }

§29 Complemento super Numeros complexo.

✻ 1. E ɯ

·0 $n\varepsilon N_1 .\supset. Cn = Cxn$ Def

·1 » $. x\varepsilon Cn . r\varepsilon 1\cdots n .\supset. E_r x = x_r$ Def

·1 » $. u,v\varepsilon$ Cls'$Cn . u\supset v . r\varepsilon 1\cdots n .\supset. E_r u \supset E_r v$

·2 » $. u\varepsilon$ Cls'$Cn . \lambda u = u . r\varepsilon 1\cdots n .\supset. \lambda E_r{}^{\backprime}u = E_r u$

Nos abbrevia in Cn, « campo ad n dimensione », numero complexo de ordine n.

Dato numero naturale n, et complexo x de ordine n, (vide p. 144), si r indica uno ex numeros inter 1 et n, tunc $E_r x$, lege : « elemento vel coordinata de loco r de x », indica ipso x_r. Signo E es signo de præfunctione (p. 73). Tunc, si u es classe de complexos, $E_r{}^{\backprime}u$ indica coordinatas de loco r de complexos u (p. 77).

Si classe u continere in v, et coordinatas de loco r de u continere in coordinatas de v. Si classe u es clauso, et classe de suo coordinatas de loco r, es clauso.

·3 $n\varepsilon N_1 . x\varepsilon Cn .\supset. mx = \text{mod}x$ Def

·4 $n\varepsilon N_1 .\supset. ɯ_n = Cn \cap x\ni(mx < 1)$ Def

Nos abbrevia symbolo mod in m.

$ɯ_n$ indica complexos de ordine n, de modulo minore de 1, vel « sphæra ad n dimensione, de centro puncto zero, et de radio 1 ».

·5 $n\varepsilon N_1 . u\varepsilon$ Cls'$Cn .\supset.$

$\lambda u = Cn \cap x\ni[r\varepsilon Q .\supset_r. \exists\, u \cap (x+rɯ_n)]$ Defp

Classe limite λ de dato classe u, jam definito supra (pag. 139 pro numeros reale, citato in pag. 145 P4 pro complexos), pote es expresso per novo symbolo ɯ: λu es classe de complexos de ordine n, et x tale que, pro omni radio r, semper existe elemento de classe u, in sphæra de centro x et de radio r.

✻ 2.

·0 $n \varepsilon \mathrm{N}_1 \,.\, f \varepsilon \,\mathrm{Cls'C}n\ \mathrm{F}\ \mathrm{N}_0 : r \varepsilon \mathrm{N}_0 \,.\supset_r.\ \exists fr \,.\, \mathrm{l'm}fr\ \varepsilon \mathrm{Q} \,.\, \lambda fr = fr \,.\, f(r+1) \supset fr :\supset.\ \iota\, \mathrm{C}n \frown x \ni \{ r \varepsilon 1 \cdots n \,.\supset_r\, x_r = \mathrm{l}_{,}[\mathrm{l'}\ \mathrm{E}_r`\{fp \frown z \ni [s \varepsilon 1 \cdots (r-1) \,.\supset_s.\ z_s = x_s]\} \mid p`\mathrm{N}_0]\}\ \varepsilon \bigcap f`\mathrm{N}_0$

·1 $\mathrm{Hyp}{\cdot}0 \,.\supset.\ \exists \bigcap f`\mathrm{N}_0$

Dato numero naturale n, si f es classe de complexos de ordine n, functione de numeros 0, 1, 2, ...; vel si $f0$, $f1$, $f2$, ... es successione de classes de complexos de ordine n; et si pro omni valore de indice r, semper:

classe fr existe, vel non es vacuo,

es limitato, vel limite supero de modulos de fr es finito,

es clauso, vel classe limite de fr coincide cum se ipso,

et classe $f(r+1)$ sequente fr es parte de fr,

tunc P·1 dice que existe elemento commune ad omni classe fr, et in modo plus præciso, P·0 dice que uno elemento commune x es ita determinato:

Sume numero p, considera classe fp, suo coordinatas primo $\mathrm{E}_1`fp$, suo limite supero $\mathrm{l'E}_1`fp$, varia p in campo de numeros, et sume limite infero de limites supero; isto es x_1, primo coordinata de x.

Sume in fp individuos que habe ut primo coordinata x_1, et opera in modo analogo super coordinatas secundo; resulta secundo coordinata de x. Et ita continua. Isto elemento x es commune ad omni classe fp pro omni numero p.

Dem.

Si $n=1$, theorema sume forma:

$f \varepsilon\, \mathrm{Cls'q\ F\ N}_0 : r \varepsilon \mathrm{N}_0 \,.\supset_r .\ \exists fr \,.\, \mathrm{l'm}fr\ \varepsilon\, \mathrm{Q} \,.\, \lambda fr = fr \,.\, f(r+1) \supset fr :\supset.$
$\mathrm{l}_{,}[(\mathrm{l'}fs) \mid s`\mathrm{N}_0]\ \varepsilon \bigcap f`\mathrm{N}_0$

Isto theorema es simile ad theorema super existentia de classe limes (Lm) de omni successione de quantitates, exposito in pag. 213, Prop. 2·1·4; et uno es reductibile ad altero. Demonstratione que seque et propositiones (1)(2)(3)(4) es analogo ad Dem. de pag. 213, et ad Prop. correspondentes.

$\mathrm{Hp} \,.\, m,s \varepsilon \mathrm{N}_0 \,.\supset.\ f(m+s) \supset fm \,.\supset.\ \mathrm{l'}f(m+s) \leqq \mathrm{l'}fm$ (1)

In hypothesi dato si m et s numeros, tunc $f(m+s)$ es parte de fm, nam primo classe seque secundo de s loco.

Ergo, limite supero de primo classe es minore aut æquale, de limite supero de secundo. Resulta ex applicatione de operatione l′, secundo regula pag. 116 Prop. 11·3.

$m\varepsilon \mathrm{N}_0 \,.\, (1) \,.\supset.\, \mathrm{l}_,[\mathrm{l}'fs \,|s`\mathrm{N}_0] = \mathrm{l}_,[\mathrm{l}'f(m+s) \,|r`\mathrm{N}_0]$ (2)

Ergo limite infero de valores sumpto per $\mathrm{l}'fs$, ubi s sume omni valore numerico, æqua limite infero de valores sumpto per fs, ubi s varia de m in post.

$m\varepsilon \mathrm{N}_0 \,.\, (2) \,.\supset.\, \mathrm{l}_,[\mathrm{l}'fs \,|s`\mathrm{N}_0] \,\varepsilon\, \lambda\, \mathrm{l}'[f(m+s) \,|s`\mathrm{N}_0]$ (2′)

Ergo limite infero dicto, es valore limite de limites supero de classes $f(m+s)$, ubi varia s; nam limite infero ($\mathrm{l}_,$) de classe es valore limite (λ) de classe, secundo pag. 140 Prop. 1·5.

$m{,}s\varepsilon \mathrm{N}_0 \,.\supset.\, \mathrm{l}'f(m+s) \,\varepsilon\, \lambda f(m+s) \,.\, \lambda f(m+s) = f(m+s) \,.\, f(m+s) \supset fm \,.\supset.$
$\mathrm{l}'f(m+s) \,\varepsilon\, fm$ (3)

Dato duo numero m et s, tunc limite supero de classe $f(m+s)$ es valore limite λ de classe $f(m+s)$; ce classe es clauso, et suo classe limite coincide cum $f(m+s)$, que continere in fm. Ergo limite supero de $f(m+s)$ es uno ex valores de fm.

$m\varepsilon \mathrm{N}_0 \,.\supset.\, \mathrm{l}'f(m+s) \,|s`\mathrm{N}_0 \supset fm$ (3′)

Ergo, dato m, classe de limites supero de classes $f(m+s)$, ubi varia s, continere in fm.

$m\varepsilon \mathrm{N}_0 \,.\, (2') \,.\, (3') \,.\supset.\, \mathrm{l}_,[\mathrm{l}'fs \,|s`\mathrm{N}_0] \,\varepsilon\, \lambda fm \,.\, \lambda fm = fm \,.\supset.\, \mathrm{l}_,[\mathrm{l}'fs \,|s`\mathrm{N}_0] \,\varepsilon\, fm$ (4)

Nunc, per Prop. (2′), limite infero considerato in theorema es limite λ de valore considerato in (3′), que continere in fm; ergo illo limite infero es elemento de omni classe fm; quod es theorema demonstrando.

Dem. pro $n=2$.

$g = \mathrm{E}_1`fr \,|r \,.\supset:\, g\varepsilon\, \mathrm{Cls}`\mathrm{q}\, \mathrm{F}\, \mathrm{N}_0 \,:\, r\varepsilon \mathrm{N}_0 \,.\supset_r .\, \exists\, gr \,.\, \mathrm{l}'\, \mathrm{m}gr \,\varepsilon \mathrm{Q} \,.\, \lambda gr = gr \,.$
$g(r+1) \supset gr \,:\supset.\, x_1 \,\varepsilon \bigcap \mathrm{E}_1`fs \,|s`\mathrm{N}_0$

In vero, si nos voca g classe de primo coordinatas de fr, ubi varia r, tunc g es successione de classe de quantitates; et pro omni indice r, ce classe existe, es limitato, clauso, et omni classe continere in præcedentes. Ergo, per theorema in casu $n=1$, quantitate vocato x_1 es elemento commune ad omni gr.

$h = \mathrm{E}_2`fr \cap z\ni(z_1 = x_1) \,|r \,.\supset:\, r\varepsilon \mathrm{N}_0 \,.\supset.\, \exists hr \,.\, \mathrm{l}'\mathrm{m}hr \,\varepsilon \mathrm{Q} \,.\, \lambda hr = hr \,.\, h(r+1) \supset$
$hr \,:\supset.\, x_2 \,\varepsilon \bigcap \mathrm{E}_2`\, fr \cap z\ni(z_1 = x_1) \,|r`\mathrm{N}_0$

Nunc, si nos voca h classe de secundo coordinatas de elementos de fr, que habe ut primo coordinata x_1, h es novo successione de classes de quantitates; omni classe es non vacuo, limitato, clauso, et continere in præcedentes. Ergo quantitate vocato x_2 es commune ad omni hr.

$r\varepsilon \mathrm{N}_0 \,.\supset.\, \exists\, fr \cap z\ni(z_1 = x_1) \cap z\ni(z_2 = x_2) \,.\supset.\, \exists\, fr \cap z\ni(z = x) \,.\supset.\, x\varepsilon\, fr$

Si nos elimina signos $\bigcap$, E, resulta: si r es numero, tunc existe elemento in fr que habe ut primo coordinata x_1, et ut secundo x_2, vel que es æquale ad x, id es, x es elemento commune ad omni fr.

In modo simile, pro omni valore de n.

Nota.

Theorema P2·1 es ab aliquo auctore demostrato ut seque:

Divide toto spatio ad n dimensione in cubos de latere 1.

Tunc existe aliquo cubo, que habe proprietate que, pro omni valore de r, aliquo elemento de classe fr es commune ad isto cubo. Divide isto cubo in novo cubos de latere 1/2, 1/4, 1/8, ...; semper existe aliquo cubo partiale, que habe idem proprietate. Resulta successione de cubos, de latere vergente ad 0; ergo cubo verge ad puncto limite, que satisfac thesi de theorema.

Isto typo de demonstratione occurre in Bolzano, a. 1817, Cauchy, *Cours d'Analyse*, a.1821 note 3 (Œuvres s. 2, t. 3 p. 378), et sæpe in Weierstrass. Vide MA. t. 23 p. 455.

In isto demonstratione occurre elige, in numero infinito de vice, cubo inter plure cubo; quod non lice, si nos non da lege generale de electione. Vide infra, §30 P4·1. Ergo nos da lege de electione, et nos scribe expressione de puncto limite, in Prop. 2·1; id es pro demonstratione de existentia de classe, nos da expressione de uno individuo que pertine ad classe.

·2 Hyp·1 $.\supset. \lambda \bigcap f`N_0 = \bigcap f`N_0$

Classe commune ad omni classe de successione $f0$ $f1$ $f2$ ... considerato in Prop. ·1, es clauso.

Dem. $n \varepsilon N_0 .\supset. \bigcap f`N_0 \supset fn .\supset. \lambda \bigcap f`N_0 \supset \lambda fn \supset fn$ (1)
$(1) .\supset. \lambda \bigcap f`N_0 \supset \bigcap f`N_0 .\supset.$ P

·21 $n \varepsilon N_1 . f \varepsilon$ Cls'Cn F $N_0 : r \varepsilon N_0 .\supset_r. \exists fr . \text{l'm} fr \, \varepsilon Q .$
$\lambda f(r+1) \supset fr :\supset. \exists \bigcap f`N_0$

Dem.

P·1 $.\supset. \exists \bigcap \lambda f`N_0$ (1)
$r \varepsilon N_0 . \S\lambda$ (pag.139) P·1 $.\supset. f(r+1) \supset \lambda f(r+1) \supset fr$ (2)
$(2) .\supset. \bigcap f(r+1) | r`N_0 \supset \bigcap \lambda f(r+1) | r`N_0 \supset \bigcap f`N_0 .$
$\bigcap f(r+1) | r`N_0 = \bigcap f`N_0 .\supset. \bigcap \lambda f`N_0 = \bigcap f`N_0$ (3)
(1) . (3) $.\supset.$ P

Theorema P·1 ad nos occurre (§30 P5·1·2) sub forma:

·3 $n \varepsilon N_1 . f \varepsilon$ Cls'Cn FQ : $h \varepsilon Q .\supset_h. \exists fh . \text{l'm} fh \, \varepsilon Q .$
$h,k \varepsilon Q . h < k .\supset_{h,k}. \lambda fh \supset fk :\supset. \exists \bigcap f`Q$

Si f es classe de complexos de ordine n; functione de quantitates positivo; et si pro omni valore de quantitate positivo h, classe fh es non vacuo et limitato, et si pro omni $h<k$, classe limite de fh continere in fk, tunc existe elemento commune ad classes fh, pro omni valore de h.

Dem. $g = [f(1/n) | n, N_1] .\supset: g \varepsilon \text{Cls'C}n \, \text{F} \, N_1 : r \varepsilon N_1 .\supset. \exists gr . \text{l'm} \, gr \, \varepsilon Q .$
$\lambda g(r+1) \supset gr : h \varepsilon \theta .\supset. g(\text{E} \, 1/h + 1) \supset fh \supset g(\text{E} \, 1/h) :$
$\text{P2·21} :\supset. \exists \bigcap g`N_1 . \bigcap g`N_1 = \bigcap f`Q .\supset. \text{P}$

Tribue ad h valores 1, 1/2, 1/3, ..., et es in casu de P·1.

Theorema P2·1, fundamentale in plure theoria de Analysi, pote sume vario forma:

·4 $n \varepsilon N_1 . f \varepsilon \text{Cls'C}n \, \text{F} \, N_0 : r \varepsilon N_0 .\supset_r. \exists fr . \text{l'm} \, fr \, \varepsilon Q .$
$f(r+1) \supset fr :\supset. \exists \bigcap (\lambda fs | s`N_0)$

Si nos tace conditione que classe fr es clauso, tunc existe elemento commune ad classes limite de classes dato.

·5 $n \varepsilon N_1 . f \varepsilon \text{Cls'C}n \, \text{F} \, N_0 : r \varepsilon N_0 .\supset_r. \exists fr . f(r+1) \supset fr :\supset.$
$\exists \bigcap (\Lambda fs | s, N_0)$

Nos supprime hypothesi que classes de successione es limitato; suffice scribe Λ (classe limite generale), in loco de λ (classe limite finito). Vide pag. 139.

·6 $n \varepsilon N_1 . f \varepsilon (\text{Cls'C}n - \iota \wedge) \, \text{F} \, N_0 .\supset. \exists \bigcap [\Lambda \bigcup f`(s+N_0) | s`N_0]$

Nos elimina hypothesi que omni classe contine sequentes; tunc nos considera signo $\bigcup$ de summa, in sensu logico, de classe que seque classe de loco s.

·7 $\text{Hyp} \; ·6 .\supset. \text{Lm}f = \bigcap [\Lambda \bigcup f`(s+N_0) | s`N_0]$ Def

Classe que figura in secundo membro de P·5 vocare « limes » (Lm) de successione f. Si omni classe de successione consta ex uno solo individuo, resulta ut casu particulare Def. de p. 211 P 1·0, pro quantitates reale; definitione generale jam occurre in Geometria pag. 237 P 71·4, et es adoptato in pag. 331, pro definitione de figura tangente. Tunc Prop. sume forma:

·8 $\text{Hyp} \; ·6 .\supset. \exists \, \text{Lm} f$

analogo ad p. 213 Prop. 2·4, pag. 231 Prop. 40·4, pag. 331 Prop. 68·3.

·9 $n \varepsilon N_1 . w \varepsilon \text{Cls'Cls'C}n : u \cup v \, \varepsilon w .=. u \varepsilon w .\cup. v \varepsilon w : u \varepsilon w .$
$\text{l'm} \, u \, \varepsilon Q :\supset. \exists \lambda u \frown x \ni [r \varepsilon Q .\supset_r. x + r\text{m}_n \, \varepsilon w]$

Dem. $f = \{\bigcup [(\text{n} + \Theta \, \text{F} \, 1 \cdots n)/2^s \frown w \frown z \ni (\exists z \frown v)] | s, N_0\} .\supset. \exists \bigcap f`N_0$ (1)
$\text{Hp}(1) . x \varepsilon \bigcap f`N_0 . r \varepsilon Q .\supset. x \varepsilon \lambda u . x + r\text{m}_n \, \varepsilon w$

Es dato numero naturale n. Nos considera w, que es classe de classes de complexos de ordine n; id es, w es proprietate de classes de complexos (non es proprietate de individuos de classe).

Nos dice que ce proprietate es distributivo, quando, si summa in sensu logico de duo classe u et v habe proprietate w, tunc uno ex classes u et v habe proprietate w; et viceversa, si uno ex classes u et v habe proprietate w, tunc summa de duo classe habe idem proprietate.

Nunc, si aliquo classe u habe proprietate w et es limitato, tunc existe aliquo individuo in classe limite de u, et x tale que, si nos fixa ad arbitrio radio r, sphæra de centro x et de radio r habe proprietate w.

In vero, figura composito ex cubos de latere $1/2^s$, que es w, et que habe aliquo puncto commune cum u, ubi varia numero integro s, habe proprietates dicto in P·0; ergo existe elemento commune ad omni figura, pro omni s. Nunc, si nos voca x elemento commune, illo satisfac thesi.

Præsente forma de theorema P2·1 occurre in G. Cantor, *Ueber unendliche Punktmannichfaltigkeiten*, Math. Ann. t. 13 a. 1884 p. 454.

Exemplo. Propositione « classe u contine infinito individuos » es proprietate de classe u, distributivo; ergo si u es classe cum infinito individuos, et es limitato, tunc existe elemento x, prope u, tale que omni sphæra de centro x contine infinito individuos de classe u. Vide p. 141 Prop. 5·1.

Alio applicationes de proprietate distributivo es in meo libro a. 1893 t. 2 p. 48.

❋ 3. Dg (derivata generale)

·0 $n\varepsilon N_1 \,.\, u\varepsilon \text{Cls'q} \,.\, f\varepsilon \text{Cn F } u \,.\, x\varepsilon u\frown\delta u \,.\supset.$
$\text{Dg} f x = \text{Lm}[\text{D}(f;x,y) \,|y,u,x]$ Def

Dato functione f, complexo de variabile reale, in hypothesi de definitione de derivata (pag. 275 P1·2, pag. 284 P15), tunc $\text{Dg} fx$, lege « derivata generale de f, pro valore x », indica classe limes (Lm) de ratione incrementale de f, pro valores x et y, dum varia y, in campo ubi es definito f et verge ad x.

Classe limes (Lm) es definito in pag. 230 P40, pag. 235 P51, et supra p. 413 P2·7.

Ergo omni functione complexo de variabile reale, pro valore x in campo de variabilitate de functione, et in suo campo derivato, habe semper derivata generale, que es classe non nullo.

Operatione D es ligato ad Dg, ut lim ad Lm (p. 214 P3·0):

·1 Hyp·0 $.\supset.$ $\text{D}fx = \iota\, \text{Dg} fx$

id es, si classe $\text{Dg}fx$ consta ex uno solo individuo, illo es derivata de fx.

Regulas de derivatione subsiste cum pauco modificatione, si nos substitue Dg ad D, vel Lm ad lim.

·2 Hyp·0 . $g \varepsilon$ Cn F u . l'm Dg fx, l'm Dg gx εQ .⊃.
Dg [$(fx+gx)$ |x, u]x ⊃ Dg fx + Dg gx

Si derivatas generale de duo functione es limitato, tunc derivata generale de summa continere in summa de derivatas. Vide §D p.279 P5·1, §Lm p.217 P6·2.

·3 Hyp·0 . $g \varepsilon$ Cn F u . D gx εq .⊃.
Dg [$(fx \times gx)$ |x, u]x = $fx \times$ Dgx + $gx \times$ Dg fx

Si uno factore habe derivata in sensu proprio, tunc derivata generale de producto es dato per regula jam explicato in p.280.

·4 Hyp·0 .⊃. Dg mod fx ⊃ $\pm\Theta$ mod Dg fx

·5 $u \varepsilon$ Cls'q . u⊃ δu . $n \varepsilon N_1$. $f \varepsilon$ Cn F u : $x \varepsilon u$.⊃$_x$. Dg fx ⊃ q
:⊃. $f \varepsilon$ cont

Si u es classe de quantitates, condensato (p.160), vel que continere in derivata, et si f es complexo de ordine n, functione definito in campo u; et si pro omni valore x in campo u, derivata generale de fx es classe de quantitates (finito), tunc functione f es continuo. Vide pag. 279 P3·3.

·6 $a,b \varepsilon$q . a -= b . $n \varepsilon N_1$. $f \varepsilon$ Cn F a⌐b .⊃.
$(fb-fa)/(b-a)$ ε Medio ∪ Dg f'a⌐b

Theorema de valore medio (pag. 288 pro functione reale, pag. 312 pro functione complexo) sume forma:
Si f es complexo de ordine n, functione definito de variabile ab a ad b, tunc suo ratione incrementale, pro valores a et b, es medio inter classe de classes de derivata generale de f, in intervallo dato.

§ 30.

Æquationes differentiale.

❋ 1. Theorema

$$n \varepsilon \mathrm{N}_1 \,.\, a \varepsilon \mathrm{C}n \,.\, x \varepsilon \mathrm{q} \,.\, r,s \varepsilon \mathrm{Q} \,.\, f \varepsilon \mathrm{C}n\, \mathrm{F}\,(a+\lambda r \text{ɯ}_n \,\vdots\, x+\lambda s \text{ɯ}_1)\mathrm{cont} \,.$$
$$l = \mathrm{l'm} f`(a+\lambda r \text{ɯ}_n \,\vdots\, x+\lambda s \text{ɯ}_1) \,.\, t = \min(\iota s \cup \iota r/l) \;.\supset.$$
$$\exists\; [\mathrm{C}n\, \mathrm{F}\,(x+t\text{ɯ}_1)] \frown g \ni \{ gx = a \,.\, \mathrm{D}g = [f(gz, z) \,|\, z, x+t\text{ɯ}_1] \}$$

n indica numero naturale; a es complexo de ordine n,; x es quantitate reale; r, s es quantitate positivo; f es complexo de ordine n, functione continuo de duo variabile, uno in interno et in superficie de sphæra ad n dimensione de centro a et de radio r, altero in intervallo de $x-s$ ad $x+s$, extremos incluso. Nos voca l limite supero de modulos de valores de f, ubi variabiles varia in campo dato (causa continuitate de f in campo clauso, l es finito); et voca t minimo de duo quantitate s et r/l.

Tunc existe complexo de ordine n, functione definito in intervallo de $x-t$ ad $x+t$, et g tale que, pro variabile $= x$, sume valore a, et que pro omni z in intervallo dato satisfac æquatione:

$$\mathrm{D}gz = f(gz, z).$$

Si nos voca $g_1 z$, $g_2 z$, ... elementos de gz, et f_1, f_2 ... elementos de f, tunc æquatione præcedente vale systema de n « æquatione differentiale »:

$$\begin{aligned} \mathrm{D}g_1 z &= f_1(g_1 z, g_2 z, \ldots g_n z, z) \\ \mathrm{D}g_2 z &= f_2(\quad \text{»} \quad\quad \text{»} \quad) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \mathrm{D}g_n z &= f_n(\quad \text{»} \quad\quad \text{»} \quad) \end{aligned}$$

Complexo g vocare in aliquo casu « puncto mobile »: variabile z vocare « tempore ». Theorema dice que existe puncto mobile functione de tempore, que pro tempore x habe positione a, et que habe velocitate $\mathrm{D}gz$ dato in functione de gz et de z

Demonstratione.

In vero, nos nosce positione $gx = a$ de puncto mobile; ergo es noto suo velocitate $\mathrm{D}gx = f(gx,x)$; vel es noto puncto successivo de gx, vel puncto ad distantia infinitesimo. Noto novo puncto, es determinato suo velocitate, vel tertio puncto successivo ad secundo; et ita continua.

Isto demoustratione, vel explicatione, plus aut minus diffuso, occurre in libros de Analysi ab Euler, *Institutiones calculi integralis*, a. 1768, t. 1 p. 493, usque ad Lacroix a. 1810 p. 2.

Sed in illo tempore, vocabulo «infinitesimo» non es definito. Cauchy defini «infinitesimo» ut «quantitate variabile que verge ad 0»; et isto definitione es secuto hodie in generale (et si aliquo puncto mane obscuro, et existe alio interpretatione). Ergo Cauchy a. 1840, *Exercices* p. 327, evolve demonstratione præcedente ut seque: Integra per approximatione æquatione dato, et transi ad limite. Analysi de demonstratione pone in evidentia plure propositione intermedio, non exposito in modo explicito ab auctores. Isto analysi, vel demonstratione completo, es multo longo.

Nos indica per B solutione approximato de æquatione dato, vel solutione cum errore minore de quantitate positivo h; et in P2-3 nos stude proprietates de B. Nos voca A limite de B, quando errore h verge ad 0; et in Prop. 4-5 nos stude proprietates de A; P7 da expressione explicito de functione que satisfac æquatione dato.

Nam demonstratione de propositione existentiale «existe aliquo a» semper consta ex constructione, per symbolos de Analysi et de Logica (vel per lingua commune), de aliquo elemento x in ciasse a, secundo regula pag. 12 Prop. 2·1.

✻ 2. B

Hyp P1 .⊃:

·0 $y \varepsilon (x + \mathrm{m}_1 s) - \iota x \,.\, k \varepsilon \mathrm{Q} \,.\supset.$

$\mathrm{B}(a,x,y,k) = \mathrm{C}n \frown b \ni \exists\, (a + r\mathrm{m}_n)\, \mathrm{F}\, (x \overline{\;\;} y) \frown g \ni [gx = a \,.\, gy = b :$
$z \varepsilon x \overline{\;\;} y \,.\supset_z.\, \mathrm{Dg}\, gz \supset f(gz, z) + k\mathrm{m}_n]$ Def

In hypothesi de theorema supra enunciato, si y es valore

in intervallo de $x-s$ ad $x+s$, differente de x, et si k es quantitate positivo, tunc signo $\mathrm{B}(a,x,y,k)$, lege « valores de solutiones de æquatione dato, que i ex a, de x ad y, cum errore minore de k » indica omni complexo b tale que existe complexo g sumente valores in sphæra de centro a et radio r, functione definito de variabile in intervallo $x \urcorner y$, que pro valores x et y de variabile sume valores a et b, et que pro omni valore z in intervallo $x \urcorner y$, satisfac æquatione differentiale dato cum errore minore de k; id es, derivata generale de gz continere in $f(gz,z)$, plus complexo de modulo minore de k.

·1 $k \varepsilon \mathrm{Q} . \supset . \mathrm{B}(a,x,x,k) = \iota a$ Def

Si $y=x$, classe B es reducto ad solo elemento a.

·2 Hyp·0 . $b \varepsilon \mathrm{B}(a,x,y,k) . \supset . a \varepsilon \mathrm{B}(b,y,x,k)$

·3 » . $z \varepsilon x \urcorner y . c \varepsilon \mathrm{B}(a,x,z,k) . b \varepsilon \mathrm{B}(c,z,y,k) . \supset . b \varepsilon \mathrm{B}(a,x,y,k)$

·4 » . » . $b \varepsilon \mathrm{B}(a,x,y,k) . \supset . \exists \mathrm{B}(a,x,z,k) \frown \mathrm{B}(b,y,z,k)$

·5 » . $h \varepsilon \mathrm{Q} . \supset . \mathrm{B}(a,x,y,k) \supset \mathrm{B}(a,x,y,k+h)$

✻ 3. Hyp P1 $. \supset :$

·1 $y \varepsilon x + \mathrm{u}_1 t . k \varepsilon \mathrm{Q} . \supset . \exists \mathrm{B}(a,x,y,k)$

Si y es valore in intervallo ab $x-t$ ad $x+t$, et k es quantitate positivo, tunc classe $\mathrm{B}(a,x,y,k)$ non es vacuo, id es existe functione, dato in intervallo $x \urcorner y$, que pro valore x de variabile sume valore a, et que satisfac æquatione differentiale dato, cum errore minore de k.

Dem.

$k \varepsilon \mathrm{Q} . \supset . \exists (h', h'') \ni \{h', h'' \varepsilon \mathrm{Q} : y,z \varepsilon x + t\mathrm{u}_1 . \mathrm{m}(y-z) < h' . b,c \varepsilon a + r\mathrm{u}_n . \mathrm{m}(b-c) < h'' . \supset_{y,z,b,c} . \mathrm{m}[f(c,z)-f(b,y)] < k\}$ (1)

In vero, functione $f(b,y)$ es continuo in campo clauso considerato pro b et pro y; ergo habe continuitate uniforme (pag. 239 Prop. 1·3), id es, nos pote determina duo quantitate positivo h' et h'' in modo que, si y et z es duo valore arbitrario in intervallo de $x-t$ ad $x+t$, sed differente in valore absoluto minus que h', et si b et c es duo valore in sphæra de centro x et de radio r, differente inter se in valore absoluto minus que h'', semper differentia inter valores de functione $f(c,z)$ et $f(b,y)$ es minore, in valore absoluto, de k.

$k \varepsilon Q$. $|h',h''\varepsilon$ ut supra$|$. $p\varepsilon N_1$. $p > t/h'$. $p > lt/h''$. $g = \imath\, |\, Cn\, F\,(x + t\mathrm{u}_1)$
$\frown g\bar{\imath}\,(\, gx = a$:
$q\varepsilon\, 0 \cdots (p-1)$. $x_q = x + qt/p$. $z\varepsilon\, \Theta t/p$ $.\supset_{q,z}.$ $g(x_q + z) = gx_q + zf(\, gx_q\, , x_q\,)$:
» . $x_{-q} = x -$ » » » . $(x_{-q} - z) = gx_{-q} - zf(gx_{-q}, x_{-q})$:
$\supset$: $g\varepsilon\, Cn\, F\,(x + t\mathrm{u}_1)$: $z\varepsilon\, x + s\mathrm{u}_1$ $.\supset_z.$ $Dg\, gz \supset f(gz, z) + k\mathrm{u}_n$ (2)

Si k,h',h'' habe valore ut in propositione præcedente, nos divide intervallo de $x - t$ ad $x + t$ in $2p$ partes æquale, cum valores

..... $x + 2t/p$, $x - t/p$, x, $x - t/p$, $x + 2t/p$,... et nos considera illo functione g, definito in intervallo de $x - t$ ad $x + t$ ut seque:

$gx = a$. Si nos nosce valore de g pro puncto de divisione $x_q = x + qt/p$, ubi q es uno ex numeros de 0 ad $p - 1$, tunc functione g, pro omni $x_q + z$, in intervallo sequente, es definito ut functione lineare

$$gx_q + zf(gx_q\, , x_q\,);$$

et in modo analogo in intervallo de $x - t$ ad x.

Nos elige p ita magno, ut intervallo $t/p < h'$, et $(lt)/p < h''$.

Tunc functione g es definito in toto intervallo de $x - t$ ad $x + t$, et ibi satisfac æquatione differentiale dato, cum errore minore de k.

Nam functione g es definito pro valore x: $gx = a$; ergo es definito in intervallo de $x_0 = x$ ad $x_1 = x + t/p$, et pro valore x_1:

$$gx_1 = gx_0 + (x_1 - x_0) f(gx_0, x_0);$$

nunc mod $f(gx_0, x_0) \leqq l$; ergo

$$gx_1\, \varepsilon\, a + (x_1 - x_0) l\mathrm{u}_n \supset a + r\mathrm{u}_n,$$

vel pertine ad nostro campo de variabilitate. Noto gx_1, resulta definito functione g in intervallo de x_1 ad x_2, et $fx_2\, \varepsilon\, a + (x_2 - x_0) l\mathrm{u}_n \supset a + r\mathrm{u}_n$, pertine ad campo de variabilitate. Et ita porro. Ergo functione f es definito in toto intervallo de $x - t$ ad $x + t$.

Nunc si nos considera aliquo valore $x_q + z$, interno ad aliquo intervallo partiale, ibi functione g habe derivata in sensu proprio, $= f(g_q\, , x_q\,)$; nunc $z < t/p < h'$, $g(x_q + z) - gx_q = zf(gx_q\, , x_q\,)$, unde

$$\mathrm{m}[g(x_q + z) - gx_q\,] < tl/p < h''.$$

Ergo $\mathrm{m}|f[g(x_q + z),\, x_q + z] - f(gx_q\, , x_q\,)| < k$, et functione g satisfac æquatione, cum errore minore de k.

In omni puncto de divisione x_q, functione g habe duo derivata, uno in intervallo ante puncto, altero in intervallo post puncto, ambo satisfaciente conditione. Ergo derivata generico de g satisfac æquatione.

(1) . (2) $.\supset.$ P

·2 $h,k\varepsilon Q$. $p\varepsilon\, \theta r$. $q\varepsilon\, \theta t$. $\mathrm{l'm}[f(a + p\mathrm{u}_n \,\vdots\, x + q\mathrm{u}_1) - f(a,x)] < h$.
$q \times [\mathrm{m}f(a,x) + h + k] < p$. $y\varepsilon\, x + p\mathrm{u}_1$ $.\supset.$
$B(a,x,y,k) \supset a + (y - x)[f(a,x) + (h + k)\mathrm{u}_n]$

Dato duo quantitate positivo h,k, et p minore de r, et q minore de t, in modo que limite supero de modulos de differentias $f(c,z) - f(a,x)$, ubi c varia in sphæra de centro a et de

radio p, et z varia in intervallo de centro x et de semi-amplitudine q, es minore de h, et que producto de q per modulo de $f(a,x)$ plus h plus k es minore de p; et si y es valore in intervallo de $x-q$ ad $x+q$, tunc classe B, pro elementos a, x, y, k, continere in sphæra de centro $a+(y-x)f(a,x)$, et de radio $h+k$.

Dem.

$$g \varepsilon (a+p\mathrm{ɯ}_n)\mathrm{F}\, x \frown y \,.\, gx=a :\, z\varepsilon\, x\frown y \,.\supset.\, \mathrm{D}g\, gz \supset f(gz,z)+k\mathrm{ɯ}_n :\supset$$
$$: \quad » \quad \mathrm{D}g\, gz \supset f(a,x)+h\mathrm{ɯ}_n+k\mathrm{ɯ}_n :$$
$$\S29\ \mathrm{P}3{\cdot}3 :\supset.\, gy\, \varepsilon\, a+(y-x)[f(a,x)+(h+k)\mathrm{ɯ}_n] \quad (1)$$

Si g es functione definito in intervallo de x ad y, que sume valores in sphæra de centro a et de radio p; que pro valore x, sume valore a; et que pro omni valore in intervallo de x ad y satisfac æquatione differentiale dato, cum errore minore de k; tunc pro omni valore z in intervallo $x\frown y$, derivata de g differ de $f(a,x)$ de quantitate minore de $h+k$. Unde, per theorema de valore medio, gy es in sphæra de centro $a+(y-x)f(a,x)$ et de radio $(y-x)(h+k)$.

$$b\varepsilon\, \mathrm{B}(a,x,y,k) \,.\, g\varepsilon (a+r\mathrm{ɯ}_n)\mathrm{F}\, x\frown y \,.\, gx=a \,.\, gy=b :$$
$$z\varepsilon\, u\frown y \,.\supset_z.\, \mathrm{D}g\, gz \supset f(gz,z)+k\mathrm{ɯ}_n :$$
$$v=[\mathrm{m}(gz-a)-p \mid z,\, x\frown y] :\supset: v\varepsilon (\mathrm{q}\,\mathrm{F}\, x\frown y)\mathrm{cont} \,.\, vx=-p<0 \quad (2)$$
$$\mathrm{Hp}(2) \,.\, x'\varepsilon\, x\frown y :\, z\varepsilon\, x\frown x' \,.\supset_z.\, vz<0 :\supset$$
$$: \quad » \quad .\, gz\, \varepsilon\, a+p\mathrm{ɯ}_n : (1) :\supset.$$
$$\mathrm{m}(gx'-a) < q\times[\mathrm{m}f(a,x)+h+k] < h \,.\supset.\, vx'<0 \quad (3)$$

Nunc si b es uno ex valores de $\mathrm{B}(a,x,y,k)$, et si nos determina g, complexo sumente valores in sphæra de centro a et de radio r, functione definito in intervallo de x ad y, que pro valores x et y sume valores a et b, et que in intervallo de x ad y satisfac æquatione differentiale cum errore minore de k; nos demonstra que valores de g es in sphæra de centro a et de radio p. Nam nos voca v functione $\mathrm{mod}(gz-a)-p$, ubi varia z, in campo $x\frown y$; tunc v es quantitate reale, functione continuo in intervallo de x ad y, que pro x sume valore negativo; et es tale que si in interno de aliquo intervallo $x\frown x'$, es semper negativo, resulta etiam negativo pro extremo x'.

$$x,y\varepsilon \mathrm{q} \,.\, x-\!=y \,.\, v\varepsilon (\mathrm{q}\,\mathrm{F}\, x\frown y)\mathrm{cont} \,.\, vx<0 \,.\therefore\, x'\varepsilon\, x\frown y :\, z\varepsilon\, x\frown x' \,.\supset_z.\, vz<0$$
$$:\supset_{x'}.\, vx'<0 \,.\therefore\supset: z\varepsilon\, x\frown y \,.\supset_z.\, vz<0 \quad (4)$$
$$\mathrm{Hp}(2) \,.\, (3) \,.\, (4) :\supset: z\varepsilon\, x\frown y \,.\supset_z.\, vz<0 :\supset.\, g\varepsilon (a+p\mathrm{ɯ}_n)\mathrm{F}\, x\frown y \,.\, (1) \,.\supset.$$
$$b\varepsilon\, a+(y-x)[f(a,x)+(h+k)\mathrm{ɯ}_n] \quad (5)$$

Ergo (per Prop. 2·1 de pag. 239, transformato), illo es semper negativo in toto intervallo; id es valores de gz es semper in sphæra de centro a et radio $p<r$; unde, per P(1), $gy=b$ satisfac conditione scripto.

$$(5) \,.\, \mathrm{Elim}\, g \,.\, \mathrm{Oper}\, b\mathrm{ɜ} \,.\supset.\, \mathrm{P}$$

Si nos elimina g, que existe in hypothesi scripto, et nos opera per $b\mathrm{ɜ}$, resulta theorema.

·3 $h,k \varepsilon Q \,.\, y \varepsilon\, x+t\text{ш}_1 \,.\supset.\, \lambda B(a,x,y,h) \supset B(a,x,y,h+k)$

Dato duo quantitate positivo h et k, et y in intervallo de centro x et amplitudo $2t$, tunc classe limite de B, pro errore h, continere in B, pro errore $h+k$.

Dem.

$$b\varepsilon\, \lambda B(a,x,y,h) \,.\, p\varepsilon Q \,.\, z\varepsilon\, x\frown y \,.\, \text{l'm}[f(b+p\text{ш}_n \,|\, z\frown y) - f(b,y)] < k/3 \,.$$
$$(y-z)\times[\text{m}f(b,y)+h+k] < p \,.\, c\varepsilon B(a,x,y,h) \,.\, \text{m}(c-b) < k(y-z)/3 \,.\, \text{P}\cdot 2 \,.$$
$$\supset.\, B(c,y,z,h) \supset c+(z-y)[f(b,y)+(h+k/3)\text{ш}_n] \,.\, c\varepsilon\, b+(z-y)k/3\,\text{ш}_n \,.\supset.$$
$$B(c,y,z,h) \supset b+(z-y)[f(b,y)+(h+2k/3)\text{ш}_n] \qquad (1)$$

Si b es valore limite de classe $B(a,x,y,h)$, et si nos determina quantitate positivo p, et valore z in intervallo de x ad y, in modo que limite supero de modulos de differentia inter valore de f pro valores in sphæra de centro b et radio p, et in intervallo $z\frown y$, et $f(b,y)$, es minore que $k/3$, et in modo que $y-z$ multiplicato per $\text{m}f(b+y)+h+k$ es minore de p (et semper lice determinatione de p et de z), et si nos sume valore c in campo $B(a,x,y,h)$, que differ de limite b minus que $k(y-z)/3$ (quod lice), tunc, per propositione præcedente, $B(c,y,z,h)$ continere in $c+(z-y)[(b,y)+(h+k/3)\text{ш}_n]$; vel, per relatione inter b et c, continere in $b+(z-y)[f(b,y)+(h+2k/3)\text{ш}_n]$.

$$\text{Hp}(1) \,.\, \text{P2}\cdot 4 \,.\supset.\, \exists\, B(a,x,z,h) \cap B(c,y,z,h) \,.\, (1) \,.\supset.$$
$$\exists\, B(a,x,z,h) \cap \{b+(z-y)[f(b,y)+(h+2k/3)\text{ш}_n]\} \qquad (2)$$

$$\text{Hp}(1) \,.\, d\varepsilon\, B(a,x,z,h) \cap \{b+(z-y)[f(b,y)+(h+2k/3)\text{ш}_n]\} \,.$$
$$g = [d+(b-d)(u-z)/(y-z) \,|u,\, z\frown y] \,.\supset:\, gz = d \,.\, gy = b \,:$$
$$u\varepsilon\, x\frown y \,.\supset_u.\, \text{D}gu = (b-d)/(y-z)\, \varepsilon\, f(b,y)+(h+2k/3)\text{ш}_n \,:\supset:$$
$$\text{m}[\text{D}gu - f(gu,u)] < h+2k/3+k/3 = h+k \qquad (3)$$

Si literas serva valore præcedente, existe valore commune ad classes B de x ad z, et de y ad z; sed secundo classe continere in classe supra scripto, ergo, si nos sume aliquo valore d in classe B, de x ad z, et tale que $\text{m}[(d-b)/(z-y)-f(b,y)] < h+2k/3$, et si nos voca g functione lineare, que pro valores z et y de variabile u sume valores d et b, isto functione satisfac æquatione differentiale cum errore minore de $h+k$.

$$\text{Hp}(3) \,.\supset.\, b\varepsilon\, B(d,z,y,h+k) \,.\, d\varepsilon\, B(a,x,z,h) \,.\supset.\, b\varepsilon\, B(a,x,y,h+k) \qquad (4)$$

Ergo b pertine ad classe B, ex d, de z ad y, pro errore minore de $h+k$; sed d pertine ad classe B, ex a, de x ad z, cum errore minore de h, ergo cum errore minore de $h+k$; ergo b es valore de solutione de æquatione dato que i ex a, de x ad y, cum errore minore de $h+k$.

(4) . Elim p,z,c,d . Oper $b\exists$ $.\supset.$ P

Si ex (4) nos elimina literas p,z,c,d, que figura in hypothesi et non in thesi, et que repræsenta elementos existente per hypothesi de theorema, et si nos opera per $b\exists$, resulta Prop.

✻ 4. A

Hyp P1 $.\supset:$

·0 $y\varepsilon\, x+\text{ɯ}_1 t \,.\supset.\, \mathrm{A}(a,x,y) = \bigcap[\mathrm{B}(a,x,y,k)\, |k\text{‘}\mathrm{Q}]$ Def

In hypothesi de Prop. 1, super f,a,x,r,t, tunc, si y es in intervallo de $x-t$ ad $x+t$, nos pone $\mathrm{A}(a,x,y) =$ parte commune ad omni classe $\mathrm{B}(a,x,y,k)$, ubi varia k, et sume valores positivo.

·1 $\mathrm{A}(a,x,x) = \iota a$ [P·0 . P2·1 $.\supset.$ P]

·2 Hyp·0 . $b\varepsilon\, \mathrm{A}(a,x,y) \,.\supset.\, a\varepsilon\, \mathrm{A}(b,y,x)$ [P2·2 $\supset$ P]

·3 Hyp·0 $.\supset.\, \lambda\mathrm{A}(a,x,y) = \mathrm{A}(a,x,y)$ [§29 P2·2 $.\supset.$ P]

·4 Hyp·0 . $z\varepsilon\, x\urcorner y \,.\, c\varepsilon\, \mathrm{A}(a,x,z) \,.\, b\varepsilon\, \mathrm{A}(c,z,y) \,.\supset.\, b\varepsilon\, \mathrm{A}(a,x,y)$
[P2·3 $\supset$ P]

✻ 5·1 Hyp P4·0 $.\supset.\, \exists\, \mathrm{A}(a,x,y)$

In hypothesi supra scripto, existe aliquo individuo in classe $\mathrm{A}(a,x,y)$.

Dem.

$k\varepsilon\mathrm{Q} \,.\, \mathrm{P3{\cdot}1} \,.\supset.\, \exists\, \mathrm{B}(a,x,y,k) \,.\, \mathrm{B}(a,x,y,k) \supset a+r\text{ɯ}_n$
» $.\, h\varepsilon\, \theta k \,.\, \mathrm{P2{\cdot}5} \,.\supset.\, \lambda\mathrm{B}(a,x,y,h) \supset \mathrm{B}(a,x,y,k)$
§29 P2·3 $.\supset.\, \exists \bigcap \mathrm{B}(a,x,y,k)\, |k\text{‘}\mathrm{Q} \,.\supset.$ P

In vero, pro omni k, classe B existe; et si $h<k$, classe limite de B pro h continere in B pro k; ergo, per propositione de § præcedente, existe elemento commune ad omni B.

·2 Hyp·1 . $b\varepsilon\, \mathrm{A}(a,x,y) \,.\, z\varepsilon\, x\urcorner y \,.\supset.\, \exists\, \mathrm{A}(a,x,z) \cap \mathrm{A}(b,y,z)$

Dem.

$k\varepsilon\mathrm{Q} \,.\, \mathrm{Def\ A} \,.\supset.\, b\varepsilon\, \mathrm{B}(a,x,y,k)$ (1)
$k\varepsilon\mathrm{Q} \,.\, \beta k = \mathrm{B}(a,x,z,k) \,.\, \beta' k = \mathrm{B}(b,y,z,k) \,.\, (1) \,.\, \mathrm{P2{\cdot}4} \,.\supset.\, \exists\, \beta k \cap \beta' k$ (2)
$\mathrm{Hp}(2) \,.\, h\varepsilon\mathrm{Q} \,.\, h<k \,.\, \S\lambda\ (\mathrm{p}.140)\ \mathrm{P1{\cdot}4} \,.\, \mathrm{P3{\cdot}3} \,.\supset.\, \lambda(\beta h \cap \beta' h) \supset \lambda\beta h \cap \lambda\beta' h \,.$
$\lambda\beta h \supset \beta k \,.\, \lambda\beta' h \supset \beta' k \,.\supset.\, \lambda(\beta h \cap \beta' h) \supset \beta k \cap \beta' k$ (3)
$(2) \,.\, (3) \,.\, \S 29\ \mathrm{P2{\cdot}3} \,.\supset.\, \exists \bigcap(\beta h \cap \beta' h\, |h\text{‘}\mathrm{Q}) \,.$
$\bigcap(\beta h \cap \beta' h\, |h\text{‘}\mathrm{Q}) \supset (\bigcap \beta\text{‘}\mathrm{Q}) \cap (\bigcap \beta'\text{‘}\mathrm{Q}) = \mathrm{A}(a,x,z) \cap \mathrm{A}(b,x,z) \,.\supset.$ P

·3 $h\varepsilon\mathrm{Q} \,.\supset.\, \exists\, \theta t \cap k\ni[y\varepsilon\, x+k\text{ɯ}_1 \,.\supset_y.$
$\mathrm{A}(a,x,y) \supset a+(y-x)[f(a,x)+h\text{ɯ}_n]$

[$\mathrm{P3{\cdot}2} \,.\supset.\, \exists\, \theta t \cap k\ni[y\varepsilon\, x+k\text{ɯ}_1 \,.\supset.\, \mathrm{B}(a,x,y,h/2) \supset a+(y-x)[f(a,x)+h\text{m}_n$
$\mathrm{A}(a,x,y) \supset \mathrm{B}(a,x,y,h/2)] \,.\supset.$ P]

✻ 6.

·1 $n \varepsilon N_1 \,.\, u \varepsilon \mathrm{Cls'C}n \,.\supset.\, \omega u = \imath\, \mathrm{C}n \frown x \ni \{ r \varepsilon 1 \cdots n \,.\supset_r.\, x_r =$ $\mathrm{l'}\,\mathrm{E}_r\text{‘}\, u \frown z \ni [s \varepsilon 1 \cdots (r-1) \,.\supset_s.\, z_s = x_s] \}$ Def

·2 $n \varepsilon N_1 \,.\, u \varepsilon \mathrm{Cls'C}n \,.\, \exists u \,.\, \mathrm{l'm}\, u\, \varepsilon \mathrm{Q} \,.\, \lambda u = u \,.\supset.\, \omega u\, \varepsilon u$

In P7 nos debe elige uno individuo in classe determinato, que contine plure individuo; et hoc per numero infinito de vice. Forma de ratiocinio, ubi occurre electione de elemento arbitrario, per numero infinito de vice, se præsenta in plure libro; observatione que isto electione non es licito, occurre in meo articulo de MA. t. 37 p. 210, et alibi. Vide RdM. t. 8 p. 145; Jourdain, *Quarterly Journal* a. 1907 p. 352, Zermelo MA, a. 1908 t. 65 p. 111, etc. etc.

Ergo nos da lege, que, ad omni classe satisfaciente aliquo conditione, fac corresponde individuo in ce classe.

·1 Si n es numero, et u es classe de complexos de ordine n, tunc ωu indica illo complexo x tale que suo primo coordinata x_1 æqua limite supero de primo coordinatas de individuos de u; suo secundo coordinata æqua limite supero de secundo coordinatas de individuos de classe u, que habe ut primo coordinata x_1; et in generale, suo coordinata de indice r vale limite supero de coordinatas de individuos de classe u, que habe omni coordinata præcedente æquale ad coordinata jam determinato de x.

·2 Si classe u es existente, vel non vacuo, et limitato et clauso, tunc ωu indica individuo in classe u.

Dem. pro $n = 2$.

$\mathrm{Hp} \,.\supset.\, \mathrm{E}_1\text{‘}u\, \varepsilon\, \mathrm{Cls'q} \,.\, \exists\, \mathrm{E}_1\text{‘}u \,.\, \mathrm{l'm}\, \mathrm{E}_1\text{‘}u\, \varepsilon\, \mathrm{Q} \,.\, \lambda \mathrm{E}_1\text{‘}u = \mathrm{E}_1\text{‘}u \,.\supset.$
$x_1 = \mathrm{l'E}_1\text{‘}u\, \varepsilon\, \mathrm{E}_1\text{‘}u \,.\supset.\, \exists\, u \frown y \ni (y_1 = x_1)$ (1)

$(1) \,.\, u_1 = u \frown y \ni (y_1 = x_1) \,.\supset.\, \exists u_1 \,.\, u_1 \supset u \,.\, \lambda u_1 = u_1 \,.\supset.$
$x_2 = \mathrm{l'E}_2\text{‘}u_1\, \varepsilon\, \mathrm{E}_2\text{‘}u_1 \,.\supset.\, \exists\, u_1 \frown y \ni (y_2 = x_2) \,.\supset.\, \exists\, u \frown y \ni (y_1 = x_1 \,.\, y_2 = x_2)$
$.\supset.\, \exists\, u \frown y \ni (y = x) \,.\supset.\, x \varepsilon u$

Si nos considera successione composito ex classes æquale ad u, tunc limite de successione es ipso classe u. In isto casu, theorema de §29 P2·0 determina uno individuo in classe u, que es ipso ωu. Sed nos præfer demonstratione directo de theorema.

·3 $m \varepsilon N_0 \,.\supset.\, Z_m = x + (y - x)(0 \cdots 2^m)/2^m$ Def

·4 $Z = \bigcup (Z_m | m\text{‘}N_0)$ Def

✻ 7.

$$\text{Hyp P1} . y \varepsilon\, x + \text{m}_1 t . b \varepsilon\, \mathrm{A}(a,x,y) . g = \iota\,(\mathrm{C}n\, \mathrm{F}\, x \frown y) \cap g\ni$$
$$\{ gx = a . gy = b :$$
$$m \varepsilon \mathrm{N}_0 . z \varepsilon\, Z_{m+1} - Z_m . z_1 = z - (y-x)/2^{m+1} . z_2 = z + (y-x)/2^{m+1} .$$
$$\supset . gz = \omega[\mathrm{A}(gz_1, z_1, z) \cap \mathrm{A}(gz_2, z_2, z)] :$$
$$z \varepsilon\, x \frown y - Z .\supset. gz = \iota \bigcap [\mathrm{A}(gu,u,z) | u \text{ ‘ } Z] \}$$
$$.\supset. g \varepsilon\, \mathrm{C}n\, \mathrm{F}\, x \frown y . \mathrm{D}g = [f(gz,z) | z, x \frown y]$$

Si n,a,x,f,r,t habe sensu ut in Prop. 1, si y es valore in intervallo de $x-t$ ad $x+t$, et b es complexo in classe $\mathrm{A}(a,x,y)$, et si nos determina g, complexo functione definito in intervallo de x ad y, per conditiones sequente:

Pro variabile $= x$ et y, g sume valores a et b.

Si nos divide intervallo de x ad y in duo parte æquale, et pone $z = x + (y-x)/2$, tunc existe classe commune ad $\mathrm{A}(gx,x,z)$ et $\mathrm{A}(gy,y,z)$; sume in isto classe individuo ω; isto es valore de gz. Nos divide intervallo partiale in novo partes; ita functione g es definito pro omni valore de classe Z; vel pro omni puncto de divisione de intervallo $x \frown y$ in numero potestate de 2, de partes æquale.

Si z es in intervallo $x \frown y$, sed non es aliquo puncto de divisione, tunc gz es elemento commune ad omni classe $\mathrm{A}(gu,u,z)$, ubi u sume omni valore in classe Z.

Tunc g es in realitate complexo functione dato in intervallo $x \frown y$, vel conditiones scripto defini functione, et g satisfac æquatione differentiale dato.

Dem.

(a) $z \varepsilon Z_0 .\supset. gz \,\varepsilon\, \mathrm{C}n$

[$z \varepsilon Z_0 :\supset: z = x .\cup. z = y : gx = a . gy = b :\supset$ Ths]

Si z es uno ex extremos de intervallo $x \frown y$, tunc gz es complexo determinato.

(a') $z, z' \varepsilon Z_0 .\supset. gz' \,\varepsilon\, \mathrm{A}(gz,z,z')$

[» $.\supset: z = x . z' = y .\cup. z = y . z' = x .\cup. z = z' = x .\cup. z = z' = y :$
P4·2 . P4·1 $:\supset$ P]

Et, si z et z' es extremos (coincidente aut non), gz' pertine ad classe A de gz; nam hoc significa aut $b \varepsilon \mathrm{A}(a,x,y)$, quod es hypothesi; aut $a \varepsilon \mathrm{A}(b,y,x)$, quod resulta ex hypothesi per P4·2; aut $a \varepsilon \mathrm{A}(a,x,x)$, aut $b \varepsilon \mathrm{B}(b,y,y)$, quod resulta ex P4·1.

(β) $m \varepsilon \mathrm{N}_0 : z \varepsilon Z_m . \supset_z . gz \,\varepsilon\, \mathrm{C}n : z,z' \varepsilon Z_m . \supset_{z,z'} . gz' \,\varepsilon\, \mathrm{A}(gz,z,z')$
$: \supset : z \varepsilon Z_{m+1} . \supset_z . gz \,\varepsilon\, \mathrm{C}n : z,z' \varepsilon Z_{m+1} . \supset_{z,z'} . gz' \,\varepsilon\, \mathrm{A}(gz,z,z')$

[$z \varepsilon Z_{m+1} - Z_m . z_1 = z - (y-x)/2^{m+1} . z_2 = z + (y-x)/2^{m+1} . \supset .$
$z_1, z_2 \varepsilon Z_m . \supset . gz_2 \,\varepsilon\, \mathrm{A}(gz_1,z_1,z_2) . \supset . \exists [\mathrm{A}(gz_1,z_1,z) \cap \mathrm{A}(gz_2,z_2,z)]$.
$\mathrm{l}'\mathrm{m}[$ » » $] \varepsilon \mathrm{Q}$.
P3·3 . $\S\lambda$ (pag.140) P1·41 $. \supset . \lambda[$ » » $] = [$ » » $]$
P6·2 $. \supset . \omega[$ » » $] \,\varepsilon\, \mathrm{C}n$
$. \supset . gz \,\varepsilon\, \mathrm{C}n . gz \,\varepsilon\, \mathrm{A}(gz_1,z_1,z) . gz \,\varepsilon\, \mathrm{A}(gz_2,z_2,z)$ (1)
$u \varepsilon Z_m \cap x \frown z . \supset . gz_1 \,\varepsilon\, \mathrm{A}(gu,u,z_1) . gz \,\varepsilon\, \mathrm{A}(gz_1,z_1,z) . \supset . gz \,\varepsilon\, \mathrm{A}(gu,u,z)$ (2)
$u \varepsilon Z_m \cap z \frown y . \supset . gz_2 \,\varepsilon$ » $_2)$ » $z_2,z_2,z) . \supset .$ » » » (3)
$u \varepsilon Z_m . (2) . (3) . \supset . gz \,\varepsilon\, \mathrm{A}(gu,u,z)$ (4)
$z,z' \varepsilon Z_{m+1} - Z_m . z'' \varepsilon Z_m \cap z \frown z' . \supset . gz'' \,\varepsilon\, \mathrm{A}(gz,z,z'') . gz' \,\varepsilon\, \mathrm{A}(gz'',z'',z')$
$. \supset . gz' \,\varepsilon\, \mathrm{A}(gz,z,z')$ (5)
(1) . (5) $. \supset .$ P]

Si functione g es definito pro punctos de divisione de intervallo $x \frown y$ in $2 \uparrow m$ partes, et semper, si z et z' es punctos de divisione, gz' es A de gz; tunc functione g es etiam definito pro punctos de divisione successivo, et satisfac idem conditione.

In vero, si z es puncto de divisione in $2 \uparrow (m+1)$ partes, distincto ab punctos de divisione in $2 \uparrow m$ partes, et si nos voca z_1 et z_2 punctos de classe Z_m que contine z, tunc gz_2 es A de gz_1; ergo classe commune ad A ex gz_1, ab z_1 ad z, et ad A ex gz_2, ab z_2 ad z, es non vacuo, limitato, clauso; ergo operatione ω da individuo determinato in classe.

(2)(3)(4)(5) demonstra secundo parte de theorema.

(γ) $z \varepsilon Z . \supset . gz \,\varepsilon\, \mathrm{C}n : z,z' \varepsilon Z . \supset . gz' \,\varepsilon\, \mathrm{A}(gz,z,z')$
[$(\alpha) . (\alpha') . (\beta)$. Inductione $. \supset .$ P]

Ergo, pro omni puncto de divisione de intervallo $x \frown y$ in partes, in numero arbitrario, semper es determinato g, et omni valore de functione g es A de omni alio, pro variabiles in campo Z.

(δ) $z \varepsilon x \frown y . \supset . gz \,\varepsilon\, \mathrm{C}n$
[$z \varepsilon x \frown y - Z . u \varepsilon Z \cap x \frown z . v \varepsilon Z \cap z \frown y . (\gamma) . \supset . gv \,\varepsilon\, \mathrm{A}(gu,u,v)$
. P5·2 $. \supset . \exists [\mathrm{A}(gu,u,z) \cap \mathrm{A}(gv,v,z)]$
$. u' \varepsilon u \frown z . \supset . \mathrm{A}(gu',u',z) \supset \mathrm{A}(gu,u,z)$
$z \varepsilon x \frown y - Z . v \varepsilon Z \cap z \frown y . \supset . \exists \{ \bigcap [\mathrm{A}(gu,u,z) | u \text{‘} (Z \cap x \frown z)] \cap \mathrm{A}(gv,v,z) \}$
$z \varepsilon x \frown y - Z . \supset . \exists \{ \bigcap [\mathrm{A}(gu,u,z) | u \text{‘} (Z \cap x \frown z)] \cap \bigcap [\mathrm{A}(gv,v,z) | v \text{‘} (Z \cap z \frown y)] \}$
$. \supset . \exists \bigcap [\mathrm{A}(gu,u,z) | u \text{‘} Z]$ (1)
$c,d \varepsilon \bigcap [\mathrm{A}(gu,u,z) | u \text{‘} Z] . u \varepsilon Z . \supset . c,d \varepsilon \mathrm{A}(gu,u,z) \supset gu + (z-u) l \mathfrak{u}_n . \supset .$
$\mathrm{m}(c-d) \,\varepsilon\, \theta l \,\mathrm{m}(z-u)$
» » » » $. \supset . \mathrm{m}(c-d) \,\varepsilon\, \theta l \times \mathrm{l}, \mathrm{m}[(z-u) | u \text{‘} Z]$
$. \supset . \mathrm{m}(c-d) = 0 . \supset . c = d$ (2)
(1) . (2) $. \supset .$ P]

Nunc nos demonstra que pro omni valore de intervallo de x ad y, functione g es determinato.

(ε) $z,z'\varepsilon\, x \urcorner y \;.\supset.\; gz' \;\varepsilon\; \mathrm{A}(gz,z,z')$

[$z\varepsilon Z \,.\, z'\varepsilon\, x\urcorner y - Z \,.\, \mathrm{Def}\, gz' \;.\supset.\; \mathrm{Ths}$ (1)

$z,z'\varepsilon\, x\urcorner y - Z \,.\, z''\varepsilon\, z\urcorner z' \cap Z \;.\supset.\; gz'' \,\varepsilon\, \mathrm{A}(gz,z,z'') \,.\, gz' \,\varepsilon\, \mathrm{A}(gz'',z'',z')$
$.\supset.\; \mathrm{Ths}$ (2)

(γ) . (1) . (2) $.\supset.$ P]

Et que omni valore de functione es A de omni alio.

(ζ) $z\varepsilon\, x\urcorner y \;.\supset.\; \mathrm{D}gz = f(gz, z)$

[$h\varepsilon\mathrm{Q} \,.\, \mathrm{P}5{\cdot}3 \;.\supset.\; \exists\, \mathrm{Q}\cap k\ni[u\varepsilon\, x\urcorner y \cap (z+kɯ_1) \;.\supset.$

$\mathrm{A}(gz,z,u) \supset gz+(u-z)[f(gz,z)+hɯ_n] \;.\supset.$

$gu\;\varepsilon$ » » » $.\supset.$

$\mathrm{m}[(gu-gz)/(u-z) - f(gz,z)] < h\} \;.\supset.\; \mathrm{P}$]

Ergo functione g satisfac æquatione differentiale dato.

(δ) . (ζ) $.\supset.$ P

❋ 8.

·1 Hyp P1 . $y\varepsilon\, x+tɯ_1 \,.\, b\varepsilon\, \mathrm{A}(a,x,y) \;.\supset.\; \exists\, (\mathrm{C}n\,\mathrm{F}\, x\urcorner y)\cap g\ni$
$\{gx=a \,.\, gy=b \,.\, \mathrm{D}g=[f(gz,z)\,|z,\, x\urcorner y]\}$ [P7 $.\supset.$ P]

In hypothesi dato, si b es uno ex valores in classe A(a,x,y), tunc existe complexo g, functione definito in intervallo ab x ad y, que pro extremos x et y sume valores a et b, et que satisfac æquatione differentiale dato. Seque de Prop. præcedente.

·2 Hyp P1 $.\supset.\; \exists\, [\mathrm{C}n\,\mathrm{F}\,(x+tɯ_1)]\cap g\ni\{gx=a \,.\, gy=b \,.$
$\mathrm{D}g=[f(gz,z)\,|z,\, x+tɯ_1]\}$

[P·1 . Elim b . P5·1 $.\supset.$ P·2 $.=.$ P1]

Ergo, existe complexo g, functione definito in intervallo ab $x-t$ ad $x+t$, que pro valore x sume valore a, et in toto intervallo satisfac æquatione differentiale dato; quod es theorema P1, demonstrando.

Si A(a,x,y), que es classe non vacuo, ut es demonstrato in Prop. 5·1, contine plure individuo, nos pote sume plure valore b in illo, et resulta plure functione que satisfac æquatione dato, et que pro variabile $=x$ sume valore a.

Ut isto functione es unico, id es, ut functione es determinato per æquatione differentiale, et per valore initiale (a,x), es necesse et suffice que classe A(a,x,y), pro omni y, contine uno solo individuo.

❋ 9.

$$\text{Hyp P1} \,.\, \text{l}'\text{m} \bigcup \text{Dg}_1 f(a+r\text{ɯ}_n \,\text{⁝}\, x+t\text{ɯ}_1)\,\varepsilon\text{Q} \,:\, g,g'\varepsilon\, \text{C}n\,\text{F}\,(x+t\text{ɯ}_1) \,.$$
$$gx = g'x = a \,.\, \text{D}g = [f(gz,z) \,|z,\, x+t\text{ɯ}_1] \,.$$
$$\text{D}g' = [f(g'z,z) \,|z,\, x+t\text{ɯ}_1] \;.\!\supset\!.\; g = g'$$

Ad hypothesi Prop. 1, nos adde novo conditione. Considera $f(a,x)$ ut functione de suo primo elemento; et fac suo derivata.

Isto derivata es derivata partiale, indicato per $\text{D}_1 f(a,x)$, in pag. 328. Sed a es complexo; ergo nos es in casu de pag. 330, et $\text{D}_1 f(a,x)$ repræsenta substitutione de ordine n, id es substitutione expresso per matrice que habe pro elementos derivatas partiale de n coordinatas de f pro n coordinatas de a.

Pro majore generalitate, nos non suppone existentia de derivatas; ergo nos considera derivata generale, supra definito.

Nos adde conditione que limite supero de modulo de omni valore in classes de derivata generale de f, pro primo elemento, ubi duo variabile varia in compos dato in Prop. 1, es finito. Hoc significa que omni derivata generico partiale de coordinatas de f pro coordinatas de a, es minore de quantitate determinabile. Id es, $\text{m}[f(d,u) - f(d',u)] \,/\, \text{m}(d-d')$
pro omni systema $d,d'\,\varepsilon\, a+r\text{ɯ}_n,\; u\varepsilon\, x+t\text{ɯ}_1$,
habe limite supero finito.

Tunc, si g et g' es duo functione, que habe idem valore a pro variabile $= x$, et ambo satisfaciente æquatione differentiale dato, illo es semper identico.

Id es esiste uno, et uno solo complexo functione definito in intervallo de $x-t$ ad $x+t$, que pro variabile $= x$, sume valore a, et que satisfac æquatione differentiale dato.

Dem.

$$\text{Hp} \,.\, p = \text{l}'\text{m} \bigcup \text{Dg}_1 f(\ldots) \,.\, d,d'\varepsilon\, a+r\text{ɯ}_n \,.\, u\varepsilon\, x+t\text{ɯ}_1 \;.\!\supset\!.$$
$$\text{m}[f(d,u) - f(d',u)] \leqq p\times\text{m}(d-d') \qquad (1)$$
$$\text{Hp}(1) \,.\, u\varepsilon\, x+\theta t \;.\!\supset\!.\; \text{Dg}\,\text{m}(gu-g'u) \supset \text{m}(\text{D}gu - \text{D}g'u) - \text{Q}_0$$
$$\supset \text{m}[f(gu,u) - f(g'u,u)] - \text{Q}_0$$
$$\supset p\times\text{m}(gu-g'u) - \text{Q}_0$$
$$.\!\supset\!.\; [\text{Dg}\,\text{m}(gu-g'u) - p\times\text{m}(gu-g'u)]\times\text{e}\!\wedge\!(-pu) \supset -\text{Q}_0$$
$$.\!\supset\!.\; \text{Dg}\,[\text{m}(gu-g'u)\times\text{e}\!\wedge\!(-pu) \,|u,\, x+\theta t]u \supset -\text{Q}_0$$
$$.\!\supset\!.\; \Delta\,[\quad \text{»} \quad \text{»} \quad \text{»} \quad |u;\, x,u] \leqq 0$$
$$.\!\supset\!.\; \text{m}(gu-g'u)\times\text{e}\!\wedge\!(-pu) \leqq 0 \;.\!\supset\!.\; \text{m}(gu-g'u) = 0$$
$$.\!\supset\!.\; gu = g'u \qquad (2)$$

$$(2) \supset \text{P}$$

In vero, si nos voca p limite supero de modulos de derivata-substitutione de f, pro primo elemento, in toto campo supra considerato de duo elemento, limite supero que es finito, tunc mod$(gu-g'u)$, ubi varia u, es nullo pro $u=x$, et suo derivata generale es classe de valores minore de $p\times$m$(gu-g'u)$. Ergo isto functione satisfac in-æquatione differentiale lineare, simile ad æquatione differentiale lineare, tractato in pag. 323.

Nos tracta in-æquatione in modo analogo.

Multiplica per $e\nwarrow(-pu)$, resulta functione que habe derivata generale negativo aut nullo; ergo suo incremento es negativo aut nullo; sed pro $u=x$, functione es nullo; ergo illo, que non sume valores negativo, es semper nullo; id es, pro omni valore de u, es $gu=g'u$; quod significa $g=g'$.

✻ 10. Exemplo.

·1 $f\varepsilon\, \mathrm{qFq}\,.\, \mathrm{D}f = [3\,(fx)\nwarrow(2/3)\ |x, \mathrm{q}]\ .=.$
$f=(\iota 0 \,\vdots\, \mathrm{q})\ .\cup.$
$\exists\ \mathrm{q}\frown a\ni\{f=(\iota 0 \,\vdots\, a-\mathrm{Q}_0)\cup[(x-a)^3\,|x, a+\mathrm{Q}_0]\}\ .\cup.$
$\exists\ \mathrm{q}\frown a\ni\{f=[(x-a)^3\,|x, a-\mathrm{Q}_0]\cup(\iota 0 \,\vdots\, a+\mathrm{Q}_0)\}\ .\cup.$
$\exists\ (a,b)\ni\{a,b\varepsilon\mathrm{q}\,.\, a<b\,.\, f=[(x-a)^3\,|x, a-\mathrm{Q}_0]\cup(\iota 0 : a\overline{\ \ }b)\cup$
$[(x-b)^3\,|x, b+\mathrm{Q}_0]\}$

Omni f, functione reale de variabile reale, que pro omni x, satisfac æquatione

$$\mathrm{D}fx=3(fx)\nwarrow(2/3),$$

aut es semper nullo,

aut es nullo de $-\infty$ usque ad aliquo valore finito a, et de isto valore in post, habe forma $(x-a)^3$,

aut habe forma $(x-a)^3$ pro valores de $-\infty$ ad a, et es nullo de a ad $+\infty$,

aut de $-\infty$ ad valore finito a habe forma $(x-a)^3$, de a ad valore maiore b, es nullo, et de b ad $+\infty$ habe forma $(x-a)^3$.

Ergo existe infinito functione, que satisfac æquatione differentiale et que pro valore arbitrario a, sume valore 0.

·2 $f\varepsilon\, \mathrm{qFq}\,.\, \mathrm{D}f=\{4x^3(fx)/[x^4+(fx)^2]\ |x, \mathrm{q}\}\ .=.$
$\exists\ \mathrm{q}\frown c\ni\{f=[c\pm\surd(c^2+x^4)\ |x, \mathrm{q}]\}$

Omni solutione de æquatione differentiale scripto in primo membro habe forma scripto in secundo, ubi c es constante arbitrario.

Si per $c>0$ nos sume radicale cum signo $-$, et per $c<0$, radicale cum signo $+$, resulta solutiones, in numero infinito, que satisfac conditione $f0=0$.

Dfx es dato ut functione continuo de x et fx, si ad expressione, pro $x=0$, $fx=0$, nos tribue ut valore suo limite 0.

Historia.

Cauchy, *Exercices* a.1840 p.327, demonstra theorema P1 et P9, in hypothesi de existentia et continuitate de $D_1 f$, id es, de derivatas partiale de coordinatas de f pro coordinatas de a.

Lipschitz, Ann. di Mat. a.1868 p.288, BD. a.1876 p.149, expone theorema æquivalente ad P9.

Me habe reducto in symbolos demonstratione in:

Démonstration de l'intégrabilité des équations différentielles ordinaires, MA. a.1890 t.37 p.182. Vide :

TorinoA. a.1886, AnnN. a.1892 p.289.

G. Mie, MA. a.1893 t.43 p.553.

Encyclopädie t.2 p.197.

Ex demonstratione resulta que, pro existentia de solutione, suffice que functione f es continuo.

Demonstratione in pag. 416-428 es reproductione de demonstratione nunc citato, post aliquo reductione de ratiocinio et de scriptura.

Ch. de la Vallée-Poussin, BruxellesM. t.47 a.1893, et

C. Arzelà, BolognaM. t.5 a.1995 p.225, t.6 a.1896 p.131,
demonstra que in hypothesi de continuitate de $f(a,x)$ pro a, et integrabilitate pro x, classe B verge ad limite A.

W. F. Osgood, Monatsh. t.9 a.1898 p.322 demonstra que solutione de æquatione differentiale es univoco, non solo in hypothesi de Prop. 9, sed
et si $\quad \mathrm{m}[f(d,u)-f(d',u)] \,/\, \mathrm{m}[(d'-d)\log \mathrm{m}(d'-d)]$
vel $\quad$ » $\quad / \,\mathrm{m}[(d'-d)\log \mathrm{m}(d'-d)\log\log \mathrm{m}(d'-d)]$
etc., ubi d,d',u varia in campos indicato in Prop. 9, habe limite supero finito.

Univocitate subsiste et in alios casu. P. ex. æquatione $\mathrm{D}gx = f(gx)$, si f non es nullo, habe solutione unico, ut resulta ex P11·2, et nullo hypothesi es facto super derivata de f.

Vide: Bliss, *The solutions of differential equations...*, Annals of Math. s.2 t.6 a.1905 p.49.

Bolza, AmericanT. a.1906 t.7 p.464.

✱ 11.

·1 $u \varepsilon \mathrm{Intv} \,.\, f \varepsilon (\mathrm{qF}u)\mathrm{cont} \,.\, x_0 \varepsilon u \,.\, y_0 \varepsilon \mathrm{q} \,.\supset:$
$g \varepsilon \mathrm{qF}u \,.\, \mathrm{D}g = f \,.\, gx_0 = y_0 \,.=.\, g = \{[y_0 + \mathrm{S}(f; x_0, x)] \,|x, u\}$

Dem. §S p.343 P12·1 $\supset$ P

Si f es functione reale, definito in intervallo u, et continuo, et es dato duo quantitate x_0 et y_0, tunc omni functione g definito in intervallo u, que pro valore x_0 sume valore y_0, et

que satisfac æquatione differentiale

$$\mathrm{D}gx = fx, \qquad \text{pro omni } x \text{ in } u,$$

habe forma $\qquad gx = y_0 + \mathrm{S}(f; x_0, x), \qquad$ » ,

ut resulta ex relatione inter integrale et derivata.

·2 $u, v \varepsilon \mathrm{Intv} \,.\, f \varepsilon (\mathrm{q} - \iota 0 \mathrm{\,F\,} v)\mathrm{cont} \,.\, x_0 \varepsilon u \,.\, y_0 \varepsilon v \,.\, g \varepsilon v \mathrm{F} u \,.\, gx_0 = y_0$

$$
\begin{aligned}
.\supset\therefore \mathrm{D}g = fg \;:=:\;& x \varepsilon u \,.\supset_x .\; \mathrm{D}gx = fgx \\
:=:\;& \text{»} \quad (\mathrm{D}gx)/(fgx) = 1 \\
:=:\;& \text{»} \quad \mathrm{D}[\mathrm{S}(1/f; y_0, gx) \,|x, u]x = 1 \\
:=:\;& \text{»} \quad \mathrm{S}(1/f; y_0, gx) = x - x_0 \\
:=:\;& \text{»} \quad [\mathrm{S}(1/f; y_0, y) \,|y, v]gx = x - x_0 \\
:=:\;& \text{»} \quad x \varepsilon [x_0 - \mathrm{S}(1/f; \mathrm{l}_{,}v, y_0)] \frown [x_0 + \mathrm{S}(1/f; y_0, \mathrm{l}'v)] \,. \\
& \qquad gx = [\mathrm{S}(1/f; y_0, y) \,|y, v]^{-1}(x - x_0) \\
:=:\;& u \supset [x_0 - \mathrm{S}(1/f; \mathrm{l}_{,}v, y_0)] \frown [x_0 + \mathrm{S}(1/f; y_0, \mathrm{l}'v)] \,. \\
& g = \{[\mathrm{S}(1/f; y_0, y) \,|y, v]^{-1}(x - x_0) \,|x, u\}
\end{aligned}
$$

u et v es intervallo; f es quantitate non nullo, functione definito in campo v, et continuo; ergo f habe signo constante; nos sume x_0 in u, et y_0 in v; et nos considera functione g, definito in campo u, que sume valores in campo v, que pro valore x_0 sume valore y_0, et que satisfac æquatione inter functiones $\mathrm{D}g = fg$. Ce æquatione vale

$$\mathrm{D}gx = f(gx) \text{ pro omni } x \text{ in } u.$$

Divide per fgx, quantitate non nullo; primo membro fi derivata de integrale de $1/fy$, ubi varia y, de y_0 ad gx. Ergo ce integrale vale $x - x_0$. Tunc primo membro contine gx, et es reductibile ad functione de gx. Ce functione es invertibile, per theorema p. 282 P10·0·1, nam suo derivata habe signo constante. Inverte functione; resulta valore de gx; opera per $|x$, et resulta functione g. Intervallo u debe es parte de intervallo inter $x_0 \pm$ integrale de $1/f$, de y_0 ad limites infero et supero de v.

Si functione f es semper continuo, nos pote sume per intervallo v, intervallo inter duo radice consecutivo de æquatione $fy = 0$, vel superiore ad maximo radice, vel inferiore ad minimo, vel toto campo de numeros reale, si functione f semper es differente de 0.

Si $fy_0 = 0$, plure casu pote occurre. Vide exemplo in P10·1.

❋ 12. ÆQUATIONE DIFFERENTIALE LINEARE.

·1 $g \varepsilon \mathrm{qFq} \,.\supset:$

$$f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}f = (gx \times fx \,|x, \mathrm{q}) \;.=.\; f = [f0\, \mathrm{e}\upharpoonright \mathrm{S}(g;\, 0,x) \,|x, \mathrm{q}]$$

Dato g functione reale de variabile reale, tunc functione f, que pro omni valore de x, satisfac æquatione

$$\mathrm{D}fx = gx \times fx,$$

es dato per formula:

$$fx = f0\, \mathrm{e}\upharpoonright \text{integrale de } g, \text{ de } 0 \text{ ad } x,$$

ubi varia x.

Dem.

$$\begin{aligned}
\mathrm{D}f = [(gx \times fx)|x, \mathrm{q}] \;&.=:\; x\varepsilon\mathrm{q} \,.\supset_x .\, \mathrm{D}fx = gx \times fx \\
&.=: \quad \text{»} \quad \{[\mathrm{e}\upharpoonright -\mathrm{S}(g;0,x)] \times (\mathrm{D}fx - gx \times fx) = 0 \\
&.=: \quad \text{»} \quad \mathrm{D}\{[\mathrm{e}\upharpoonright -\mathrm{S}(g;0,x)] \times fx \,|x, \mathrm{q}\}x = 0 \\
&.=: \quad \text{»} \quad \varDelta\{[\mathrm{e}\upharpoonright -\mathrm{S}(g;0,x)] \times fx \,|x;\, 0,x\} = 0 \\
&.=: \quad \text{»} \quad \mathrm{e}\upharpoonright -\mathrm{S}(g;0,x) \times fx = f0 \\
&.=: \quad \text{»} \quad fx = (f0) \times \mathrm{e}\upharpoonright \mathrm{S}(g;0,x) \\
&.=.\; f = [f0 \times \mathrm{e}\upharpoonright -\mathrm{S}(g;0,x) \,|x, \mathrm{q}]
\end{aligned}$$

In vero, transporta in primo membro, et multiplica per $\mathrm{e}\upharpoonright -\mathrm{S}(g;0,x)$; primo membro fi derivata de expressione scripto; unde incremento de functione es nullo, vel valore de functione vale suo valore pro $x = 0$; unde nos deduce valore de fx, et infine functione f.

·2 $g,h \varepsilon \mathrm{qFq} \,.\supset: f \varepsilon \mathrm{qFq} \,.\, \mathrm{D}f = [(gx \times fx + hx)|x, \mathrm{q}] \;.=.$

$$f = \{\mathrm{e}\upharpoonright \mathrm{S}(g;\, 0,x)[f0 + \mathrm{S}\{[\mathrm{e}\upharpoonright -\mathrm{S}(g;\, 0,x)] \times hx|x;\, 0,x\}] \,|x, \mathrm{q}\}$$

{ LEIBNIZ A.Erud. a.1694 Math.S. t.5 p.313 }

Dato duo functione g et h, æquatione, identitate in numero reale x, conditione in functione f:

$$\mathrm{D}fx = gx \times fx + hx$$

vocare in f « æquatione differentiale lineare ». Si gx es constante, et hx es nullo, resulta æquatione considerato in p.323.

Calculo de functione f: $\quad x\varepsilon\mathrm{q} \,.\supset_x .\, \mathrm{D}fx = gx\, fx - hx$

Transporta: $\quad x\varepsilon\mathrm{q} \,.\supset_x .\, \mathrm{D}fx - gx\, fx = hx$

Multiplica per $\mathrm{e}\upharpoonright -\mathrm{S}(g;0,x)$:

$$x\varepsilon\mathrm{q} \,.\supset_x .\, \mathrm{D}[\mathrm{e}\upharpoonright -\mathrm{S}(g;0,x) \times fx|x,\mathrm{q}]x = \mathrm{e}\upharpoonright -\mathrm{S}(g;\, 0,x) \times hx$$

Integra ab 0 ad x:

$$x\varepsilon\mathrm{q} \,.\supset_x .\, \mathrm{e}\upharpoonright -\mathrm{S}(g;0,x) \times fx - f0 = \mathrm{S}[\mathrm{e}\upharpoonright -\mathrm{S}(g;0,x) \times hx|x;0,x]$$

Trahe fx:

$$x\varepsilon\mathrm{q} \,.\supset_x .\, fx = \mathrm{e}\upharpoonright \mathrm{S}(g;0,x)\{f0 + \mathrm{S}[\mathrm{e}\upharpoonright -\mathrm{S}(g,0,x) \times hx|x;0,x]\}$$

Unde nos deduce $f =$ secundo membro, ubi varia x in campo de numeros reale.

·3 $g, h \varepsilon$ qfq . $m \varepsilon$ q-ι0 .Ɔ: $f \varepsilon$ qFq . Df = ⎨[$gx \times fx + hx(fx)^{m+1}$] | x, q⎬ .=. ⎨[$(fx)^{-m}$e↑[mS(g; 0,x)]—mS⎨$hx \times$e↑[mS(g; 0,x)] | x; 0,x⎬] | x, q⎬ ε (qFq)const

⎨ Jac. Bernoulli AErud. a.1695 p.553; Joh. Bernoulli Opera t.1 p.175 ⎬

Æquatione in functione f:

$$Dfx = gx \times fx + hx(fx)^{m+1} \text{ pro omni } x,$$

vocare « æquatione differentiale de Bernoulli », nam tractato per ce fratres.

Solutione. Divide per $(fx)^{m+1}$; æquatione fi

$$(fx)^{-m-1}Dfx = (gx)(fx)^{-m} + hx$$

vel

$$D(fx)^{-m} = -m(gx)(fx)^{-m} - mhx$$

que es æquatione differentiale de forma ·2, si nos sume $(fx)^{-m}$ ut functione.

·4 $n \varepsilon N_1$. $a \varepsilon$ Subst n . $h \varepsilon$ Cn f q .Ɔ: $f \varepsilon$ CnFq . Df = [$afx + hx$ | x, q] .=. f = [e^{ax}⎨$f0$ + S[$e^{-ax}hx$ | x; 0,x]⎬ | x, q]

Si n es numero naturale, et a es substitutione de complexos de ordine n, et h es complexo de ordine n functione de numeros reale, tunc functione f complexo de variabile reale, que pro omni x satisfac æquatione

$$Dfx = a(fx) + hx$$

es dato per formula scripto.

Casu $hx = 0$ es tractato in pag. 327 P65.

Dem. [(ιa ¦ q) | g] P·2 Ɔ P

·5 $n \varepsilon N_1$. $u \varepsilon$ (Subst$n$ F q)cont . $a \varepsilon$ Cxn .Ɔ: $x \varepsilon$ Cxn F q . Dx = ux . $x0$ = a .=. x = Σ[⎨[S(uv; 0,z) | z] | v⎬^{r} | r, N_0]a

n es numero naturale. u es substitutione de ordine n functione continuo de variabile reale. a es numero complexo de ordine n. Nos considera x, numero complexo de ordine n functione de variabile reale, que satisfac æquatione Dx = ux, id es systema de n æquationes inter variabiles reale:

$$\begin{aligned} Dx_1 &= u_{11}x_1 + u_{12}x_2 + \ldots + u_{1n}x_n \\ Dx_2 &= u_{21}x_1 + u_{22}x_2 + \ldots + u_{2n}x_n \\ &\ldots\ldots\ldots\ldots \\ Dx_n &= u_{n1}x_1 + u_{n2}x_2 + \ldots + u_{nn}x_n. \end{aligned}$$

Si a es valore de x, respondente ad valore 0 de variabile, tunc valore de xz respondente ad valore z de variabile $=$

$$a + \mathrm{S}(ua; 0,z) + \mathrm{S}[u\mathrm{S}(ua; 0,z)|z; 0,z] +$$
$$\mathrm{S}\{u\mathrm{S}[u\mathrm{S}(ua; 0,z)|z; 0,z]|z; 0,z\} + \ldots$$

id es, xz es summa de serie. Primo termine es a. Pone a in loco de x in secundo membro de æquatione dato, et integra de 0 ad z; resulta secundo termine. Pone secundo termine in loco de x in secundo membro de æquatione dato et integra de 0 ad z; resulta tertio termine; et ita continua.

In generale, si v et w es duo termine successivo de serie de x, vel vz et wz es duo termine successivo de serie de xz, resulta:

$$wz = \mathrm{S}(uv; 0,z), \quad \text{vel} \quad w = \mathrm{S}(uv; 0,z) \,|z,$$

et w es functione de v:

$$[\mathrm{S}(uv; 0,z)|z]|v.$$

Dem.

$z \varepsilon \mathrm{q} \,.\, m = \max \mathrm{mod} u \text{‘} \Theta z \,.\, x = \Sigma[\{[\mathrm{S}(uv; 0,z)|z]|v\}^r \,|r, \mathrm{N}_0]a \;.\supset.$
$\mathrm{mod} xz \leqq \Sigma[m^r (\mathrm{mod} z)^r / r! | r, \mathrm{N}_0] \mathrm{mod} a = \mathrm{e}\text{↖}(m \,\mathrm{mod} z) \,\mathrm{mod} a \,.$
$\mathrm{D}x = \Sigma[u\{[\mathrm{S}(uv; 0,z)|z]|v\}^{r-1} | r, \mathrm{N}_1]a =$
$u\Sigma[\{[\mathrm{S}(uv; 0,z)|z]|v\}^r \,| r, \mathrm{N}_0]a = ux$

In vero, si z es valore de variabile, et m es maximo inter valores de modulo de substitutione u pro valores inter 0 et z, et si nos voca x summa de serie considerato, tunc ce serie, pro valore z de variabile habe suo termines non majore de termines de serie que exprime $\mathrm{e}\text{↖}(mz)a$, que es convergente. Unde serie considerato es convergente.

Derivata de primo termine de serie x vale 0. Derivata de secundo termine vale primo termine multiplicato per u. Derivata de tertio vale secundo multiplicato per u. Et ita porro. Unde $\mathrm{D}x = ux$.

Functio x es evoluto in serie semper convergente, ope «integrationes successivo» vel «approximationes successivo».

Methodo de approximationes successivo occurre in Astronomia, ab longo tempore. Cauchy expone theorema præcedente in casu particulare (Moigno Traité t.2 p.702). Coqué JdM. a.1864 t.9 p.185, et Fuchs AdM a.1870 t.4 p.36 extende ad alios casu. Me enuntia illo in generale in;

TorinoA. a.1887, MA. a.1888 t.32 p.450, TorinoA. a.1897. Vide:

Encyclopädie t.2 p.199.

M. Bôcher, AmericanT. a.1902 t.3 p.196.

Picard JdM. t.6 a.1890 p.145 da novo applicationes de theorema.

§31 Integrale elliptico.

·1 $a,b \varepsilon \mathrm{Q} \,.\, n = (a^2-b^2)/(a^2+b^2) \,.\supset.\, \mathrm{S}[\surd(a^2 \mathrm{c}x^2 + b^2 \mathrm{s}x^2) | x, 2\Theta\pi] =$
$2\pi\surd[(a^2+b^2)/2] \} 1 - 4^{-2}n^2 - \Sigma[n^{2r}(4r)^{-2}\Pi[1-(4s)^{-2}|s, 1\cdots(r-1)]|r, \mathrm{N}_1+1] \{$
$= 2\pi\surd[(a^2+b^2)/2] \} 1 - n^2 4^{-2} - n^4 8^{-2}(1-4^{-2}) -$
$n^6 12^{-2}(1-4^{-2})(1-8^{-2}) - n^8 16^{-2}(1-4^{-2})(1-8^{-2})(1-12^{-2}) - \ldots$
$= 2\pi\surd[(a^2+b^2)/2] \} 1 - \cdot 0625 n^2 - \cdot 0146 n^4 - \cdot 0064 n^6 - \cdot 0036 n^8 - \ldots$]

} Euler, PetrNC. t.18 a.1773 p.71 (edito a.1774) {

Integrale considerato vocare « integrale elliptico de secundo specie »; vide p.392 P4·2-·5

Si $a > b$, et si nos pone $\sin y = \surd(a^2-b^2)/a$ = « eccentricitate », vel $\cos y = b/a$, tunc perimetro de ellipsi de semi-axi a et b vale $4\mathrm{E}y$, ubi $\mathrm{E}y$ es dato per tabula sequente :

Tabula de $\mathrm{E}y = \mathrm{S}\}\surd[1-(\sin y)^2 (\sin x)^2] | x, \Theta\pi/2 \{$

$y =$	0^0	5^0	10^0	15^0	20^0	25^0	30^0	35^0	40^0	45^0
$\mathrm{E}y =$	1·570	1·567	1·558	1·544	1·523	1·498	1·467	1·432	1·393	1·350
$y =$	50^0	55^0	60^0	65^0	70^0	75^0	80^0	85^0	90^0	
$\mathrm{E}y =$	1·305	1·258	1·211	1·163	1·119	1·076	1·040	1·012	1·000	

·2 $a,b \varepsilon \mathrm{Q} \,.\, c = (a+b)/2 \,.\, d = \surd(ab) \,.\supset.$
$\mathrm{S}\}/\surd[(a\mathrm{c}x)^2 + (b\mathrm{s}x)^2] | x, \Theta\pi/2 \{ = \mathrm{S}\}/\surd[(c\mathrm{c}x)^2 + (d\mathrm{s}x)^2] | x, \Theta\pi/2 \{$

} Lagrange TorinoM. a.1784 t.2; Œuvres t.2 p.272 {

Integrale dato es « integrale elliptico de primo specie ».

Gauss t.3 p.360, voca $\pi/\mathrm{S}\}/\surd[(a\,\mathrm{c}x)^2 + (b\,\mathrm{s}x)^2] | x, \Theta\pi/2 \{$ « Arithmetisch-geometrisches Mittel » inter a et b. Ce medio ne varia si nos substitue ad a et b valore medio arithmetico et geometrico de illos.

·3 $a,b \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{S}\}\mathrm{s}(2x)\surd[(a\,\mathrm{s}x)^2 + (b\,\mathrm{c}x)^2] | x, \Theta\pi/2 \{ =$
$2(a^2+ab+b^2)/[3(a+b)]$

§32 Producto de duo serie.

$a,b \varepsilon \mathrm{q'f}\mathrm{N}_0 \,.\, \Sigma(\mathrm{mod}a, \mathrm{N}_0), \Sigma(\mathrm{mod}b, \mathrm{N}_0) \; \varepsilon \mathrm{Q} \,.\supset.$
$\Sigma(a \times b, \mathrm{N}_0) = \mathrm{S}\}\Sigma(a_r \mathrm{e}^{rix}|r, \mathrm{N}_0) \times \Sigma(b_s \mathrm{e}^{-six}|s, \mathrm{N}_0) | x, 2\Theta\pi \{ /(2\pi)$

} Parseval a.1805 ParisSE. t.1 p.639; IdM. a.1894 p.196 {

[Hp $.\supset.\, \mathrm{S}\}[\Sigma(a_r \, \mathrm{e}^{rix}|r, \mathrm{N}_0) \times \Sigma(b_s \, \mathrm{e}^{-six}|s, \mathrm{N}_0)]|x, 2\Theta\pi\{$
$= \mathrm{S}\Sigma\}[a_r \, b_s \, \mathrm{e}^{(r-s)ix}|(r;s), (\mathrm{N}_0 \mathrm{:} \mathrm{N}_0)]|x, 2\Theta\pi\{$
$= \Sigma\}a_r \, b_s \, \mathrm{S}[\mathrm{e}^{(r-s)ix}|x, 2\Theta\pi]|(r;s), (\mathrm{N}_0 \mathrm{:} \mathrm{N}_0)\{ = \Sigma(a_r \, b_r \, 2\pi \,|r, \mathrm{N}_0)$]

§33 Functione de variabile imaginario.

·1 $r \varepsilon Q . f, Df \varepsilon q' F [q' \frown x \ni (\text{mod} x \leq r)] . x \varepsilon q' . \text{mod} x < r . \supset .$
$fx = /(2\pi) S[re^{it} f(re^{it})/(re^{it} - x) | t, 2\Theta\pi]$

Si f es functione imaginario de variabile imaginario, cum derivata, pro omni valore de variabile de modulo minore, aut æquale ad r, id es in circulo de radio r, si x indica valore interno ad circulo, tunc formula exprime fx ope integrale extenso ad circumferentia de circulo.

·2 Hp·1 . $n \varepsilon N_1 . \supset . D^n fx = n!/(2\pi) S[re^{it} f(re^{it})/(re^{it} - x)^{n+1} | t, 2\Theta\pi]$

·3 Hp·1 $. \supset . fx = \Sigma[x^n D^n f0 / n! | n, N_0]$

{ ·1-·3 Cauchy a.1841, œuvres s.1 t.6 p.348 }

Hp de continuitate de Df ne es necessario. Vide
Goursat, AM. t.4 a.1884 p.197-200.
Pringsheim, AmericanT. a.1901 p.413.

§34 Serie de Fourier.

·1 $f \varepsilon q F 2\Theta\pi . \text{Arcu}(f, 2\Theta\pi) \varepsilon Q . gx =$
$S(f, 2\Theta\pi) / (2\pi) + \Sigma\{S[fz\, c(nz - nx) | z, 2\Theta\pi] | n, N_1\} / \pi . \supset :$
$x \varepsilon 2\theta\pi . \supset . gx = [\lim(f, 0\,\overline{}\,x, x) + \lim(f, x\,\overline{}\,2\pi, x)]/2 :$
$g0 = g(2\pi) = [\lim(f, 2\theta\pi, 2\pi) + \lim(f, 2\theta\pi, 0)] /2$

Si f es functione reale definito in intervallo de 0 ad 2π, et ibi suo variatione (indicato per Arcu in p.371 P47) es finito, et si nos voca gx serie scripto, tunc, si x es interno ad intervallo considerato, gx æqua valore medio arithmetico inter limites de f, pro variabile tendente ad x, per valores minore, aut majore de x.

{ Fourier ParisM. a.1807 ; a.1822 p.212 :

« Ces théoremes conviennent à toutes les fonctions possibles, soit que l'on en puisse exprimer la nature par les moyens connus de l'analyse, soit qu'elles correspondent à des courbes tracées arbitrairement ». }

Hypothesi de « variatione limitato » es in Jordan.

·2 $f \varepsilon q' F 2\Theta\pi . f(2\pi) = f0 . \text{Arcu}(f, 2\Theta\pi) \varepsilon Q . x \varepsilon 2\Theta\pi . \supset .$
$fx = \Sigma[e^{rix} S(e^{-rix} fz | z, 2\Theta\pi) | r, n]$

f indica numero imaginario functione de variabile reale in intervallo $0\,\overline{}\,2\pi$. Functione sume valores æquale in extremos

de intervallo, et suo variatione es limitato. Tunc fx es summa de serie scripto. Si nos repræsenta numeros imaginario per punctos in plano, fx repræsenta puncto mobile. Motu de puncto $a_r e^{rix}$ vocare « motu harmonico ». Ergo omni motu periodico es summa de motus harmonico.

·3 $h \varepsilon Q . 0 < h \leqq \pi/2 : f \varepsilon \text{ q F } \Theta h . \text{Arcu}(f, \Theta h) \varepsilon Q :\supset.$
$\lim S[(fx) s(nx)/sx \mid x, \Theta h] \mid n = \pi/2 \lim(f, \Theta h, 0)$

} Dirichlet a.1829 JfM. t.4; Werke t.1 p.127:
« Quelle que soit la fonction $f(\beta)$, pourvu qu'elle reste continue entre les limites 0 et h (h étant positive et tout au plus égale a $\frac{\pi}{2}$), et qu'elle croïsse, ou qu'elle décroisse depuis la première de ces limites jusqu'à la seconde, l'intégrale $\int_0^h f(\beta) \frac{\sin i\beta}{\sin \beta} d\beta$ finira par différer constamment de $\frac{\pi}{2} f(0)$ d'une quantité moindre que tout nombre assignable, lorsqu'on y fait croitre i au delà de toute limite positive ». {

·4 Hp ·3 . $k \varepsilon \theta h .\supset. \lim S[(fx)s(nx)/sx \mid x, k-h] \mid n = 0$
} Dirichlet id. p.128 {

§35 Limite de integrale.

·1 $u \varepsilon \text{Cls'q} . u = \delta u . a \varepsilon u . v \varepsilon \text{Intv} . f \varepsilon [\text{q f}(u:v)]\text{cont} .\supset.$
$\lim\{ S[f(x,y) \mid y, v] \mid x, u, a\} = S[f(a,y) \mid y, v]$ Comm(lim,S)

u es classe de quantitates, perfecto; a es puncto in u; v es intervallo; f indica quantitate reale functio continuo de systema de duo variabile, uno in campo u, altero in intervallo v. Tunc si nos integra $f(x,y)$ pro y, in intervallo v, limite de integrale, ubi x varia in u et verge ad a, vale integrale, ubi in loco de x nos pone a. Exprime proprietate commutativo de operationes limite et integrale.

Dem. $x \varepsilon u$. §S P18·1 $.\supset. \text{mod}\{S[f(x,y) \mid y,v] - S[f(a,y) \mid y,v]\} \leqq$
$S\{\text{mod}[f(x,y) - f(a,y)] \mid y,v\}$ (1)

$h \varepsilon Q$. §cont P1·3 $.\supset. \exists\, Q \frown k \ni \{x \varepsilon u . \text{mod}(x-a) < k . y \varepsilon v .\supset_{x,y}.$
$\text{mod}[f(x,y) - f(a,y)] < h$ (2)

$h \varepsilon Q . (1) . (2) .\supset. \exists\, Q \frown k \ni (x \varepsilon u . \text{mod}(x-a) < k .\supset_x .$
$\text{mod}\{S[f(x,y) \mid y,v] - S[f(a,y) \mid y,v]\} < h \times \text{Long}v) .\supset. \text{P}$

In vero, modulo de differentia de integrales de $f(x,y)$ et de $f(a,y)$ es minore de integrale de modulo de differentia $f(x,y) - f(a,y)$.

Nunc, dato quantitate positivo h, causa theorema de continuitate uniforme (pag. 239), lice determina alio quantitate positivo k, ut pro omni

valore de x in u, differente de a minus que k, in valore absoluto, et pro omni valore de y in v, differentia $f(x,y) - f(a,y)$ habe modulo inferiore ad h.

Tunc, modulo de differentia de duo integrale es minore que h multiplicato per longore de intervallo de integratione ; ce producto es parvo ad arbitrio ; unde seque theorema.

·2 $\text{Hp}\cdot 1 \;.\supset.\; \text{S}[f(x,y) \,|y, v] \,|x \;\varepsilon\; (\text{qf}u)\text{cont}$ [P·1 $\supset$ P]

Ergo, integrale de functione continuo de systema de duo variabile, facto pro uno, es functione continuo de altero.

Nos suppone continuitate composito de $f(x,y)$. Non suffice suo continuitate in sensu diviso, pro x, et pro y.

Exemplo de integrale discontinuo (pag. 355 P22·1) :

$a\varepsilon\text{Q} \;.\supset.\; \text{S}[a/(a^2+x^2) \,|x, \text{q}] = \pi$

Si $a<0$, integrale $= -\pi$; ergo es functione de a discontinuo pro $a=0$.

$\text{Lm}[a/(a^2+x^2) \,|(x,a), (\text{q}{:}\text{q}), (0,0)] = \text{q} \cup \iota\pm\infty$

Classe limes de functione integrando, si varia dyade (x,a) in campo de dyades de quantitate, et verge ad dyade $(0,0)$, es classe totale de quantitates finito et infinito. Functione integrando non es continuo in $(0,0)$.

Integrale (pag 361 P34·1) $\quad a\varepsilon\text{q} \;.\supset.\; 1/\pi\, \text{S}[\sin(ax)/x \,|x, \text{q}] = \text{sgn}a$
es functione discontinuo de a. Campo de integratione es infinito.

·3 $(n\varepsilon\text{N}_1 \,.\, u\varepsilon \text{Cls'C}n) \,|\, (u\varepsilon\, \text{Cls'q})$ §S P51

Definitione de extremos oscillatorio, dato in pag. 374, pro functione reale de variabile reale, subsiste pro functione reale de variabile complexo.

·4 $u\varepsilon\, \text{Cls'q} \,.\, a\varepsilon\, \delta u \,.\, v\varepsilon\, \text{Intv} \,.\, f\varepsilon\, \text{qf}(u{:}v) \,.\, \text{l}'f`(u{:}v), \text{l}_{,}f`(u{:}v)\, \varepsilon\text{Q}$

$$.\supset.\; \begin{matrix}\max\\ \min\end{matrix}\; \text{Lm}\{\text{S}[f(x,y)|y,v] \,|x, u, a\} \;\begin{matrix}\leqq \text{S}'[\text{l}'f(a,y)\,|y, v] \\ \geqq \text{S}_{,}[\text{l}_{,}\text{———}]\end{matrix} \;.$$

Si u es classe de quantitates, a es valore, finito aut non, in classe derivata de u, v es intervallo, f es functione reale de duo variabile, uno in u, altero in v, cum limites supero et infero finito, tunc classe limes de integrale es inter integrales supero et infero de extremos oscillatorio (que P·3 defini pro functiones de duo variabile) de functione f pro (a,y), ubi varia y in v. Si ce duo integrale coincide, existe limite de integrale dato, æquale ad valore commune de illos.

Si f es summa de primos x termine de serie, et termines depende de y, resulta novo conditione pro integrabilitate de serie, plus generale de regula in pag. 364, P40. Vide :

Osgood, *Non-uniform convergence and the Integration of Series Term by Term*, AmericanJ. t.19 a.1896 p.155.

Richardson, American B. a.1907 p.431.

§36 Derivata de integrale.

$u,v\varepsilon$ Intv . $f\varepsilon$qF$(u{:}v)$. $D_2 f\varepsilon$ qF$(u{:}v)$ cont . $y\varepsilon v$.⊃.
$D\{S[f(x,y)\,|x,u]\,|y,v\}y = S[D_2 f(x,y)\,|x,u]$

Comm(D,S)

Si f es functione de duo variabile in intervallos u et v, cum derivata pro y continuo, tunc derivata pro y de integrale pro x de $f(x,y)$, æqua integrale pro x de derivata pro y de $f(x,y)$.

Dem.

$a,x\varepsilon u$.⊃. $D\{S[f(x,y)|x;\,a,x]\,|x,u\}x = f(x,y)$
.⊃. $D(D\{S[f(x,y)|x;\,a,x]\,|x,u\}x\,|y,v)y = D_2 f(x,y)$
.⊃. $D(D\{S[f(x,y)|x;\,a,x]\,|y,v\}y\,|x,u)x = D_2 f(x,y)$
.⊃. $D\{S[f(x,y)|x;\,a,x]\,|y,v\}y = S[D_2 f(x,y)|x;\,a,x]$

Sume integrale inter duo limite a et x. Tunc derivata pro x de integrale pro x de $f(x,y)$ vale $f(x,y)$. Vide pag. 348 P12·1. Ergo derivata pro y de derivata pro x de integrale pro x vale derivata pro y. Commuta derivatas (pag. 329 P·4) : Derivata pro x de derivata pro y de integrale pro x vale derivata pro y. Integra : Derivata pro y de integrale pro x vale integrale pro x de derivata pro y.

§37 Commutatione de integrationes.

$u,v\varepsilon$ Intv . $f\varepsilon$ [qf$(u{:}v)$]cont .⊃. $S\{S[f(x,y)\,|x,u]\,|y,v\} =$
$S\{S[f(x,y)\,|y,v]\,|x,u\}$

Dem.

$a,x\varepsilon u$.⊃. $D(S\{S[f(x,y)|x;\,a,x]\,|y,v\}\,|x,u)x =$
$S(D\{S[f(x,y)|x;\,a,x]\,|x,u\}x\,|y,v) = S[f(x,y)|y,v]$.⊃. P

Sume integrale pro x ab a ad x. Tunc derivata pro x de integrale pro y de integrale pro x vale integrale pro y de derivata pro x de integrale pro x, id es vale integrale pro y. Integra pro x, et resulta propositione.

Si hypothesi non es satisfacto, commutatione de integrales pote es non licito. Exemplo (Cauchy a.1814 Œuvres t.1) :

$S\{S[(y^2-x^2)/(x^2+y^2)^2\,|x,\theta]\,|y,\theta\} = +\pi/4$
» » » » $|y,\theta]\,|x,\theta\} = -\pi/4$

Vide alios theorema in :
O. Stolz, Grundzüge der Differential- und Integralrechnung, t.3 a.1899.

§38 Integrale multiplo.

$n \varepsilon \mathrm{N}_1 \,.\supset.\, \Theta_n = \Theta\, \mathrm{F}(1 \cdots n)$ Df

Si n es numero naturale, Θ_n indica successione de n quantitate inter 0 et 1, id es numero complexo de ordine n, de omni coordinata es Θ; id es cubo ad n dimensione de latere 1.

✻ 1.

$n \varepsilon \mathrm{N}_1 \,.\, f \varepsilon\, \mathrm{qF\,C}n \,.\, \mathrm{l'mod}\, x \ni (fx -= 0)\, \varepsilon \mathrm{Q} \,.\, \mathrm{l'mod}\, f\text{‘}\mathrm{C}n\, \varepsilon \mathrm{Q} \,.\supset:$

·0 $h \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{s'}(f,h) = h^n \Sigma[\mathrm{l'}f\text{‘}\, h(p+\Theta_n) \,|\, p, \mathrm{nF}\, 1 \cdots n]$ Df

·01 ——— $\mathrm{s}_{,}$———— $\mathrm{l}_{,}$———————— —

·1 $\mathrm{S'}f = \mathrm{l}_{,}\, \mathrm{s'}(f,h) \,|\, h\text{‘}\mathrm{Q}$ Df

·11 $\mathrm{S}_{,}$—— $\mathrm{l'}$ $\mathrm{s}_{,}$——— —

·2 $\mathrm{S}f = \imath(\iota \mathrm{S'}f \frown \iota \mathrm{S}_{,}f)$ Df

Es dato numero naturale n, et f, quantitate reale functione definito de complexos de ordine n, vel in spatio ad n dimensione. Id es, f es functione reale de n variabile reale. Nos suppone que, pro valores satis magno de variabiles, functione es sempre nullo; id es, limite supero de modulos de complexos x tale que fx non es nullo, es finito. Et nos suppone que limite supero de valores absoluto de f es finito.

Tunc fixa quantitate positivo h, et divide spatio ad n dimensione in cubos de latere h. Uno vertice de uno cubo es

$$(p_1 h, p_2 h, \ldots. p_n h) = h(p_1, p_2, \ldots p_n) = hp$$

ubi $p_1, p_2, \ldots. p_n$ es numero integro, positivo aut negativo aut nullo; id es $p \varepsilon\, \mathrm{nF}1 \cdots n$, p es successione de n·numero integro. Cubo, que habe pro vertice de coordinatas minimo, puncto præcedente habe pro expressione $h(p+\Theta_n)$.

Nos indica per $\mathrm{s'}(f,h)$ producto de h^n per summa de limites supero de functione f in singulo cubo.

$\mathrm{s}_{,}(f,h)$ es producto de h^n per summa de limites infero.

$\mathrm{S'}f$, « integrale supero de functione definito f » es limite in fero de $\mathrm{s'}(f,h)$, ubi varia h, et sume omni valore positivo. In modo analogo, nos defini integrale infero, et integrale proprio.

Functione f es « definito », repræsentato per symbolo F; ergo nos pote loque de campo de variabilitate de f (vide p.80): Variab $f = \mathrm{C}n$.

Si in idea de functione (f) non es implicito suo campo de variabilitate, ut es casu commune in Analysi, nos pone defitiones sequente:

❋ 2. $n \varepsilon \mathrm{N}_1 \,.\, u \varepsilon \mathrm{Cls'C}n \,.\, \mathrm{l'mod}u \, \varepsilon \mathrm{Q} \,.\, f\varepsilon \, \mathrm{q}f u \,.\, \mathrm{l'mod}`f`u \, \varepsilon \mathrm{Q} \,.\supset.$

·0 $\mathrm{S}(f,u) = \mathrm{S}[(f,u) \cup (\iota 0 \,\mathrm{:}\, \mathrm{C}n - u)]$ Df

·1 $\mathrm{S}' \mid \mathrm{S}$ P·0 . $\mathrm{S}_, \mid \mathrm{S}$ P·0

Si u es classe de complexos de ordine n, vel campo in spatio ad n dimensione, et es limitato; et si f es quantitate functione dato in campo u, et limite supero de valores absoluto de f in campo u es finito, tunc:

$\mathrm{S}(f,u)$, « integrale de functione f, in campo u » es integrale de illo functione definito, que in campo u vale f, et in campo de complexos non u vale 0. In modo analogo pro integrales supero et infero.

Integrale in campo ad n dimensione vocare « integrale multiplo de ordine n ». Plure theorema super integrales simplice mane pro integrale multiplo:

§S (p.341) P3·2 (p.342) P4·1·2 (p.345) P7·0·1·2 (p.346) P·3·4·5 (p.347) P9 (p.348) P10·1·2 (p.375) P51·1

P6·0 (p.344) sume forma (vide P3):

$v,w \varepsilon \mathrm{Cls}'u \,.\, \mathrm{S}(\lambda v \cap \lambda w) = \wedge \,.\supset.\, \mathrm{S}'(f, v \cup w) = \mathrm{S}'(f,v) + \mathrm{S}'(f,w)$

❋ 3. Hyp P2 $.\supset.$

·0 $\mathrm{S}u = \mathrm{S}(\iota 1 \,\mathrm{:}\, u)$ $\mathrm{S}', \mathrm{S}_, \mid \mathrm{S}$ P·0 Df

Si, ut in P2, u es classe de complexos de ordine n, limitato, integrale de functione que habe valore constante 1 in campo u, (et 0 ex u) responde ad volumen (p.379) pro spatio geometrico. Me indica illo cum notatione $\mathrm{S}u$ « integrale de campo u ». Si illo non existe, tunc nos considera integrales, supero et infero.

·1 $\mathrm{S}'u - \mathrm{S}_,u = \mathrm{S}'\mathrm{am}u$

Responde ad P48·3 (pag.373).

·2 $\mathrm{S}'u = \mathrm{S}'\lambda u \;.\; \mathrm{S}_{,}u = \mathrm{S}_{,}\mathrm{in}u$

·3 $\mathrm{S}u\,\varepsilon\mathrm{Q} \,.\supset.\, \mathrm{S}\,\lambda u = \mathrm{S}\delta u = \mathrm{S}\,\mathrm{in}u = \mathrm{S}\,u$

Volumen supero de u æqua volumen supero de campo limite; volumen infero æqua illo de campo interno. Si u habe volumen proprio, isto æqua volumen de campo limite, de campo derivato, et de campo interno.

Signo de integrale multiplo habe forma S (non ∫) in Lagrange Œuvres t.11 p.85, et in Jordan *Cours d'Analyse* a.1892 t.1 p.37.

Suo definitione symbolico es in Formul. t.3 a.1901. Novo propositiones, que nunc (a.1907) me adde, es in:

J. Pierpont, *On multiple integrals,* American T. a.1905, t.6, p.416-434.

* 4. Hyp P1 $.\supset\therefore$

·1 $\mathrm{S}f\,\varepsilon\mathrm{q} \,.=:\, h\varepsilon\mathrm{Q} \,.\supset_h.\, \mathrm{S}'\,x\ni(\mathrm{O}fx > h) = 0$

Ut functione f definito de complexos de ordine n, es integrabile, es necesse et suffice que, dato quantitate positivo h, volumen supero de punctos x tale que oscillatione de functione f in puncto x supera h, vale zero. Responde ad criterio de integrabilitate de Riemann et Du Bois-Reymond, scripto in pag. 375 P51·2.

·2 $p,q\varepsilon\mathrm{N}_1 \,.\, p+q=n \,.\supset.\, \mathrm{S}'f \geqq \mathrm{S}'\{\mathrm{S}'[f(x,y)\,|x,\,\mathrm{C}p]\,|y,\,\mathrm{C}q\}$

$\qquad .\, \mathrm{S}_{,}f \leqq \mathrm{S}_{,}\{\mathrm{S}_{,}$ ———————————

Decompone numero naturale n in duo parte p et q. Complexo de ordine n pote es scripto (x,y), ubi x es complexo de ordine p, et y de ordine q.

Integrale supero de f, in campo de complexos de ordine $p+q$ es majore aut æquale ad integrale supero, pro y in campo $\mathrm{C}q$, de integrale supero pro x in campo $\mathrm{C}p$, de $f(x,y)$. In modo simile pro integrales infero.

Resultato de plure integratione successivo vocare « integrale iterato ». Propositione liga integrale multiplo cum integrale iterato.

Dem.

$h\varepsilon\mathrm{Q} \,.\, r\varepsilon\,\mathrm{nF}1\cdots p \,.\, s\varepsilon\,\mathrm{nF}1\cdots q \,.\, y\varepsilon\, h(s+\Theta_q) \,.\supset.$

$\mathrm{S}'[f(x,y)\,|x,\,h(r+\Theta_p)] \leqq h^p\,\mathrm{l}'f`[h(r+\Theta_p),\,y]$

$\leqq h^p\,\mathrm{l}'f`[h(r+\Theta_p) \,;\, h(s+\Theta_q)] \,.\supset.$

$\mathrm{S}'\{\mathrm{S}'[f(x,y)\,|x,\,h(r+\Theta_p)]\,|y,\,h(s+\Theta_q)\} \leqq h^n\,\mathrm{l}'f`[$ » $]$

$.\supset.\ \mathrm{S}\,|\,\mathrm{S}'[f(x,y)\,|x,\ \mathrm{C}p]\,|y,\ \mathrm{C}q\,|\ \leqq \mathrm{s}'(f,h)$
$.\supset.\ \mathrm{S}'|\mathrm{S}'[\qquad »\qquad |\ \leqq \mathrm{S}'f$

Sume h quantitate positivo, r successione de p integro, s successione de de q integro. Tunc

$$[(r+\Theta_p)h \,\vdots\, (s+\Theta_q)h]$$

repræsenta cubo ad $n=p+q$ dimensione de basi ad p dimensione et de altitudine ad q dimensione.

Sume y in altitudine de ce cubo. Integrale $f(x,y)$ ubi x varia in basi es minore de dimensione de basi h^p per limite supero de valores de $f(x,y)$ ubi x varia in basi; ergo es minore de h^p per limite supero de valores de $f(x,y)$ ubi x varia in basi, et y in altitudine.

Integra pro y in altitudine; integrale iterato es minore de producto de duo dimensione $h^q \times h^p = h^n$, per limite supero de valores de f in cubo.

Summa pro r et s:

Integrale iterato es minore de summa, indicato (pag.417) per $\mathrm{s}'(f,h)$.

Sume limite infero pro h; resulta theorema.

❋ 5·1

$m,n\varepsilon\mathrm{N}_1 \,.\, f\varepsilon\,\mathrm{C}m\,\mathrm{F}\,\mathrm{C}n \,.\, \mathrm{l}'\mathrm{mod}\,x\ni(fx=0)\ \varepsilon\mathrm{Q} \,.\, \mathrm{l}'\mathrm{mod}f`\mathrm{C}n\ \varepsilon\mathrm{Q}\ .\supset.$
$\mathrm{S}f = \iota\,\mathrm{Cx}m \frown z\ni\{h\varepsilon\mathrm{Q}\ .\supset_h.\ z\varepsilon\, h^n\Sigma[\mathrm{Med}f`[h(p+\Theta_n)]\,|p,\ \mathrm{nF}\,1\cdots n]\}$ Df

Si f es numero complexo de ordine m functione de complexos de ordine n, integrale supero et infero non habe sensu. Tunc pro defini integrale, fixa quantitate positivo h, parvo ad arbitrio, et divide spatio ad n dimensione in cubos de latere h. Forma summa de valores medio (Med) de functione in totos cubo, et multiplica per h^n, volumen de cubo. Si existe uno et uno solo valore z pertinente ad ce summas, z es integrale quæsito.

·2 $a\varepsilon\,\mathrm{Subst}\,n\ .\supset.\ \mathrm{S}\,a`\Theta = \mathrm{mod\ Dtrm}\,a$

Si a es substitutione de complexos de ordine n, vel homographia (pag.148), tunc volumen de parallelepipedo que responde in homographia a ad cubo de latere 1 vale modulo de determinante de homographia.

·3 $m,n\varepsilon\mathrm{N}_1 \,.\, u\varepsilon\,\mathrm{Cls'C}n \,.\, \mathrm{l'mod}u\ \varepsilon\mathrm{Q} \,.\, g\varepsilon\,\mathrm{C}n\,\mathrm{F}\,u\ \mathrm{sim}\,.$
$\mathrm{D}g\ \varepsilon\ (\mathrm{Subst}\,n)\mathrm{F}u\ \mathrm{cont}\,.\ f\varepsilon\,\mathrm{C}m\,\mathrm{f}(g`u)\,.\ \mathrm{S}(f, g`u)\ \varepsilon\ \mathrm{C}m\ .\supset.$
$\mathrm{S}(f, g`u) = \mathrm{S}(fgx \times \mathrm{mod\ Dtrm\ D}gx\ |x,\ u)$

{ LAGRANGE BerlinM. a.1773, pro $n=3$;

JACOBI a.1833. JfM. t.12 p.1, Werke t.3 p.233:

Sint ξ_1, ξ_2, ... datae functiones quaelibet variabilium v_1, v_2, ..., habetur: $d\xi_1\, d\xi_2 \ldots = \left(\Sigma \pm \frac{\partial\xi_1}{\partial v_1}\frac{\partial\xi_2}{\partial v_2}\ldots\right) dv_1\, dv_2 \ldots$ ».}

Si u es campo ad n dimensione limitato, et si g es correspondentia simile inter complexos u et alios complexo, que habe derivata, substitutione de complexos de ordine n, ut es definito in pag. 330, continuo, et si f es complexo de ordine arbitrario m functione de campo $g`u$, id es in campo imagine de $u$ in repræsentatione $g$, et es integrabile, tunc integrale de $f$ in campo $g`u$ vale integrale de $fgx \times$ mod Determinante $\mathrm{D}gx$, ubi varia x, in campo u.

Dtrm Dg dicere « determinante Jacobiano ».

✻ 6. vct S

·1 $u \varepsilon \mathrm{Cls'p} \,.\, \mathrm{l'm}(u-u) \varepsilon \mathrm{Q} \,.\, f \varepsilon \mathrm{qf}u \,.\, \mathrm{l'm}f`u \,\varepsilon \mathrm{Q} \,.\supset.$
$\mathrm{S}(f,u) = \iota\, \mathrm{q} \cap z \ni [o \varepsilon \mathrm{p} \,.\, i,j,k \varepsilon \mathrm{v} \,.\, i^2 = j^2 = k^2 = 1 \,.\, i \times j = j \times k = k \times i = 0$
$.\, h \varepsilon \mathrm{Q} \,.\supset_{o,i,j,k,h}.\, z \varepsilon h^3 \Sigma \{ \mathrm{Med}f` u \cap [o + (x+\theta)hi + (y+\theta)hj + (z+\theta)hk]$
$|(x,y,z), (x,y,z) \ni \{ x,y,z \varepsilon \mathrm{n} \,.\, \exists\, u \cap [o + (x+\theta)hi + (y+\theta)hj + (z+\theta)hk] \}$
Def

Si u es classe de punctos, vel figura, et si distantia inter duo puncto de u habe limite supero finito, vel si figura u non se extende ad infinito, et si f es functione reale dato in campo u, et limite supero de suo valores absoluto es finito, tunc nos defini integrale de functione f, in campo u ut seque.

Sume puncto-origine o, tres vectore-coordinato i,j,k unitario et orthogonale, et quantitate positivo h. Divide toto spatio in cubos de forma $o+(x+\Theta)hi+(y+\Theta)hj+(z+\Theta)hk$, ubi x,y,z es numero integro, positivo aut negativo.

Considera classe de valores medio inter valores sumpto per f in parte de campo u que es in cubo præcedente. Summa ce classes, ubi varia x,y,z, et sume omni terna de valores tale que cubo correspondente habe aliquo puncto commune cum u.

Si existe uno et uno solo quantitate z, que pertine ad classe præcedente, pro omni electione de elementos o,i,j,k,h, illo z es $\mathrm{S}(f,u)$.

·2 $\mathrm{Hyp} \cdot 1 \,.\supset.\, \mathrm{S}'(f,u) = \mathrm{l}_{,}\, z \ni \{ \exists\, (o,i,j,k,h) \ni [o \varepsilon \mathrm{p} \,.\, i,j,k \varepsilon \mathrm{v} \,.\, i^2 = j^2$
$= k^2 = 1 \,.\, i \times j = j \times k = k \times i = 0 \,.\, h \varepsilon \mathrm{Q} \,.\, z = h^3 \Sigma \{ \mathrm{l'}f` u \cap [o + (x+\theta)hi +$
$(y+\theta)hj + (z+\theta)hk] \,|(x,y,z)$, ut in P·1$\}$ Def

·3 $(\mathrm{S}_{,}, \mathrm{l'}, \mathrm{l}_{,}) \mid (\mathrm{S}', \mathrm{l}_{,}, \mathrm{l'})$ P·2 Def

·4 (pnt, vct, Cx) | q P·1

In modo analogo pote es definito integrale supero, et integrale infero. Definitione P·1 de integrale proprio subsiste si f es puncto, aut vectore, aut complexo, functione in campo u.

·5 Hyp·1 . $o\varepsilon$p . $i,j,k\varepsilon$v . $i^2=j^2=k^2=1$. $i\times j=j\times k=k\times i=0$.⊃. $S(f,u) = S[f(o+xi+yj+zk) |(x,y,z), (x,y,z)\ni (x,y,z\varepsilon q . o+xi+yj+zk \ \varepsilon\ u)]$

✻ 7·1 $u\varepsilon$ Cls'p . l'm$(u-u)\varepsilon$Q . $o\varepsilon$p .⊃.
$S_{,}\{$area $(\lambda u)\frown x\ni[d(x,o)=r]\ |r,$ Q$\}$ ≧ Volum'u ≧ Volum$_{,}u$ ≧
S'$\{$area in$u\frown x\ni[d(x,o)=r]\ |r,$ Q$\}$

Volumen de figura u pote es calculato ut seque. Centro in puncto ad arbitrio o, nos imagina superficie de sphæra de radio r. Integrale infero de area suo sectione cum figura limite de u, volumen externo, volumen interno, et integrale supero de area sectione cum figura interno ad u, ubi varia r, et sume omni valore positivo, es disposito per ordine de magnitudine. Si primo et ultimo integrale coincide, valore commune es volumen de figura. Solido es decomposito in stratos sphærico.

·2 $o\varepsilon p_2$. $u\varepsilon$ Cls'p .⊃. Ths P·1

Decompositione in stratos cylindrico (*tunicae cylindraceae* de Kepler a.1615 t.4 p.584).

$c\varepsilon$p . $h\varepsilon$Q . $u\varepsilon$ Cls'p .⊃:

·3 arc u εQ .⊃. arc [Homot(c,h)'u] = h arc u

·4 area$u$ εQ .⊃. area[Homot(c,h)'u] = h^2 area$u$

·5 Volum$u$ εQ .⊃. Volum[Homot(c,h)'u] = h^3 Volum$u$

{ Cavalieri a.1635 l.2 P15, P17:

« Omnes figurae planae similes sunt inter se in dupla ratione linearum, sive laterum homologorum, earundem.

Omnia similia solida sunt in tripla ratione linearum, vel laterum homologorum, quae sunt in eorundem homologis figuris ». }

·6 $a\varepsilon p_2$. $o\varepsilon a$. $r\varepsilon$Q . $u\varepsilon$ Cls'p$\frown x\ni[d(x,a)=r]$. area$u$ εQ .⊃.
Volum$(o^- u) = r \times$ area$u$ /3

Volumen de projectione de area u in superficie de cylindro ab puncto de axi vale radio per area de u, diviso 3.

·7 $o\varepsilon$p . $a,b,c\varepsilon$ v-ι0 .⊃. area{[$(o+\mathrm{Q}a+\mathrm{Q}b+\mathrm{Q}c)\cap x\ni$[d$(x,o)=1$]}
$=\mathrm{ang}(a;b,c)+\mathrm{ang}(b;c,a)+\mathrm{ang}(c;a,b)-\pi$

{ Harriot, a.1603; LoriaB. a.1902, p.1:

« Inveni rationem accuratam mensurandi superficies triangulorum sphaericorum 18: sep. 1603.

« Et est talis; Adde simul omnes angulos trianguli inde detrahe 180 quod superest fac numeratorem ad 360. Dico quod illa fractio exprimit partes haemisphaerii quae continet triangul: vel tot gradus numera in circulo magno quot sunt in numeratore et a polo illius circuli descendunt duo quadrantes terminantes illos gradus dico quod hoc triangulum aequatur triangulo sphaerico praedicto ».

Girard, a.1629 fol.38v; Cavalieri, a.1632 p.316 }

Dato puncto o, et tres vectore a,b,c, non nullo, tunc area de figura commune ad triedro $o+\mathrm{Q}a+\mathrm{Q}b+\mathrm{Q}c$, id es ad triedro de vertice o, et de lateres parallelo ad a,b,c, et ad superficie de sphæra de centro o et de radio 1, id es area de triangulo sphærico, vale summa de angulos de triangulo minus 180°. Ce differentia vocare « excessu sphærico » de triangulo.

* 8·1 $o\varepsilon$p . $i,j,k\,\varepsilon$v . $i^2=j^2=k^2=1$. $i\times j=j\times k=k\times i=0$.
$a,b,c\,\varepsilon$Q . $a \geqq b \geqq c$.⊃. Volum{$(o+x_1i+x_2j+x_3k)|x$ ‘ Cx3 $\cap x\ni$
$[(x_1/a)^2+(x_2/b)^2+(x_3/c)^2\leqq 1]$} $=4\pi abc/3$

·2 Hp·1 . $E(a,b,c)=$ area{$(o+x_1i+x_2j+x_3k)|x$‘ Cx3 $\cap x\ni[(x_1/a)^2$
$+(x_2/b)^2+(x_3/c)^2=1]$} . $E_1(a,b,c)=4\pi ab$. $E_2(a,b,c)=$
$4\pi b\,(a+c)/2$. $E_3(a,b,c)=4\pi\,(ab+ac+bc)/3$.⊃.
$E_1(a,b,c) \leqq E(a,b,c) \leqq /2E_1(a,b,c)$. $E(a,b,c)>43/48\ E_2(a,b,c)$.
$4/15\ \pi b(a-c)^2/c \leqq E(a,b,c)-E_3(a,b,c) \leqq /5\ \pi b(a-c)^2/c$

Volumen et area de ellipsoide; vide LinceiR. a.1890 s.4 t.6 p.317.
Ugo Dainelli, BattagliniG. a.1878 t.16, p.291 da E_3 ut valore approximato de area.

·3 $o\varepsilon$p . $a,b\varepsilon$v . mod$a$ = mod$b$ = 1 . $a\times b=0$.⊃.
Volum p $\cap x\ni$[d$(x,o)<1$. d[x, recta$(o+a/2,\,b)$] $</2$] $=2\pi-8/9$.
area p $\cap x\ni$[» $=1$. » » $</2$] $=2\pi-4$.
area p $\cap x\ni$[» <1 . » » $=/2$] $=4$.
arc p $\cap x\ni$[» $=1$. » $=/2$] = S[√(cz^2+2sz^2) | z, 2Θπ]

{ Viviani, Vide ActaErud. a.1692 p.274 }

Dato puncto o, et duo vectore unitario et orthogonale a et b, tunc nos considera solido loco de punctos que dista ab o minus

que uno, vel in sphæra de centro o et de radio 1, et que dista ab recta que i per puncto $o+a/2$, et es parallelo ad b, minus que 1/2, vel es in cylindro de dato axi. Tunc formula exprime volumen de solido, area de sphæra interno ad cylindro, area de cylindro interno ad sphæra, et arcu de curva intersectione de cylindro cum sphæra.

✻ 9. Centro de Gravitate.

·0 $u \varepsilon \mathrm{Cls'p} \,.\, \mathrm{l'm}(u-u)\, \varepsilon \mathrm{Q} \,.\, f \varepsilon \mathrm{qf}u \,.\, \mathrm{S}(f,u)\, \varepsilon\, \mathrm{q}-\iota 0 \;.\supset.$
$\mathrm{G}(f,u) = \mathrm{S}[(fx)x \,|x, u]/\mathrm{S}(f,u)$ Def

Si ad omni puncto x de figura limitato u responde numero fx, que nos voca « densitate de figura in puncto x », tunc barycentro, vel centro de gravitate de figura u, cum densitate f, indicato per $\mathrm{G}(f,u)$, vale integrale de puncto x, cum massa fx, ubi x varia in campo u, diviso per massa totale de figura, que nos suppone non nullo.

·1 $u \varepsilon \mathrm{Cls'p} \,.\, \mathrm{l'm}(u-u)\, \varepsilon \mathrm{Q} \,.\, \mathrm{Volum}u\, \varepsilon \mathrm{Q} \;.\supset.$
$\mathrm{G}u = \mathrm{G}(\iota 1 \,\vdots\, u) = \mathrm{S}(\mathrm{idem}, u) / \mathrm{Volum}u$ Def

« barycentro de solido homogeneo u ».

·2 $u \varepsilon \mathrm{Cls'p} \,.\, \mathrm{l'm}(u-u)\, \varepsilon \mathrm{Q} \,.\, \mathrm{Volum}u = 0 \;.\supset.$
$\mathrm{G}u = \lim[\mathrm{G}\{\mathrm{p} \frown x \ni [\mathrm{d}(x, u) < h]\} | h, \mathrm{Q}, 0]$ Def

« barycentro de superficie et de linea », es barycentro de stratu de spissore infinitesimo $2h$, vel de filo de sectione infinitesimo de radio h, circa superficie aut linea dato.

·3 $a,b \varepsilon \mathrm{p} \;.\supset.\; \mathrm{G}(a\frown b) = (a+b)/2$

·4 $o \varepsilon \mathrm{p} \,.\, u,v \varepsilon \mathrm{v} \;.\supset.\; \mathrm{G}(o+\theta u+\theta v) = o+(u+v)/2$

Barycentro de parallelogrammo super vetice o et vectores u et v.

·5 $a,b,c \varepsilon \mathrm{p} \;.\supset.\; \mathrm{G}(a\frown b\frown c) = (a+b+c)/3$

Barycentro de triangulo, ut figura, es barycentro de vertices.

·4 ·5 Archimede *Κέντρα βαρῶν* P 10, 14.

·6 $a,b,c,d \varepsilon \mathrm{p} \;.\supset.\; \mathrm{G}(a\frown b\frown c\frown d) = (a+b+c+d)/4$

·7 $o \varepsilon \mathrm{p} \,.\, r \varepsilon \mathrm{Q} \,.\, i \varepsilon \mathrm{v} \,.\, \mathrm{m}i = 1 \;.\supset.$
$\mathrm{G}\; \mathrm{p} \frown x \ni [(x-o) \times i > 0 \,.\, (x-o)^2 < r^2] = o+3ri/8$

Barycentro de semisphæra de centro o, radio r, secundo vectore i.

·8 $a\varepsilon p_3 . u\varepsilon \mathrm{Cls}'a . \mathrm{Area}\, u\, \varepsilon Q . i\varepsilon v .\supset. G(u+\theta i) = Gu+i/2$

·9 » » » $. o\varepsilon p .\supset. G(o^- u) = (o+3Gu)/4$

GALILEI, *dialoghi* t.13 p.283 :

Cujuslibet coni, vel pyramidis centrum gravitatis axem dividit, ut pars ad verticem reliquæ ad basin sit tripla.

❋ 10. MOMENTO DE INERTIA.

·1 $u\varepsilon \mathrm{Cls}'p . f\varepsilon \mathrm{qF}u . a\varepsilon p \cup p_2 \cup p_3 .\supset.$
$\mathrm{Inertia}(f,u,a) = S\{fx[d(x,a)]^2 | x, u\}$ Df

« momento de inertia pro puncto aut axi aut plano a, de figura u, cum densitate fx in puncto x».

} EULER a.1765; a.1790 p.166:

« *Momentum inertiae* corporis respectu cujuspiam axis est summa omnium productorum, quae oriuntur, si singula corporis elementa per quadrata distantiarum suarum ab axe multiplicentur ». {

·2 $u\varepsilon \mathrm{Cls}'p . f\varepsilon \mathrm{qf}u . S(f,u)\, \varepsilon\, q\text{-}\iota 0 . a\varepsilon p .\supset.$
$\mathrm{Inertia}(f,u,a) = \mathrm{Inertia}[f,u, G(f,u)] + d[a, G(f,u)]^2\, S(f,u)$

·3 $\mathrm{Hp}\cdot 2 . i\varepsilon v\text{-}\iota 0 .\supset. \mathrm{Inertia}(f, u, a+qi) =$
$\mathrm{Inertia}[f, u, G(f,u)+qi] + d[a, G(f,u)+qi]^2 \times S(f\, u)$

} EULER id. p.168:

« Si... detur momentum inertiae respectu cujuspiam axis per centrum inertiae corporis transeuntis, momentum inertiae respectu alius cujusvis axis illi paralleli superat illud producto ex massa in quadratum distantiae hujus axis a centro inertiae ». {

·4 $\mathrm{Hp}\cdot 2 .\supset. \exists(i,j,k)\ni[i,j,k\varepsilon U'v\text{-}\iota 0 . i\times j = j\times k = k\times i = 0 :$
$h\varepsilon U'v\text{-}\iota 0 .\supset_h . \mathrm{Inertia}(f, u, a+qh) =$
$(h\times i)^2\, \mathrm{Inertia}(f,u,a+qi) +$
$(h\times j)^2$ » » $j) +$
$(h\times k)^2$ » » $k)\,]$

} EULER id. p.176 {

Euler p.175, voca « axi principale de inertia » rectas $a+qi$, ... $a+qk$.

·5 $o\varepsilon p . r\varepsilon Q . i\varepsilon v\text{-}\iota 0 .\supset.$
$S\{[d(x, o+qi)]^2 | x, p \cap x\ni[d(x,o) \leqq r]\} = 8\pi r^5/15$
$S\{(x-o)^2 | x, p \cap x\ni[d(x,o) \leqq r]\} = 4\pi r^5/5$

❋ 11. Variatione de Integrale

·1 $f, \mathrm{D}_1 f, \mathrm{D}_2 f, \mathrm{D}_3\mathrm{D}_1 f, \mathrm{D}_3\mathrm{D}_2 f, \mathrm{D}_3^2 f \; \varepsilon \; \mathrm{qF}(\mathrm{q}\text{'}\mathrm{q}\text{'}\Theta)\mathrm{cont}$.
$u, \mathrm{D}_1 u, \mathrm{D}_2 u, \mathrm{D}_1^2 u \; \varepsilon \; \mathrm{qF}(\Theta\text{'}\Theta)\,\mathrm{cont} \,.\, y \varepsilon \Theta \;.\supset.$

$$\mathrm{D}[\mathrm{S}\{f[u(x,y), \mathrm{D}_1 u(x,y), x] \,|x, \Theta\} \,|\, y, \Theta]y$$
$$= \Delta\{\mathrm{D}_2 f[u(x,y), \mathrm{D}_1 u(x,y), x] \times \mathrm{D}_2 u(x,y) \,|x; 0, 1\}$$
$$+ \mathrm{S}([\mathrm{D}_1 f[u(x,y), \mathrm{D}_1 u(x,y), x] - \mathrm{D}\{\mathrm{D}_2 f[u(x,y), \mathrm{D}_1 u(x,y), x] | x, \Theta\}x] \times \mathrm{D}_2 u(x,y) \,|x, \Theta)$$

f es quantitate reale functio continuo de tres variabile, que nos infra voca u, u', x. Variabile u et u' sume omni valore reale, et x varia in intervallo determinato, per exemplo de 0 ad 1. Nos suppone existentia et continuitate de omni derivata de f que occurre in calculo sequente. Nos substitue ad u functio reale de duo variabile x et y, ambo in intervallo de 0 ad 1; et ad u' derivata de u pro x. Nos suppone continuitate de u, et de suo derivatas que occurre in calculo. y indica aliquo valore in intervallo Θ.

Nos considera integrale de f de variabiles $u(x,y)$, de suo derivata pro x, et de x, ubi integrale es pro variabile x, in intervallo de 0 ad 1; illo depende de valore de y. Derivata pro y de integrale præcedente, vale summa de duo quantitate:

1°. fac derivata de $f(u,u',x)$, pro u'; multiplica per derivata de $u(x,y)$ pro y; et calcula incremento si x varia de 0 ad 1.

2°. fac differentia inter duo quantitate A et B:

A: deriva $f(u,u',x)$ pro u.

B: deriva $f(u,u',x)$ pro u'; considera u et u' ut functiones de x, et deriva pro x. Multiplica differentia A—B per $\mathrm{D}_2 u(x,y)$, et integra pro x, in intervallo Θ.

Variabile y, pro que nos deriva integrale, vocare « parametro ». Derivata de u pro y, et derivata de integrale, in ce casu, sume nomen de « variatione ».

Dem.

$$\mathrm{D}[\mathrm{S}\{f\ldots |x, \Theta\} |y, \Theta]y = \mathrm{S}\{\mathrm{D}[f\ldots |y, \Theta]y \,|x, \Theta\}$$
$$= \mathrm{S}\{[\mathrm{D}_1 f\ldots \times \mathrm{D}_2 u.. + \mathrm{D}_2 f\ldots \times \mathrm{D}_2\mathrm{D}_1 u..] \,|x, \Theta\}$$
$$= \mathrm{S}([\quad \text{»} \quad + \mathrm{D}\{\mathrm{D}_2 f\ldots \times \mathrm{D}_2 u.. |x, \Theta\}x - \mathrm{D}\{\mathrm{D}_2 f\ldots |x, \Theta]x \times \mathrm{D}_2 u..] \,|x, \Theta)$$
$$= \Delta\{\mathrm{D}_2 f\ldots \times \mathrm{D}_2 u.. |x; 0, 1\} + \mathrm{S}([\mathrm{D}_1 f\ldots - \mathrm{D}\{\mathrm{D}_2 f\ldots |x, \Theta\}x] \times \mathrm{D}_2 u.. \,|x, \Theta)$$

In vero, derivata pro y de integrale pro x de $f(u, u', x)$ vale (pag. 438) integrale pro x de derivata pro y de f, et per regula de derivatione de

functione composito (p. 329), vale integrale de derivata pro u de f per derivata pro y de u, plus derivata pro u' de f per derivata pro y de u'.

Nunc, per regula de derivata de producto, et per commutatione de derivationes (p. 329), secundo termine vale derivata pro x de producto de derivata pro u' de f per derivata pro y de u, minus derivata pro x de derivata pro u' de f, multiplicato per derivata pro y de u.

Integra ; primo et tertio termine mane invariato ; in secundo, S pro x de D pro x vale incremento de functione, unde theorema seque.

·2 $a,b \varepsilon \mathrm{q} \,.\, f, \mathrm{D}_1 f, \mathrm{D}_2 f, \mathrm{D}_3\mathrm{D}_1 f, \mathrm{D}_3\mathrm{D}_2 f \;\varepsilon\; \mathrm{qf(q{:}q{:}}a\frown b)\mathrm{cont} \,.\, y, \mathrm{D}y \;\varepsilon\; (\mathrm{qF}a\frown b)\mathrm{cont} \,.\, \mathrm{S}[f(yx, \mathrm{D}yx, x)|x, a,b] = \max \mathrm{S}[f(gx, \mathrm{D}gx, x)|x; a,b] \mid g \text{‘}\, g\ni\{g, \mathrm{D}g \varepsilon \mathrm{qF}a\frown b \text{ cont} \,.\, ga = ya \,.\, gb = yb\} \;.\supset.$
$\{\mathrm{D}_1 f(yx, \mathrm{D}yx, x)|x, a\frown b\} = \mathrm{D}\{\mathrm{D}_2 f(yx, \mathrm{D}yx, x)|x, a\frown b\}$

{ Euler a.1744 p.42:

« Si Z fuerit functio ipsarum x,y et p [x, yx, Dyx] determinata, ita ut sit $\mathrm{d}Z = M\mathrm{d}x + N\mathrm{d}y + P\mathrm{d}p$; invenire inter omnes curvas eidem abscissæ respondentes, eam in qua sit $\int Z\mathrm{d}x$ maximum vel minimum.

Solutio :... Æquatio pro curva quæsita... erit... $N - \dfrac{\mathrm{d}P}{\mathrm{d}x} = 0$. » }

Nos considera $\mathrm{S}[f(yx, \mathrm{D}yx, x)|x; a,b]$, ubi y es functio reale definito in intervallo $a\frown b$, et f es functio reale de tres variabile, cum derivatas que occurre in calculo. Si in loco de y nos pone plure functione g, que coincide cum y pro valore extremo a et b, integrale sume plure valore. Si valore de integrale respondente ad functio y es maximo inter valores de integrale respondente ad omni functio g, tunc derivata de $f(yx, \mathrm{D}yx, x)$, facto pro yx, ut si Dyx et x es constante, vale quod resulta, si nos deriva f pro Dyx, ut si yx et x es constante, et postea deriva pro x, que occurre implicito in yx et in Dyx, et explicito in tertio loco.

Dem.

$f_1 = \mathrm{D}_1 f(yx, \mathrm{D}yx, x)|x \,.\, f_2 = \mathrm{D}_2 f(yx, \mathrm{D}yx, x)|x$.
$h = \{(x-a)(1-x)(f_1 x - \mathrm{D}f_2 x)\}|x$.
$g = \mathrm{S}[f(yx + thx, \mathrm{D}yx + t\mathrm{D}hx, x)|x; a,b]|t \;.\supset.$
$h0 = h1 = 0 \,.\, g0 = \max h\text{‘}\mathrm{q} \;.\supset.\; \mathrm{D}g0 = 0$.
$\mathrm{D}g0 = \mathrm{S}[x(1-x)(f_1 x - \mathrm{D}f_2 x)^2|x; a,b] \;.\supset.\; f_1 - \mathrm{D}f_2 = (\iota 0 : a\frown b)$

In vero, nos voca f_1 et f_2 derivatas partiale de f pro primo et secundo variabile respondente ad valores yx, Dyx, x, quando varia x, et nos considera valore de integrale respondente ad functio $gx = yx + thx$, ubi $hx = (x-a)(b-x)(f_1 x - \mathrm{D}f_2 x)$. Illo depende de parametro t. Pro $t = 0$, functio $g = y$, et integrale fi maximo. Ergo derivata de illo vale 0. Derivata, secundo P·1, vale $\mathrm{S}[(x-a)(b-x)(f_1 x - \mathrm{D}f_2 x)^2|x; a, b]$, que es nullo solo quando $f_1 x - \mathrm{D}f_2 x = 0$ pro omni valore de x in intervallo de integratione.

✻ 12. VARIATIONE DE ARCU.

·1 $p \varepsilon \mathrm{pF}(\Theta{:}\Theta) \,.\, \mathrm{D}_1 p, \mathrm{D}_1^2 p, \mathrm{D}_2 p \varepsilon (\mathrm{vF}\Theta)\mathrm{cont} \,.\, y \varepsilon \Theta \;.\!\supset.$

$$\mathrm{D}\{\mathrm{Arcu}[p(x,y)|x,\Theta] \mid y, \Theta\}y = \mathrm{D}\{\mathrm{S}[\mathrm{mod}\mathrm{D}_1 p(x,y)|x,\Theta]|y, \Theta\}y$$
$$= \mathrm{S}\{\mathrm{D}[\mathrm{mod}\mathrm{D}_1 p(x,y)|y, \Theta]y \,|x, \Theta\}$$
$$= \mathrm{S}[\mathrm{UD}_1 p(x,y) \times \mathrm{D}_2\mathrm{D}_1 p(x,y)|x,\Theta]$$
$$= \mathrm{S}(\{\mathrm{D}_1[\mathrm{UD}_1 p(x,y) \times \mathrm{D}_2 p(x,y)] - [\mathrm{D}_1\mathrm{UD}_1 p(x,y)] \times \mathrm{D}_2 p(x,y)\}|x,\Theta)$$
$$= \varDelta[\mathrm{UD}_1 p(x,y) \times \mathrm{D}_2 p(x,y) \,|x, \Theta]$$
$$- \mathrm{S}\{\mathrm{curvatura}[\,p(x,y) \mid x]x \times \mathrm{D}_2 p(x,y) \times \mathrm{mod}\mathrm{D}_1 p(x,y) \,|x, \Theta\}$$

Si p es puncto mobile, functio de variabile x et de parametro y, tunc variatione de arcu descripto per puncto, id es, derivata pro y de arcu descripto ab $p(x,y)$ dum varia x, vale incremento de projectione super tangente de variatione de extremos de arcu, minus integrale de producto interno de vectore-curvatura de curva per variatione de puncto, per valore absoluto de derivata de puncto.

In vero, arcu vale integrale de modulo de derivata de p pro x (p. 370 P·3); ergo D pro y de arcu, vel D pro y de S pro x, vale (p. 438) S pro x de D pro y de mod. de D pro x de p. Calcula derivata de modulo (p. 285 P16·3); commuta derivationes. Tunc uno termine es integrale pro x de derivata pro x, que vale incremento de functione. Secundo termine es expresso per curvatura de curva (p. 318 P49).

Nos pote etiam deduce ce theorema de P11·1; nos præfer demonstratione directo.

·2 $u \varepsilon\, \mathrm{Cls}'p \,.\, p \varepsilon\, u\mathrm{F}\Theta \,.\, \mathrm{Arcu}(p,\Theta) = \min\{\mathrm{Arcu}(q,\Theta)|\, q\text{‘}(u\mathrm{F}\Theta)$
$\frown q\ni(q0 = p0 \,.\, q1 = p1)\} \,.\, x \varepsilon \Theta \,.\, a \varepsilon\, \mathrm{Tang}(u,px) \;.\!\supset.$
$(a - px) \times \mathrm{curvatura}\, px \leqq 0$

Si u es figura, et p puncto mobile in u, et si arcu descripto per p es minimo inter arcus que pote es descripto per puncto q, mobile in u, que habe commune extremos cum præcedente, tunc vectore-curvatura de px fac angulo recto aut obtuso cum omni vectore $a - px$, ubi a es puncto arbitrario de figura tangente ad u in px.

Vectore $a - px$, ubi a es puncto arbitrario de figura tangente ad u in px vocare « variatione virtuale » de puncto.

Si figura u es superficie regulare, id es, es loco de punctos que satisfac ad Hp de p.332 §D P70, tunc habe in omni suo puncto plano tangente, et omni vectore in plano tangente es variatione virtuale de puncto.

Linea inter duo puncto de superficie, et de longore minimo, vocare «geodætica», nam occurre in Geodæsia.

Ergo vectore-curvatura vel normale principale, de geodætica es normale ad superficie.

{ JOH. BERNOULLI t. 4, p. 108 }

geodaetica, A geodetical, D geodätische, F géodésique, I geodetica.
⊂ G ge = Terra. Vide: geometria, pag. 202.
\+ -o-, litera de unione in G. Vide p. 202.
\+ dæ-, G δαίε = divide.
\+ -tico (vide: aritme-tico p. 65) — -o
\+ -a (p. 21 N. 37), intellige: linea.

geodaesia, G γεωδαισία, A geodesy, D Geodäsie, F géodésie, HI geodesia, R geodeziá. = divisione de Terra.

·3 $f \varepsilon\, \mathrm{qFp} \,.\, \mathrm{D}f \,\varepsilon\, (\mathrm{vFp})\mathrm{cont} \,.\, x \varepsilon\, \mathrm{pF}(\Theta{:}\Theta) \,.\, \mathrm{D}_1 x, \mathrm{D}_1^2 x, \mathrm{D}_2 x \,\varepsilon$ $\mathrm{vF}(\Theta{:}\Theta)\,\mathrm{cont} \,.\, v \varepsilon \Theta \,.\supset.\, \mathrm{D}\{\mathrm{S}[fx(u,v) \times \mathrm{mod}\mathrm{D}_1 x(u,v) \,|u,\Theta] \,|v, \Theta\}v$
$= \Delta[fx(u,v)\, \mathrm{U}\mathrm{D}_1 x(u,v) \times \mathrm{D}_2 x(u,v) \,|u; 0, 1]$
$+ \mathrm{S}[\{[\mathrm{comp} \perp \mathrm{D}_1 x(u,v)]\mathrm{D}fx(u,v) - fx(u,v)\, \mathrm{curvatura}[x(u,v)|u, u]\}$
$\times \mathrm{D}_2 x(u,v)\, \mathrm{mod}\mathrm{D}_1 x(u,v) \,|u,\Theta]$

·4 $f \varepsilon\, \mathrm{qFp} \,.\, \mathrm{D}f \,\varepsilon\, (\mathrm{vFp})\mathrm{cont} \,.\, x \varepsilon\, \mathrm{pF}\Theta \,.\, \mathrm{D}x, \mathrm{D}^2 x \,\varepsilon\, \mathrm{vF}\Theta \,.$
$\mathrm{S}[(fxt \times \mathrm{mD}xt) \,|t, \Theta] = \max\{\mathrm{S}[(fyt \times \mathrm{mD}yt) \,|t, \Theta] \,|y' (\mathrm{pF}\Theta) \cap y \ni$
$(y0 = x0 \,.\, y1 = x1 \,.\, \mathrm{D}y \,\varepsilon\, \mathrm{vF}\Theta)\} \,.\, t \varepsilon \Theta \,.\supset.$
$(\mathrm{cmp} \perp \mathrm{D}xt)\mathrm{D}(f, xt) = fxt \times \mathrm{curvatura}\, xt$

Si f es quantitate functione de positione puncto, vel potentiale (p. 334), que habe ut derivata vectore functione continuo de puncto, et si x es puncto functione dato de tempore in aliquo intervallo, p. ex. Θ, et si integrale de potentiale fxt per valore absoluto de velocitate, dum varia t in dato intervallo es maximo inter valores de idem integrale respondente ad omni alio motu y, que pro extremos de tempore coincide cum x, tunc pro omni valore de tempore t, sempre componente normale ad curva de vectore-derivata de f in puncto xt æqua fxt per vectore-curvatura de linea xt.

P. ex. si fx es pretio per metro lineare de constructione in puncto x, de ferro-via, et si puncto x describe linea tale que pretio de constructione de ferrovia per linea x es minimo inter pretios respondente ad omni alio via y, cum extremo identico ad priore, tunc, in omni puncto xt de linea, componente normale ad linea de vectore derivata de pretio, vale pretio multiplicato curvatura de via. Id es, curvatura de via æqua componente normale de derivata de logarithmo de pretio.

§ 39. Substitutione de vectores

✻ 1·0 H= (vFv)lin Df

H, lege « homographia », indica « vectore functione lineare de vectores », vel « substitutione de vectores ». Symbolo « lin » es definito in pag. 148. Satisfac conditiones :

$a\varepsilon$H . $u,v\varepsilon$v . $h\varepsilon$q .⊃.

·1 $au\varepsilon$v . $a(u+v) = au+av$. $a(hu) = h(au)$

Conditiones 1 et 2 es scripto in definitione. Conditione 3 seque de definitione de « lin », sed non de 1 et 2.

Si i,j,k es vectore non complanare, id es $iajak -= 0$, et si $i' = ai, j' = aj, k' = ak$ es vectores respondente in functione a, tunc a pote es scripto (pag. 149) :

$$a = (i',j',k')/(i,j,k),$$

ut ratione de duo triade de vectores correspondente.

Substitutione de vectores, ut substitutione de complexos de ordine 3, habe tres invariante.

Determinante de a, vel invariante de gradu 3, pote es definito ut ratione de duo trivectore (parallelepipedo) correspondente. Ergo nos sume tres vectore x,y,z non complanares:

$$x,y,z\varepsilon\text{v} .\ xayaz -= 0,$$

et considera ratione

$$(ax\ a\ ay\ a\ az)/(xayaz).$$

Si nos varia x,y,z et tribue ad illos omni triade de valores non complanare, ratione præcedente habe valore constante, determinante de substitutione a:

·2 Dtrm $a = \imath[(ax\ a\ ay\ a\ az)/(xayaz)\ |(x,y,z)`\,(x,y,z)\ni$
$(x,y,z\varepsilon\text{v} .\ xayaz -= 0)]$ Dfp

Si h es quantitate reale, determinante de substitutione $a+h$ es expresso (vide pag. 151 Prop. 5·4) per:

·3 $\mathrm{Dtrm}(a+h) = \mathrm{Dtrm}a + h\mathrm{Inv}_2 a + h^2\mathrm{Inv}_1 a + h^3$

Coefficientes de h et de h^2 es invariantes de gradu 2 et 1 de a. Nos deduce definitiones possibile de ce invariantes:

·5 $\mathrm{Inv}_2 a = \mathrm{D}[\mathrm{Dtrm}(a+h)|h, \mathrm{q}]0$ Dfp

·6 $\mathrm{Inv}_1 a = (1/2)\mathrm{D}^2[\mathrm{Dtrm}(a+h)|h, \mathrm{q}]0$ Dfp

Si nos evolve $\mathrm{Dtrm}(a+h)$ cum regula ·3, post facile reductione, nos deduce alio definitione possibile:

·7 $\mathrm{Inv}_2 a = \imath[(ax\, a\, ay\, az + ay\, a\, az\, ax + az\, a\, ax\, ay)/(xayaz)$
$|(x,y,z)$ ‘ ut in P·2 Dfp

·8 $\mathrm{Inv}a = \imath[(axayaz + ayazax + azaxay)/(xayaz)$
$|(x,y,z)$ ‘ etc. Dfp

Ex tres invariante, illo de secundo gradu habe pauco applicatione. Determinante occurre sæpe in Geometria. Invariante de primo gradu, vel « invariante » es de uso continuo in Physica-Mathematica. Me indica illo cum signo plus breve:

·9 $Sa = \mathrm{Inv}\, a$ Df

Lege Sa « scalare de a ». Vocabulo « scalare » es de Hamilton; vide pag. 186. Hic habe usu plus generale, nam nos applica illo ad omni substitutione de vectores in spatio, et non solo ad quaterniones, que es substitutiones speciale.

* 2. $i,j,k \varepsilon \mathrm{v}\, .\, iajak -= 0\, .\, u \varepsilon \mathrm{qf}(1\cdots3 \,\dot{:}\, 1\cdots3)\, .\, a =$
$(u_{11}i+u_{12}j+u_{13}k\, ,\, u_{21}i+u_{22}j+u_{23}k\, ,\, u_{31}i+u_{32}j+u_{33}k)/(i,j,k)\, .\supset.$

·1 $Sa = u_{11}+u_{22}+u_{33}$

·2 $\mathrm{Inv}_2 a = u_{11}u_{22}-u_{12}u_{21}+u_{22}u_{33}-u_{23}u_{32}+u_{33}u_{11}-u_{13}u_{31}$

·3 $\mathrm{Dtrm}a = \mathrm{Dtrm}[u_{11},u_{12},u_{13}\, ;\, u_{21},u_{22},u_{23}\, ;\, u_{31},u_{32},u_{33}]$

Si i,j,k es vectore non complanare, et si u es quantitate functione de duo indice què ambo sume valores 1, 2, 3, et si nos voca a substitutione que ad tres vectore dato fac corresponde functiones lineare de illos, cum coefficientes u, tunc formulas præcedente exprime invariantes de a, ope 9 numeros u, que vocare « coordinatas » de substitutione.

※ 3. V

·0 $a \varepsilon \mathrm{H} . \supset . Va = \iota\, \mathrm{v} \frown x \ni [u, v \varepsilon \mathrm{v} . \supset_{u,v} .$
$(au) \times v - (av) \times u = xauav/\psi]$ Df

Dato homographia a, tunc, si u et v es vectore, quantitate $(au)\times v-(av)\times u$ es functione lineare de u et de v, et functione alterno de (u,v). In generale, omni functione lineare et alterno de duo vectore u et v, es functione lineare de bivectore uav.

In nostro casu, expressione es reductibile ad ratione de trivectore $xauav$ ad trivectore unitario (pag. 196); ubi x es vectore que depende de a.

Nos indica vectore x per Va, lege « vectore de substitutione a ». Ita nos produce coincidentia cum notationes de Hamilton.

·1 $a,b \varepsilon (\mathrm{vFv})\mathrm{lin} . \supset . V(a+b) = Va + Vb . S(a+b) = Sa + Sb$

Operationes S et V super substitutiones es distributivo pro summa.

·2 Hyp P2 . $i^2 = j^2 = k^2 = 1 . i \times j = j \times k = k \times i = 0 . \supset .$
$Va = (u_{23} - u_{32})i + (u_{31} - u_{13})j + (u_{12} - u_{21})k$

Expressione de Va per coordinatas orthogonale de a.

·3 $u \varepsilon \mathrm{v} . \supset . V[\mathrm{I}(uax) | x, \mathrm{v}] = 2u$.
S——————— $= 0$.
Inv_2——————— $= u^2$.
Dtrm——————— $= 0$

Si u et x es vectore, uax es bivectore; suo indice $\mathrm{I}(uax)$ es vectore (vide pag. 198), dicto « velocitate de vectore x, in motu rotatorio repræsentato per vectore a ». Motu hic considerato es relativo ad vectores. Si nos introduce origine o, resulta casu particulare de velocitate et de motu considerato in pag. 269.

·4 $2a = \mathrm{D}[(au \times u) | u, \mathrm{v}] + [\mathrm{I}(Va)au | u, \mathrm{v}]$

Si u,v es vectore, exprime identitate

$$2(au)\times v = [(au)\times v + (av)\times u] + [(au)\times v - (av)\times u],$$

id es, decompone functione bilineare $2(au)\times v$ in functione symmetrico et in functione alterno de duo vectores.

Ergo omni homographia es summa de derivata de functione de secundo gradu $(au)\times u$, plus velocitate de rotatione.

Primo parte, que habe vectore nullo, repræsenta substitutione vocato « dilatatione ». Punctos $o+u$ tale que $au \times u =$ constante, forma superficie de secundo gradu de centro o.

✻ 4.

Nos habe definito, in pag. 330, derivata de numero reale aut complexo de ordine m, functione de complexo de ordine n, secundo JACOBI et GRASSMANN.

Si complexo variabile es p. ex. de ordine 3, et si u es functione, tunc derivata es functione lineare

$$(D_1 u,\ D_2 u,\ D_3 u)$$

que habe ut elementos derivatas partiale de u pro tres variabile.

Si functione u es quantitate reale, vel si u es « functione scalare » secundo Hamilton, et variabile independente es vectore, vel puncto, tunc derivata de u es vectore, ut es definito in pag. 334.

Si u es vectore functione de vectore, vFv, vel vectore functione de puncto, vFp, vel puncto functione de puncto, pFp, tunc Du es substitutione de vectores, et sæpe occurre SDu et VDu.

Nomenclatura et notationes super entes nunc considerato es multo vario in differente Auctores, ut semper fi in omni quæstione que se præsenta in idem tempore ad Auctores que tracta theoria differente. Varietate in notationes non depende solo de substitutione de uno signo ad alio, sed in modo speciale, de differente campo ubi functiones es definito.

Si u es functione de puncto de coordinatas (x,y,z), et i,j,k es vectores-quaternione coordinato (vide p.200), Hamilton a.1845 introduce operatione

$$\nabla = iD_1 + jD_2 + kD_3.$$

Derivata de Grassmann a. 1862, pote es definito per

$$D = (D_1,\ D_2,\ D_3).$$

Si u es numero, vel si u es functione scalare, tunc

$$\nabla u = -Du.$$

Du, jam vocato « parametro differentiale » et « Nabla » (vide pag. 334), habe accepto ab Maxwell a. 1871 nomen de « slope » (vocabulo Anglo, ligato ad A. « slip » = « gradi, es lubrico ». Es vocabulo de origine Germanico, parallelo ad Teutonico « schleifen ». Deriva secundo Fick p. 577 de Indo-Europæo, que habe producto L. « lub-rico »), postea mutato in L. « gradiente » nomen adoptato ab numeroso Auctore moderno de Electricitate, et reducto ad symbolo « grad u ».

Si u es vectore functione de puncto, tunc ∇u de Hamilton es quaternione, que habe 4 coordinata et non coincide cum Du de Grassmann, que habe 9 coordinata, id es 9 derivatas partiale de 3 coordinatas de vectore functione pro 3 coordinatas de puncto variabile. Sed subsiste relationes sequuente:

SDu = −S∇u de Hamilton = − convergentia de u, secundo Maxwell = divergentia de u, secundo Clifford a. 1878, nomen adoptato per Electricistas, et reducto ad symbolo « div u ».

VDu = V∇u de Hamilton, = « curl », « versione » de Maxwell = « rotatione » de Clifford, nomen adoptato ab Electricistas, et reducto ad symbolo « rot u ».

* 5. $m,n \varepsilon \mathrm{qFp} \,.\, u,v \varepsilon \mathrm{vFp} \,.\supset.$

·0 $S\mathrm{D}\,mu = mS\mathrm{D}u + \mathrm{D}m \times u$

·1 $V\mathrm{D}\,mu = mV\mathrm{D}u + \mathrm{I}(\mathrm{D}m)au$

·2 $S\mathrm{D}\,\mathrm{I}\,uav = u\times V\mathrm{D}v - v\times V\mathrm{D}u$

·3 $\mathrm{D}^2 m \;\varepsilon\; \mathrm{HFp\ cont} \,.\, p\varepsilon \mathrm{p} \,.\supset.\; V\mathrm{D}^2 mp = 0$

·4 $S\mathrm{D}\,V\mathrm{D}\,u = 0$

·5 $S\mathrm{D}^2(mn) = m\,S\mathrm{D}^2 n + n\,S\mathrm{D}^2 m + 2(\mathrm{D}m)\times(\mathrm{D}n)$

Vide Heaviside, *Electromagnetic Theory* a. 1893 p. 195, 201.

Indicationes historico et bibliographico de præsente § es tracto ex:
C. Burali-Forti e R. Marcolongo, *Per l'unificazione delle notazioni vettoriali*, PalermoR. a.1907 t.23 p.324, Nota I, et sequentes.

·6 $a\varepsilon \mathrm{p} \,.\, x\varepsilon \mathrm{p}-\iota a \,.\supset.\; S\mathrm{D}^2[1/\mathrm{d}(x,a)\,|x,\mathrm{p}]x = 0$

·7 $a\varepsilon \mathrm{p}_2 \,.\, x\varepsilon \mathrm{p}-a \,.\supset\; S\mathrm{D}^2\log \mathrm{d}(x,a)\,|x,\mathrm{p}]x = 0$

* 6. Integrale de linea et de superficie.

Si u es vectore functione de puncto, vel u es « campo vectoriale » et si p es puncto functione de uno variabile x, in intervallo h, tunc

S[$(upx)\times \mathrm{D}px|x, \Theta$] = labore de fortia u, vel
= circuitatione de vectore u, pro linea descripto per p, in intervallo h (Heaviside).

Si p es puncto functione de duo numero, in duo dato intervallo h et k, tunc

$$\mathrm{S}[up(x,y)\ \alpha\ \mathrm{D}_1 p(x,y)\ \alpha\ \mathrm{D}_2 p(x,y) \mid (x,y),\ h{:}k]$$

$=$ fluxu de vectore u, trans superficie $(p, h{:}k)$.

·1 $u \varepsilon \mathrm{vFp} . \mathrm{D}u\ \varepsilon\ \mathrm{HFp\ cont} . a,b,c,d\varepsilon \mathrm{q} . p\varepsilon\ \mathrm{pF}(a\frown b : c\frown d) .$ $\mathrm{D}_1 p, \mathrm{D}_2 p, \mathrm{D}_1\mathrm{D}_2 p\ \varepsilon\ \mathrm{vF}(a\frown b : c\frown d)\ \mathrm{cont} . \supset .$

$$\varDelta\{\mathrm{S}[up(x,y) \times \mathrm{D}_2 p(x,y) | y;\ c,d]\ |x;\ a,b\}$$
$$- \varDelta\{\mathrm{S}[up(x,y) \times \mathrm{D}_1 p(x,y) | x;\ a,b]\ |y;\ c,d\}$$
$$= \mathrm{S}\{\mathrm{D\,S}[up.. \times \mathrm{D}_2 p..\ |y;\ c,d]\ |x;\ a,b\}$$
$$- \mathrm{S}\{\mathrm{D\,S}[up.. \times \mathrm{D}_1 p..\ |x;\ a,b]\ |y;\ c,d\}$$
$$= \mathrm{S}\{\mathrm{S}\,[(\mathrm{D}up..\mathrm{D}_1 p..) \times \mathrm{D}_2 p.. + up..\ \mathrm{D}_1\mathrm{D}_2 p..\ |y;\ c,d]\ |x;\ a,b\}$$
$$- \mathrm{S}\{\mathrm{S}\,[(\mathrm{D}up..\mathrm{D}_2 p..) \times \mathrm{D}_1 p.. + up..\ \mathrm{D}_2\mathrm{D}_1 p..\ |x;\ a,b]\ |y;\ c,d\}$$
$$= \mathrm{S}\{\mathrm{S}\,[(\mathrm{D}up..\mathrm{D}_1 p..) \times \mathrm{D}_2 p.. - (\mathrm{D}up..\mathrm{D}_2 p..) \times \mathrm{D}_1 p..\ |x;\ a,b]\ |y;\ c,d\}$$
$$= \mathrm{S}\{\mathrm{S}[V\mathrm{D}up(x,y)\ \alpha\ \mathrm{D}_1 p(x,y)\ \alpha\ \mathrm{D}_2 p(x,y)\ /\psi\ |x;\ a,b]\ |y;\ c,d\}$$

{ Stokes, Cambridge University Calendar a.1854 }

Si u es campo vectoriale, et si p es puncto functione de duo numero, primo in intervallo de a ad b, secundo in illo de c ad d, tunc circuitatione de vectore u pro perimetro de superficie de p, in ordine (a,c) (b,c) (b,d) (a,d), vale fluxu de vectore de derivata de u, id es fluxu de rotatione de u, trans superficie de p. Transforma integrale pro perimetro in integrale pro superficie.

Dem. resulta ex calculo. Incremento de functione inter duo limite vale integrale de suo derivata (p.349 P12·2). Commuta derivata cum secundo integrale (p.438), commuta integrationes et derivationes. Differentia de duo termine es nullo. Differentia que remane, per definitione de vectore de homographia, habe valore scripto.

·2 $u\varepsilon \mathrm{vFp} . \mathrm{D}u\ \varepsilon\ \mathrm{HFp\ cont} . p\varepsilon\ \mathrm{pF}(\Theta{:}\Theta{:}\Theta) . \mathrm{D}_1 p, \mathrm{D}_2 p, \mathrm{D}_3 p,$ $\mathrm{D}_1\mathrm{D}_2 p, \mathrm{D}_1\mathrm{D}_3 p, \mathrm{D}_2\mathrm{D}_3 p\ \varepsilon\ \mathrm{vF}(\Theta{:}\Theta{:}\Theta)\ \mathrm{cont} . \supset .$

$$\varDelta\{\mathrm{S}[(up\ \alpha\ \mathrm{D}_2 p\ \alpha\ \mathrm{D}_3 p)(x,y,z)\ /\psi\ |y;\ 0,1\ |z;\ 0,1]\ |x;\ 0,1\}$$
$$+ \varDelta\{\mathrm{S}[(up\ \alpha\ \mathrm{D}_3 p\ \alpha\ \mathrm{D}_1 p)\quad \text{»}\quad |z\ \text{»}\quad |x\ \text{»}\quad |y\ \text{»}$$
$$+ \varDelta\{\mathrm{S}[(up\ \alpha\ \mathrm{D}_1 p\ \alpha\ \mathrm{D}_2 p)\quad \text{»}\quad |x\ \text{»}\quad |y\ \text{»}\quad |z\ \text{»}$$
$$= \mathrm{SSS}[(\mathrm{D}up\ \mathrm{D}_1 p)\ \alpha\ \mathrm{D}_2 p\ \alpha\ \mathrm{D}_3 p + (\mathrm{D}up\ \mathrm{D}_2 p)\ \alpha\ \mathrm{D}_3 p\ \alpha\ \mathrm{D}_1 p +$$
$$(\mathrm{D}up\ \mathrm{D}_3 p)\ \alpha\ \mathrm{D}_1 p\ \alpha\ \mathrm{D}_2 p / \psi,\ \Theta{:}\Theta{:}\Theta]$$
$$= \mathrm{S}[(S\mathrm{D}up)(\mathrm{D}_1 p\ \alpha\ \mathrm{D}_2 p\ \alpha\ \mathrm{D}_3 p)\ /\psi,\ \Theta{:}\Theta{:}\Theta]$$

{ Ostrogradsky, Mém. Ac. Petersb. t.1 p.39, a.1828 }

{ Gauss a.1840, Werke t.5 p.195 }

Si u es campo vectoriale, et si p es puncto functione de tres variabile in dato intervallos, p. ex. Θ, Θ, Θ, tunc fluxu de de vectore u trans sex superficie que limita parallelepipedo curvilineo descripto per p, id es sex superficie descripto per $p(x,y,z)$, ubi singulo variabile sume valores 0 et 1, vale integrale de divergentia (SD) de u, in volumen campo de p.

Transformatione de integrale pro superficie de solido $p'(\Theta{:}\Theta{:}\Theta)$ in integrale pro solido.

✻ 7. $c \varepsilon \text{Cls'p} \,.\supset.$

·1 $c \varepsilon \text{cont}_1 \,.=:\, x,y \varepsilon c \,.\supset_{x,y}.\, \exists\, (c\text{F}\Theta)\text{cont} \frown h\ni(h0 = x \,.\, h1 = y)$ Df

Classe de punctos, vel figura, c, vocare « continuo de ordine 1 », vel « continuo lineare » vel « continuo », si nos pote uni duo suo puncto arbitrario per linea continuo h in figura c.

·2 $c \varepsilon \text{cont}_2 \,.=:\, f,g \varepsilon (c\text{F}\Theta)\text{cont} \,.\, g0 = f0 \,.\, g1 = f1 \,.\supset_{f,g}.$
$\exists\, [c\text{F}(\Theta{:}\Theta)]\text{cont} \frown h\varepsilon\{[h(x,0)|x,\Theta] = f \,.\, [h(x,1)|x,\Theta] = g :$
$y\varepsilon\Theta \,.\supset_y.\, [h(x,y)|x, \iota0\cup\iota1] = (f, \iota0\cup\iota1)$ Df

Figura c vocare « cóntinuo de ordine 2 », vel « continuo superficiale » vel « acyclico », si, dato duo linea f et g, in c, cum extremos commune, nos pote uni ce duo linea per serie continuo de lineas, formante superficie in c, cum extremos commune cum f.

·3 $c \varepsilon \text{cont}_3 \,.=:\, f,g \varepsilon [c\text{F}(\Theta{:}\Theta)]\text{cont} \,.\, [g, (\iota0\cup\iota1 {:}\Theta)\cup(\Theta{:}\, \iota0\cup\iota1)] =$
$[f, (\iota0\cup\iota1 {:}\Theta)\cup(\Theta{:}\, \iota0\cup\iota1)] \,.\supset_{f,g}.$
$\exists\, [c\text{F}(\Theta{:}\Theta{:}\Theta)]\text{cont} \frown h\ni\{[h(x,y,0)|(x,y),\Theta{:}\Theta] = f \,.$
$[h(x,y,1)|(x,y),\Theta{:}\Theta] = g : z\varepsilon\Theta \,.\supset_z.\, [h(x,y,z)|(x,y),(\iota0\cup\iota1 {:}\Theta)\cup$
$(\Theta{:}\, \iota0\cup\iota1)] = [f, (\iota0\cup\iota1{:}\Theta)\cup(\Theta{:}\, \iota0\cup\iota1)]$ Df

Figura c vocare « continuo de ordine 3 », si dato duo superficie in c, cum peripheria commune, semper lice uni duo superficie cum serie continuo de superficie in c, cum peripheria commune cum præcedentes

·4 $\text{Med}c = c \,.\supset.\, c \,\varepsilon\, \text{cont}_1 \frown \text{cont}_2 \frown \text{cont}_3$

Si omni puncto medio inter c pertine ad c, id es, si figura c es convexo, tunc es continuo de omni ordine.

·5 $o \varepsilon \mathrm{p} \,.\, i \varepsilon \mathrm{v}{-}\iota 0 \,.\, a,b \varepsilon \mathrm{Q} \,.\, a > b \,.\, u = \mathrm{p} \frown x \ni [(x-o) \times i = 0 \,.$
$\mathrm{d}(x,o) = a] \,.\supset.\, \mathrm{p} \frown x \ni [\mathrm{d}(x,u) < b] \; \varepsilon \; \mathrm{cont}_1 \frown {-}\mathrm{cont}_2 \frown \mathrm{cont}_3$

Si nos voca u circumferentia de centro puncto o, et de radio a, tunc anulo vel toro formato de punctos que dista de u minus que $b < a$, es continuo de ordine 1 et 3, et non de ordine 2.

·6 $o \varepsilon \mathrm{p} \,.\, a \varepsilon \mathrm{Q} \,.\supset.\, \mathrm{p} \frown x \ni [\mathrm{d}(x,o) > a] \; \varepsilon \; \mathrm{cont}_1 \frown \mathrm{cont}_2 \frown {-}\mathrm{cont}_3$

Solido ex sphæra de centro o et radio a, es continuo de ordine 1 et 2, et non de ordine 3.

·7 $a,b \varepsilon (\mathrm{Cls'p})\mathrm{cont}_1 \,.\, \exists a \frown b \,.\supset.\, a \smile b \; \varepsilon \; \mathrm{cont}_1$

✻ 8·1 $c \varepsilon (\mathrm{Cls'p})\mathrm{cont}_2 \,.\, u \varepsilon \mathrm{vF}c \,.\, V \mathrm{D}u = (\iota 0 ; c) \,.$
$f,g \varepsilon (c\mathrm{F}\Theta)\mathrm{cont} \,.\, f0 = g0 \,.\, f1 = g1 \,.\supset.\, \mathrm{S}[uf \times \mathrm{D}f, \Theta] = \mathrm{S}[ug \times \mathrm{D}g, \Theta]$

Si figura c es continuo superficiale, si ad omni puncto de campo c responde vectore, vel fortia u, et si vectore de derivata de u es semper nullo in campo c, tunc $\mathrm{S}(uf \times \mathrm{D}f, \Theta)$, que vocare «labore» de fortia u, quando puncto de applicatione f sume motu arbitrario, non varia si nos muta trajectoria de puncto, servato suo extremos. In ce casu $uf \times \mathrm{D}f$ vocare «differentiale exacto», et campo c «irrotationale».

·2 $c \varepsilon (\mathrm{Cls'p})\mathrm{cont}_3 \,.\, u \varepsilon \mathrm{vF}c \,.\, S \,\mathrm{D}u = (\iota 0 ; c) \,.\, f,g \varepsilon [c\mathrm{F}(\Theta ; \Theta)]\mathrm{cont} \,.$
$[f, (\iota 0 \smile \iota 1 ; \Theta) \smile (\Theta ; \iota 0 \smile \iota 1)] = [g, (\iota 0 \smile \iota 1 ; \Theta) \smile (\Theta ; \iota 0 \smile \iota 1)] \,.\supset.$
$\mathrm{S}[uf \; a \; \mathrm{D}_1 f \; a \; \mathrm{D}_2 f, \Theta ; \Theta] = \mathrm{S}[ug \; a \; \mathrm{D}_1 g \; a \; \mathrm{D}_2 g, \Theta ; \Theta]$

Si campo c es continuo de ordine 3, et si ad omni suo puncto responde vectore u, et si scalare de derivata de u es semper nullo in c, tunc si puncto f describe superficie, $\mathrm{S}[uf \, a \, \mathrm{D}_1 f \, a \mathrm{D}_2 f, \Theta ; \Theta]$, jam vocato «fluxu de u trans superficie», non varia si nos muta superficie, servato perimetro. In ce casu, functio u, vel distributione de vectore $\boldsymbol{u}$, vocare «solenoidale».

solenoidale (Maxwell, A treatise on Electricity and Magnetism, Oxford a.1881 p. 20) indica motu de liquido secundo canales impermeabile. $\subset$ solenoide — -e + -ale.

solenoide = in forma de tubo. $\subset$ solen + -o- + -ide.
solen, G σωλήν = canale, tubo.

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